

# PHASE DIAGRAMS FOR LARGE N GAUGE THEORIES ON COMPACT SPACES

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hep-th/0506???

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## INTRODUCTION:

- Want to understand thermodynamic properties of gauge theories (e.g. QCD)

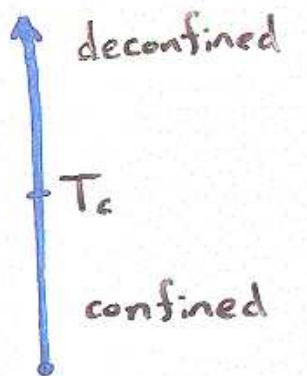
WHY?

- Relevant for:
  - Early universe
  - R.H.I.C.
  - Neutron stars
- Different d.o.f. important at different temperatures
  - high T: QGP  
↓  $10^{-5}$  s
  - low T: hadrons
- large  $N_c$ : learn about strings, black holes, ... via gauge/gravity duality

## THIS TALK:

Pure  $SU(N)$  Yang-Mills thermodynam.  
↑  
No quarks      ↑  
# of colours      (later: + adjoint matter)

Infinite volume:



numerical work:

$N=2 \rightarrow$  2nd order  
 $N \geq 3 \rightarrow$  1st order

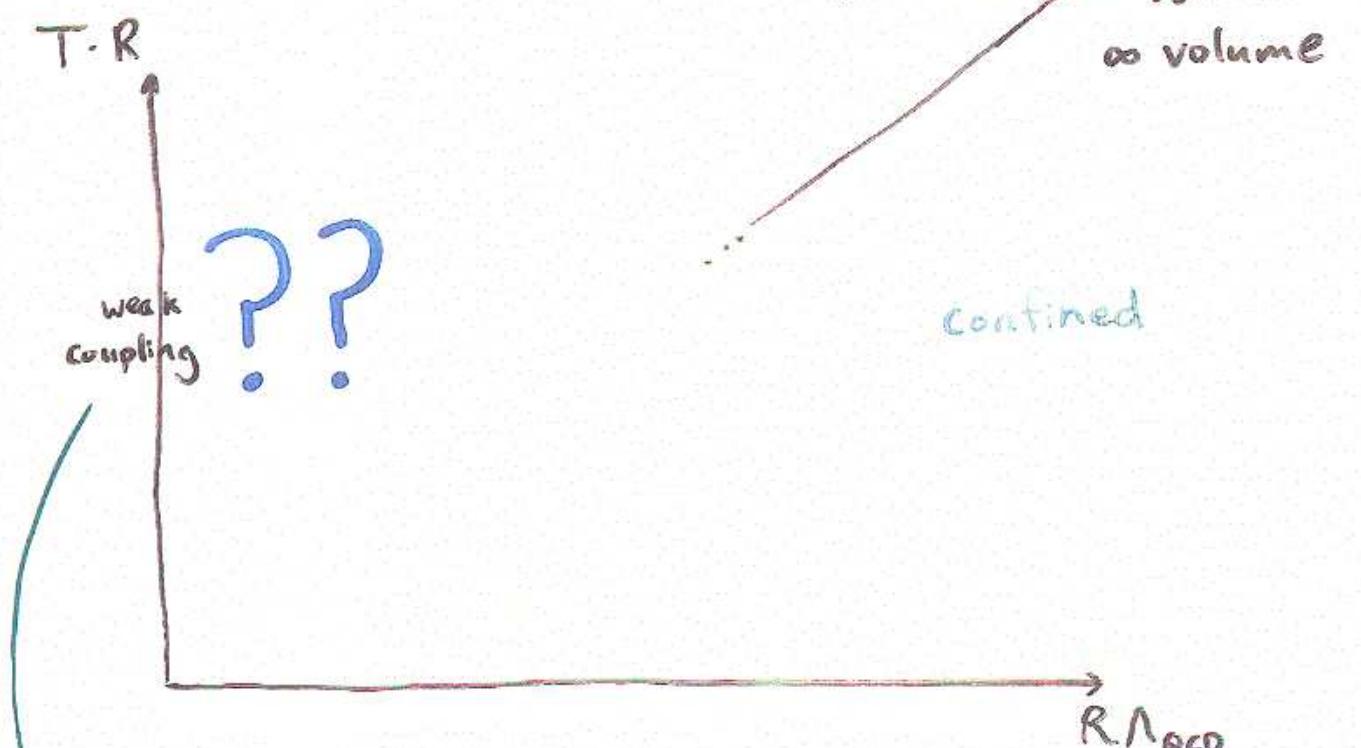
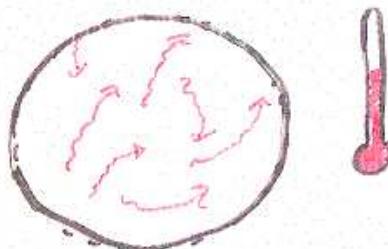
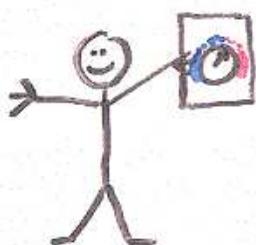
strong coupling at  $T_c$   
→ No available analytic methods

ASYMPTOTIC FREEDOM → simpler at small volume

$R\Lambda_{QCD} \ll 1 \rightarrow$  weak coupling

Transitions smoothed for finite volume, but large  $N_c$  behavior remains sharp!

$\therefore$  Study large  $N$   $U(N)$  Yang-Mills Theory on  $S^3$



Study partition function:

$$Z = \sum_{\text{States on } S^3} e^{-\beta E_i} = \int [dA_\mu] e^{-\int_{S^3 \times S^1} \mathcal{L}_{\text{Eucl}}}$$

# Order Parameters for Confinement

① Free energy of external quark

$$Z_{\text{quark}} = e^{-\beta F_{\text{quark}}} \begin{cases} 0 & \text{confined} \\ \neq 0 & \text{deconfined} \end{cases}$$

$$= \langle W \rangle \quad W = \text{Tr} \left( P e^{i \int_0^{\beta} A_0 dt} \right)$$

POLYAKOV LOOP

compact space:  
look at  $\langle |W|^2 \rangle$

② Free energy/ $N^2$

confined:  $F \sim \mathcal{O}(1)$       glueballs

deconfined:  $F \sim \mathcal{O}(N^2)$       gluons

## The Calculation:

- fix gauge: ①  $\vec{\nabla} \cdot \vec{A} = 0$

$$\textcircled{2} \quad \partial_t \int_{S^3} A_0 = 0 \Rightarrow \int_{S^3} A_0 = \text{constant}$$

III  
 $\alpha$

- all other modes massive  $A_\mu = \sum Y_{lm}^\mu(\Omega) A_{lm}(t)$

↓ INTEGRATE OUT

effective action for  $\alpha$

$$Z = \underbrace{\int [d\alpha] \Delta_{F.P.}^{\textcircled{1}} \int [dA'] \Delta_{F.P.}^{\textcircled{1}} e^{-S[\alpha, A']}}_{\int [du] \text{ Haar Measure}} \exp(-S_{\text{eff}}(u))$$

$$U = e^{i\alpha\beta} = e^{i \int_0^\beta A_0 dt} \text{ (avg over sphere)}$$

Partition function reduces to unitary matrix model

## Results: One loop ( $\lambda \rightarrow 0$ )



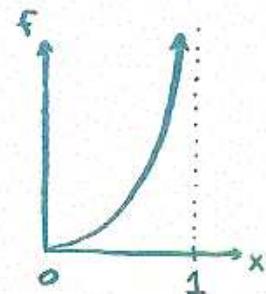
define  $x = e^{-\frac{1}{T}}$

↑  
0 low T

↓ 1 high T

$$Z = \int [du] e^{\sum f(x^n) \text{tr}(u^n) \text{tr}(u^{+n})}$$

$$f(x) = \sum_{\text{1 particle}} e^{-\beta E} = \frac{6x^2 - 2x^3}{(1-x)^3}$$



large  $N$ : saddle point techniques

$\rho(\theta)$  eigenvalue density

$$\text{Define } u_n = \int \rho(\theta) e^{in\theta} d\theta = \frac{1}{N} \text{tr}(u^n)$$

$$Z = \int du_n \bar{du}_n e^{-N^2(1-f(x^n))|u_n|^2}$$

↑  
from Jacobian

$\lambda \rightarrow 0$  model reduces to Gaussian Integral

# Order Parameters for $\lambda=0$

$$Z = \int du_n d\bar{u}_n e^{-N^2(1-f(x^n))|u_n|^2}$$

low T:

$f(x) < 1$  : saddle pt  $u_n = 0$

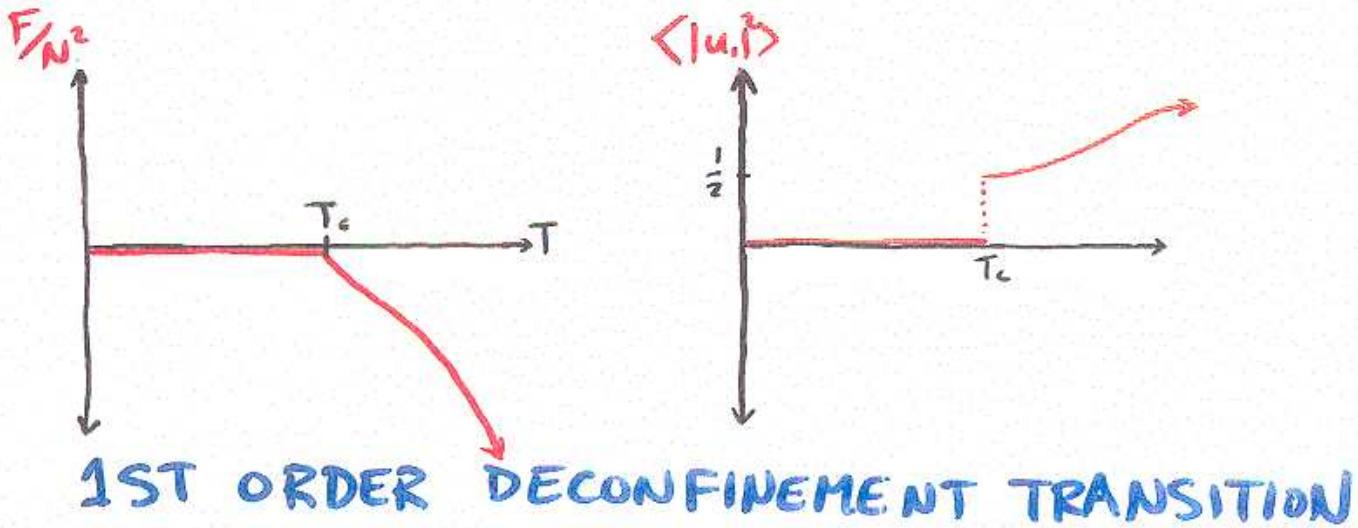
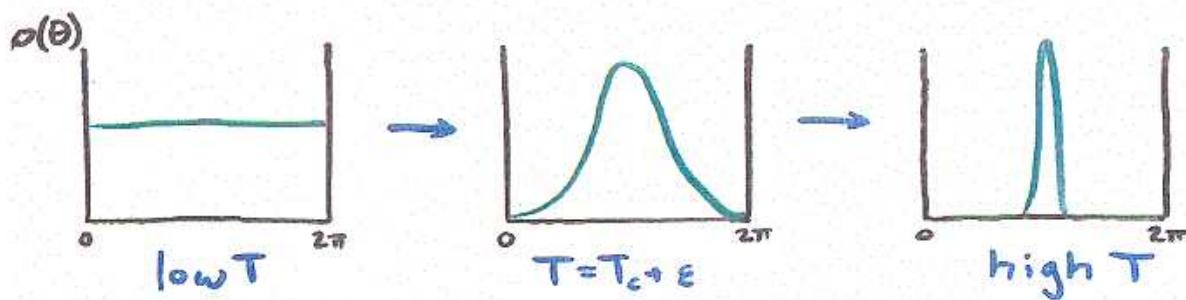
$$F/N^2 = 0$$

$$\langle |u_i|^2 \rangle = 0$$

$f(x) > 1$  :  $u$ , unstable  $\rightarrow$  condenses

$$F = \Theta(N^2)$$

$$\langle |u_i|^2 \rangle \neq 0$$



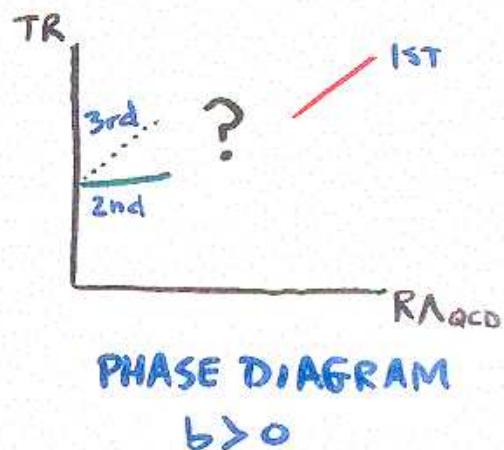
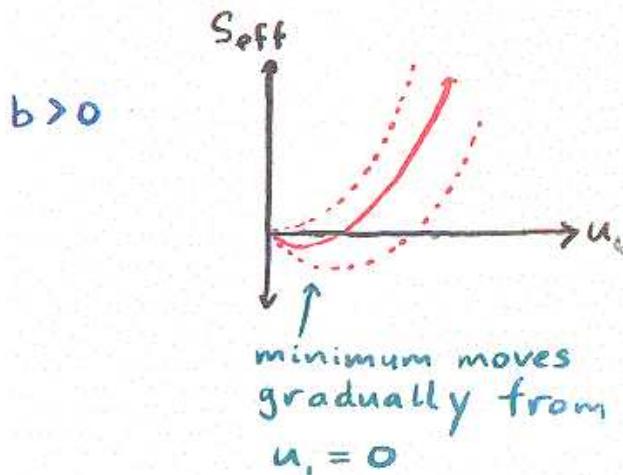
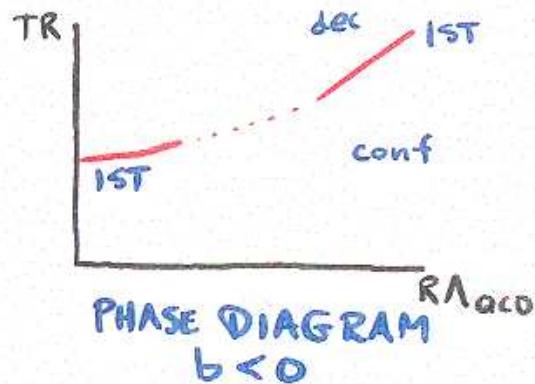
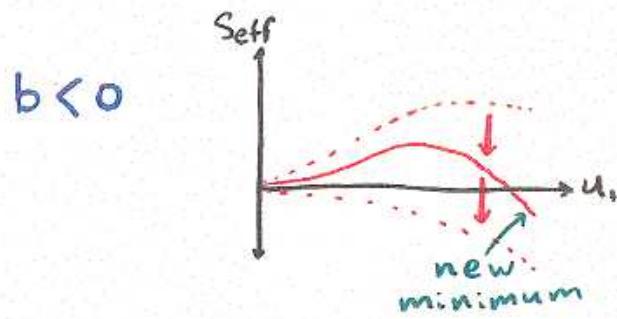
BUT: Behavior sensitive to perturbative corrections

$$8 \odot \rightarrow 2 \text{ loops} \rightarrow 3 \text{ traces} \rightarrow \begin{matrix} \text{cubic terms} \\ \text{in } S(u_n) \end{matrix}$$

$$000 \dots \rightarrow 3 \text{ loops} \rightarrow 4 \text{ traces} \rightarrow \begin{matrix} \text{quartic} \\ \text{terms in} \\ S(u_n) \end{matrix}$$

Focus on  $u_+$ :

$$S_{\text{eff}}(u_+) = \mu(T)|u_+|^2 + \lambda^2 b|u_+|^4 + O(\lambda^4)$$



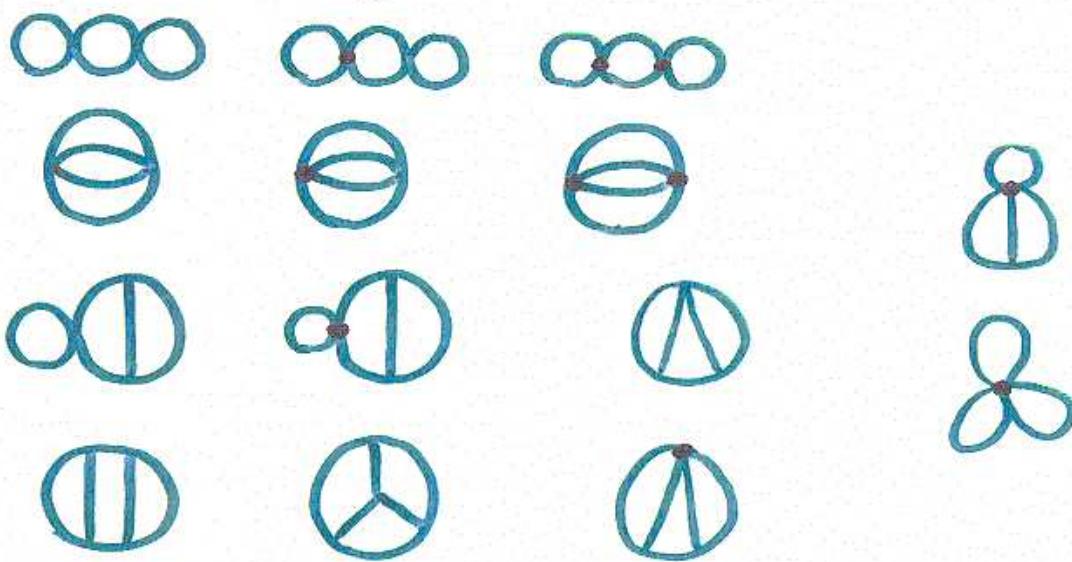
# Calculating $b$ for pure YM

$b$  determined by

3 2-loop diagrams



14 3-loop diagrams

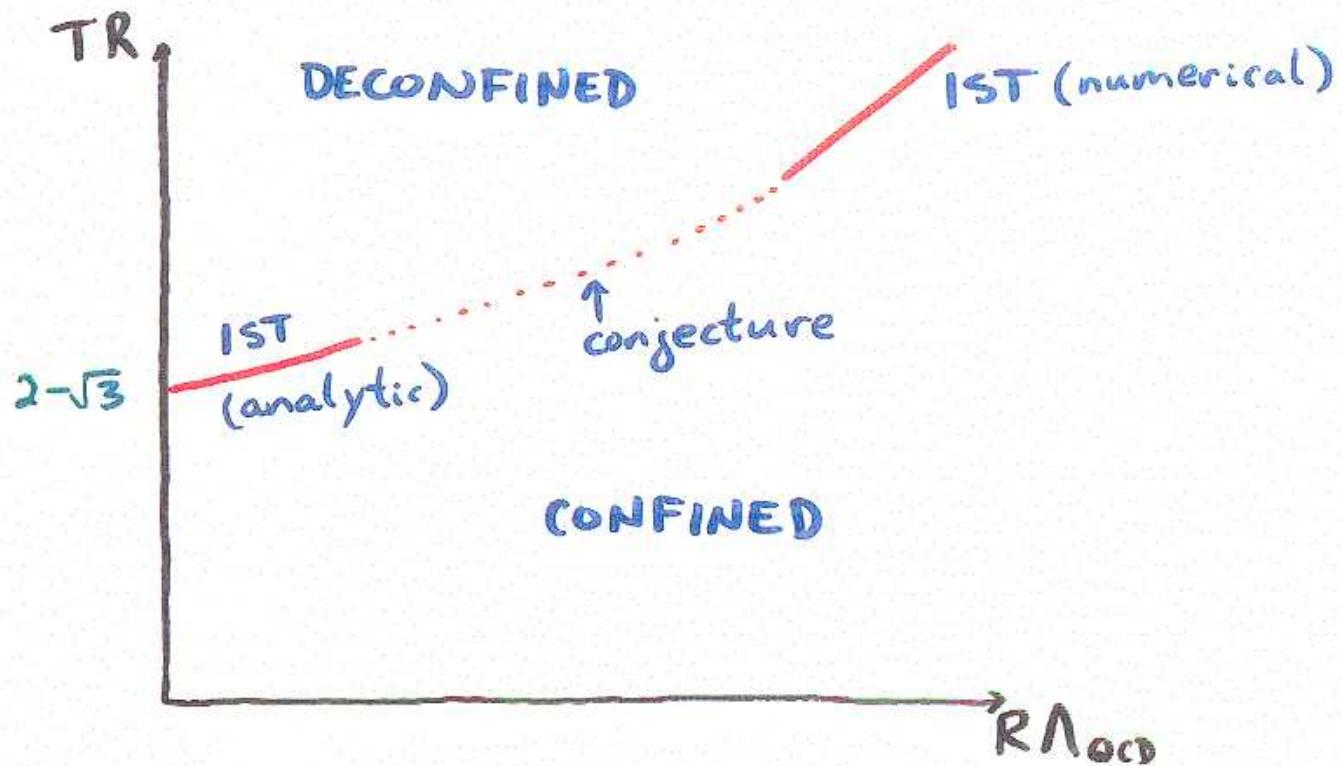


Counter terms



Result :  $b \approx -5.8 \times 10^{-4}$

**CONCLUSION:** The phase diagram for large  $N$  Yang-Mills on  $S^3$



Conjecture: phase structure independent  
of size of  $S^3$

GENERALIZATION: Arbitrary adjoint matter  
Any space w. no zero modes

$\lambda = 0$ : results qualitatively unchanged  
- 1st order transition when

$$f(x) = \sum_{\text{modes}} x^E = 1$$

$0 < \lambda \ll 1$ : behavior depends on details of theory  
- need 3-loop calculation

GENERAL QUESTION: Which spaces + matter content give  $b > 0$ ?  $b < 0$ ?



## EXAMPLES WITH $b > 0$

Consider a 0+1d theory:

$$\mathcal{L} = \text{tr} \left[ |D_0 \phi|^2 + \frac{1}{2} m^2 |\phi|^2 + \lambda |\phi|^4 + \lambda^2 c_6 |\phi|^6 + \lambda^3 c_8 |\phi|^8 \right]$$

$c_8 > 0$

Then  $b$  is linear in  $c_6$ :

$$\text{coeff} (\text{tr}(U)^4) = \text{Diagram}_1 + \text{Diagram}_2 + \text{Diagram}_3$$

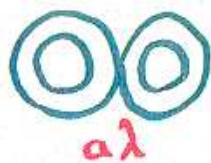
$c_6$

$$b = ac_6 + \dots$$

Can choose  $c_6$  to make  $b$  positive

Similar story: theories with double trace terms

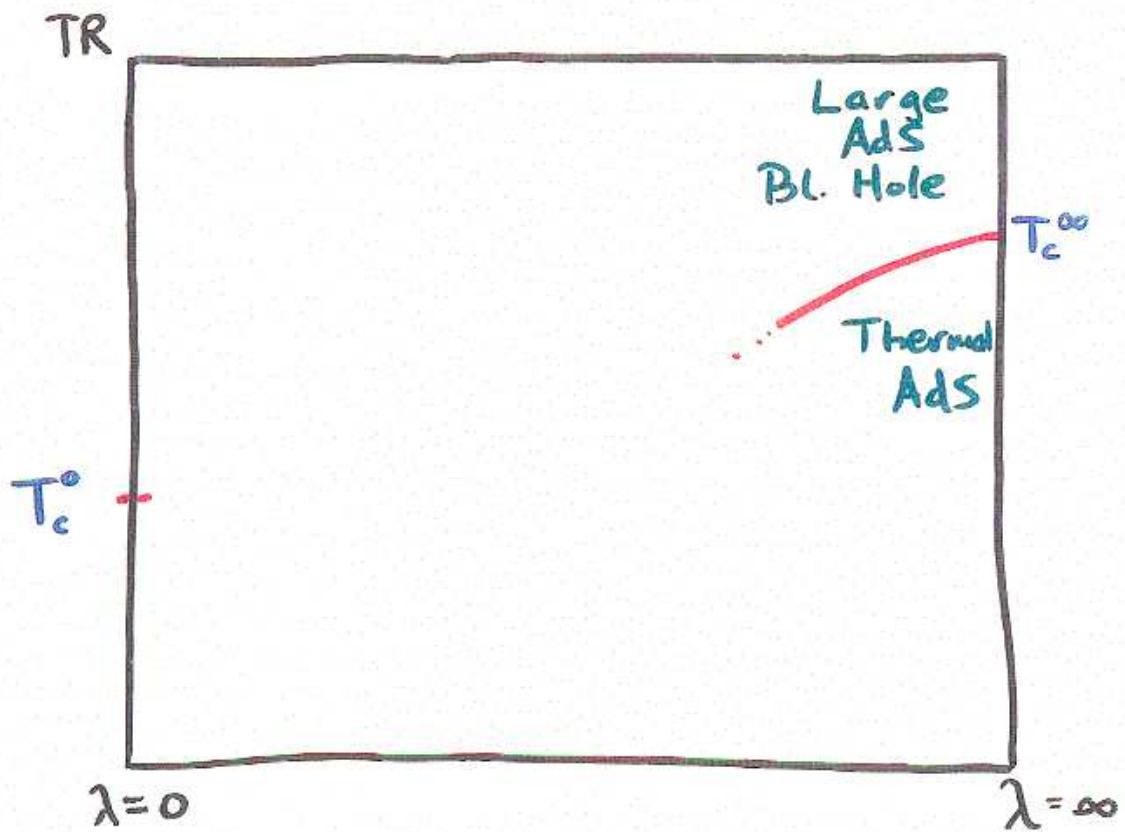
$$\mathcal{L} = \dots + a \frac{\lambda}{N} \text{tr}(\phi^2) \text{tr}(\phi^2) + c \lambda \text{tr}(\phi^4) + \dots$$



→ linear contribution to  $b$

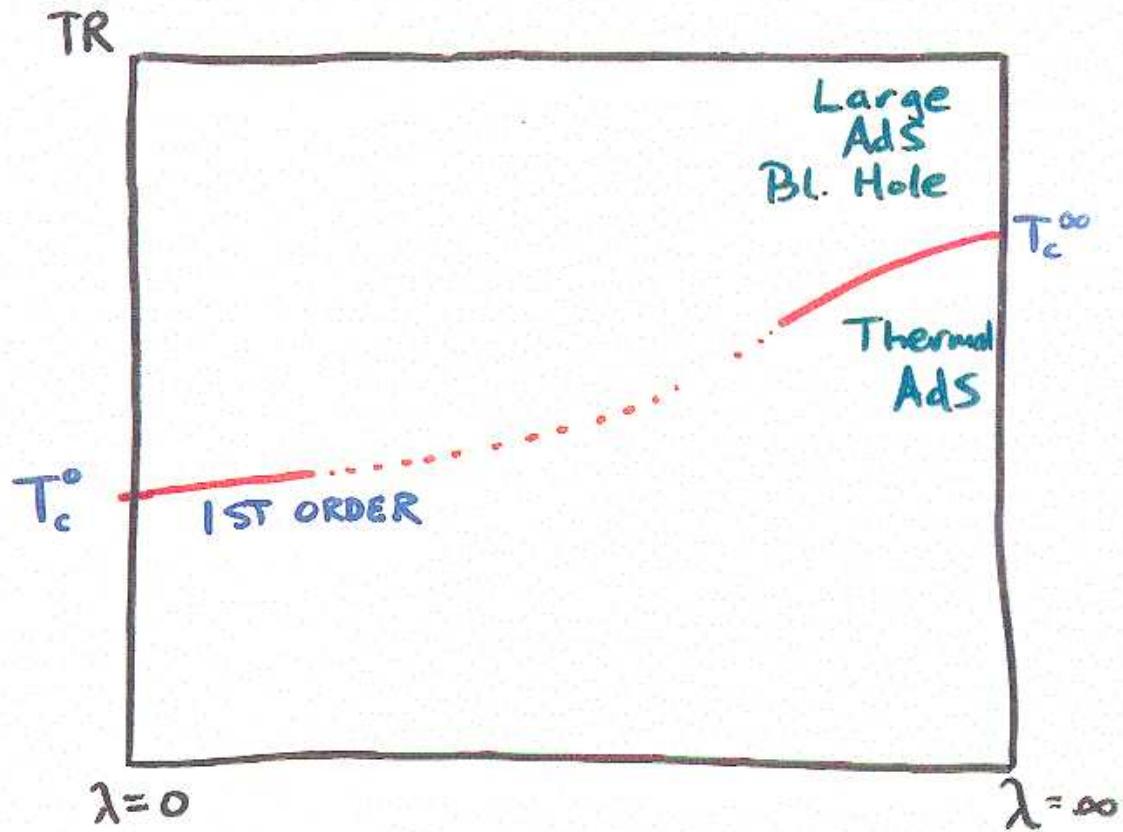
# WHAT ABOUT $N=4$ SYM ON $S^3$ ?

Large  $\lambda$ : analyze via supergravity on  
global  $AdS^5 \times S^5$



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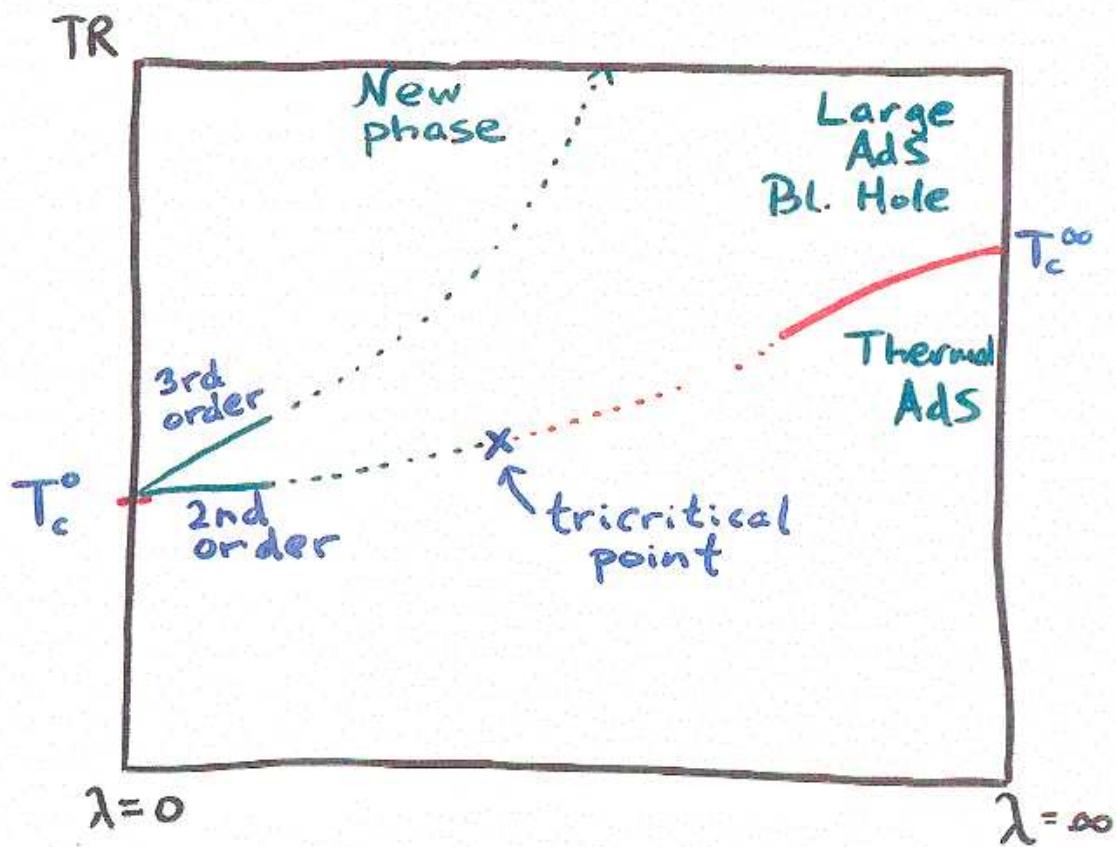


1st possibility:  $b < 0$

Probe black hole phase with weakly  
coupled high  $T$  field theory?

# WHAT ABOUT $N=4$ SYM ON $S^3$ ?

Large  $\lambda$ : analyze via supergravity on global  $AdS^5 \times S^5$



2nd possibility:  $b > 0$

- Tricritical point at special  $\lambda$
- At least one new phase

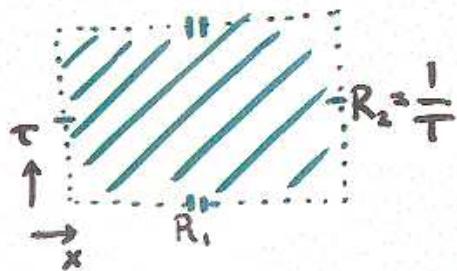
# NONTRIVIAL SPATIAL TOPOLOGY

e.g. torus



- Extra zero modes associated with Wilson lines about noncontractible spatial cycles
- Multi-matrix model at small volume

Simplest example: 1+1d YM + adjoint matter on  $S^1$



$$\left. \begin{aligned} U &= Pe^{i \int A_0 d\tau} \\ V &= Pe^{i \int A_1 dx} \end{aligned} \right\}$$

2 order parameters  
4 possible phases

# BOSONIC 2D YM + VERY MASSIVE SCALARS

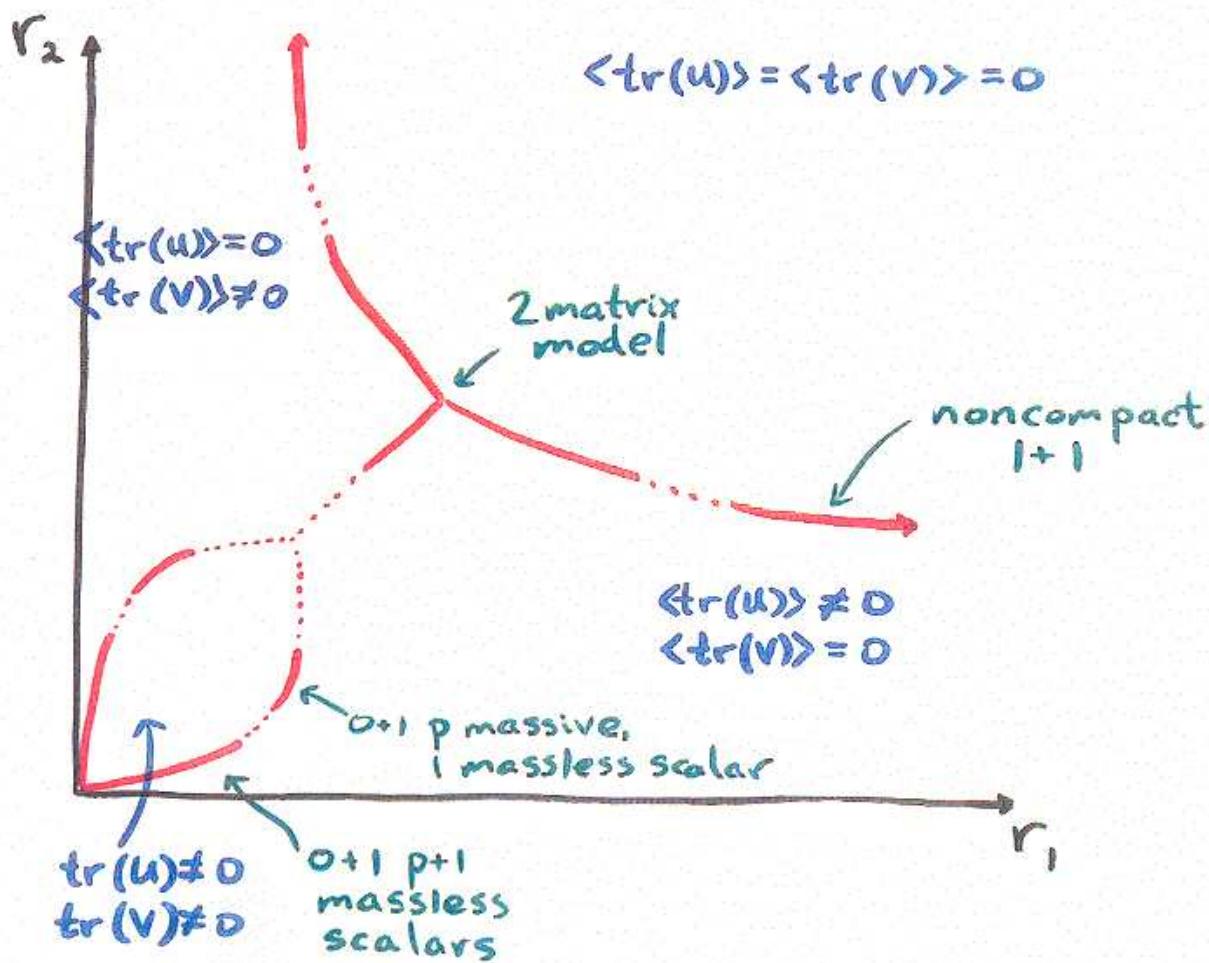
$$\mathcal{L} = \frac{N}{\lambda} \text{Tr} \left( \frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2} (\partial_\mu X^i)^2 + \frac{1}{2} M^2 (X^i)^2 - \frac{1}{4} [X^i, X^j]^2 \right)$$

dimensionless parameters:

$$\begin{aligned} r_1 &= R_1 \sqrt{\lambda} \\ r_2 &= R_2 \sqrt{\lambda} \\ m &= M / \sqrt{\lambda} \end{aligned}$$

$m = \infty$  : Pure 2D YM ,  $\langle \text{tr}(U) \rangle = \langle \text{tr}(V) \rangle = 0$

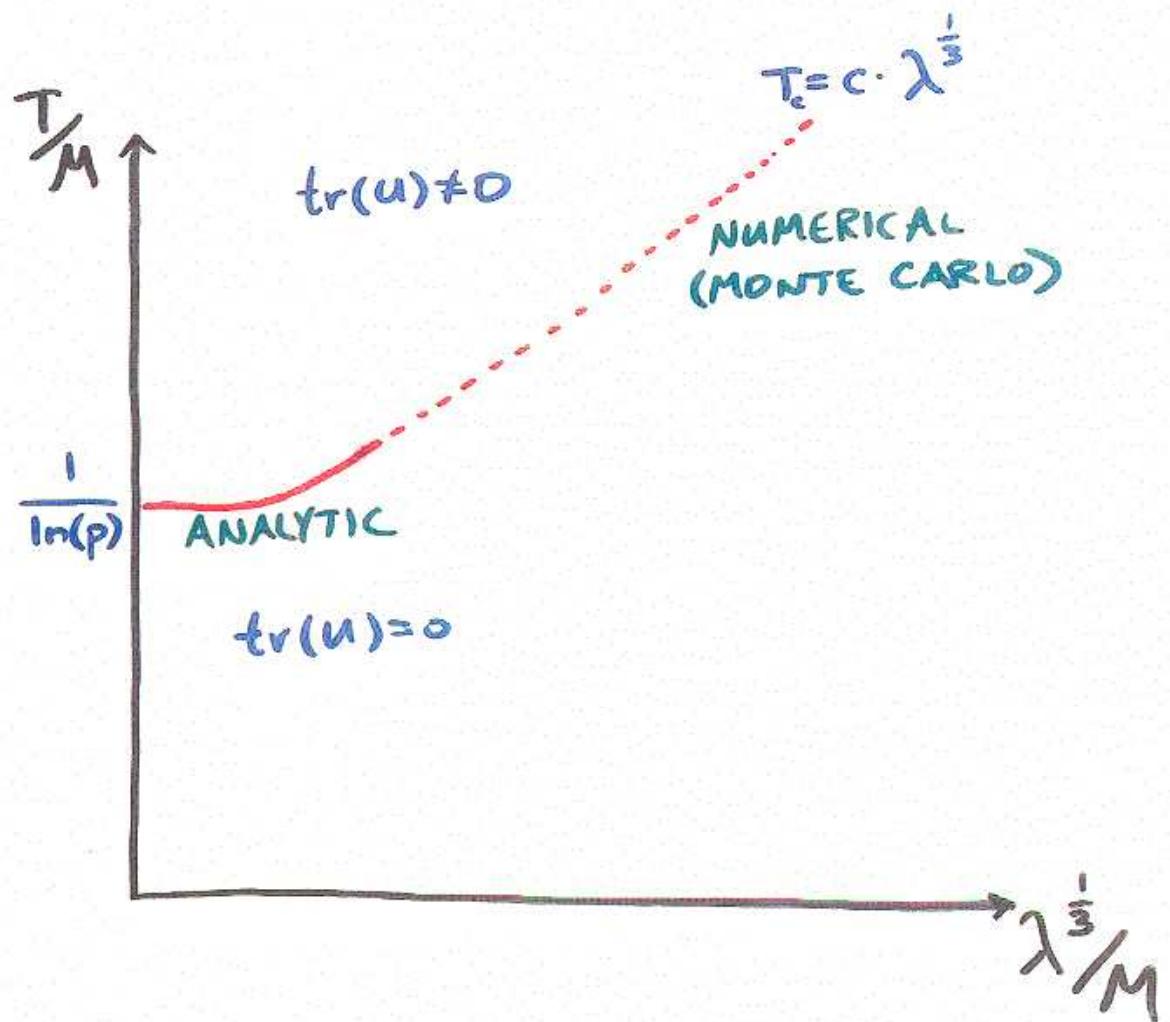
$1 \ll m < \infty$ :



# 0+1 DIMENSIONAL YM. + ADJOINT SCALARS

$$\mathcal{L} = \frac{N}{\lambda} \text{Tr} \left( i D_\alpha X^i D_\alpha X^i + \frac{1}{2} M^2 X^i X^i - \frac{1}{4} [X^i, X^j]^2 \right)$$

→ like YM on  $S^3$  truncated to finite # of KK modes

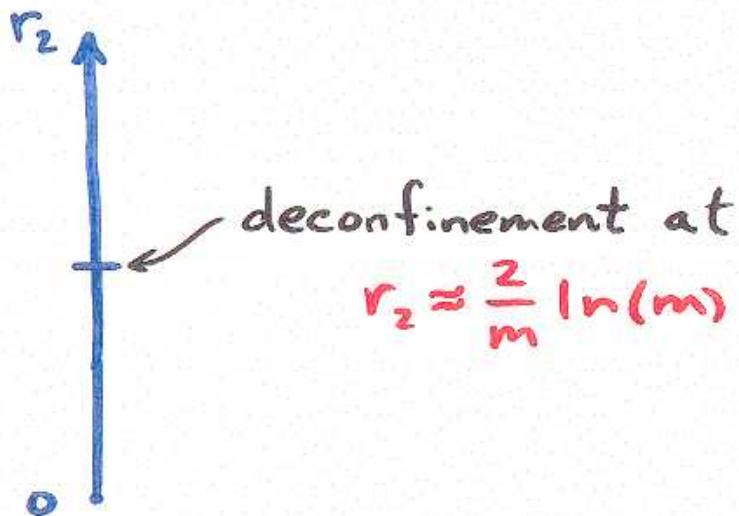


## NONCOMPACT LIMIT

$$S \rightarrow \int_0^{R_2} dx \left\{ \frac{N}{2\lambda R_2} \text{tr}(|\partial_x U|^2) + P \sqrt{\frac{M}{2\pi R_2}} e^{-MR_2} \text{tr}(U(z)) \text{tr}(U^\dagger(z)) \right\}$$

Semenoff, Tirkkonen, Zarembo

- Solvable using collective field theory methods



## THE 2-MATRIX MODEL

If  $r_i$  small  $\rightarrow$  KK modes weakly coupled

$r_i \gg \frac{1}{m}$   $\rightarrow$  scalar zero modes  
weakly coupled

 INTEGRATE OUT

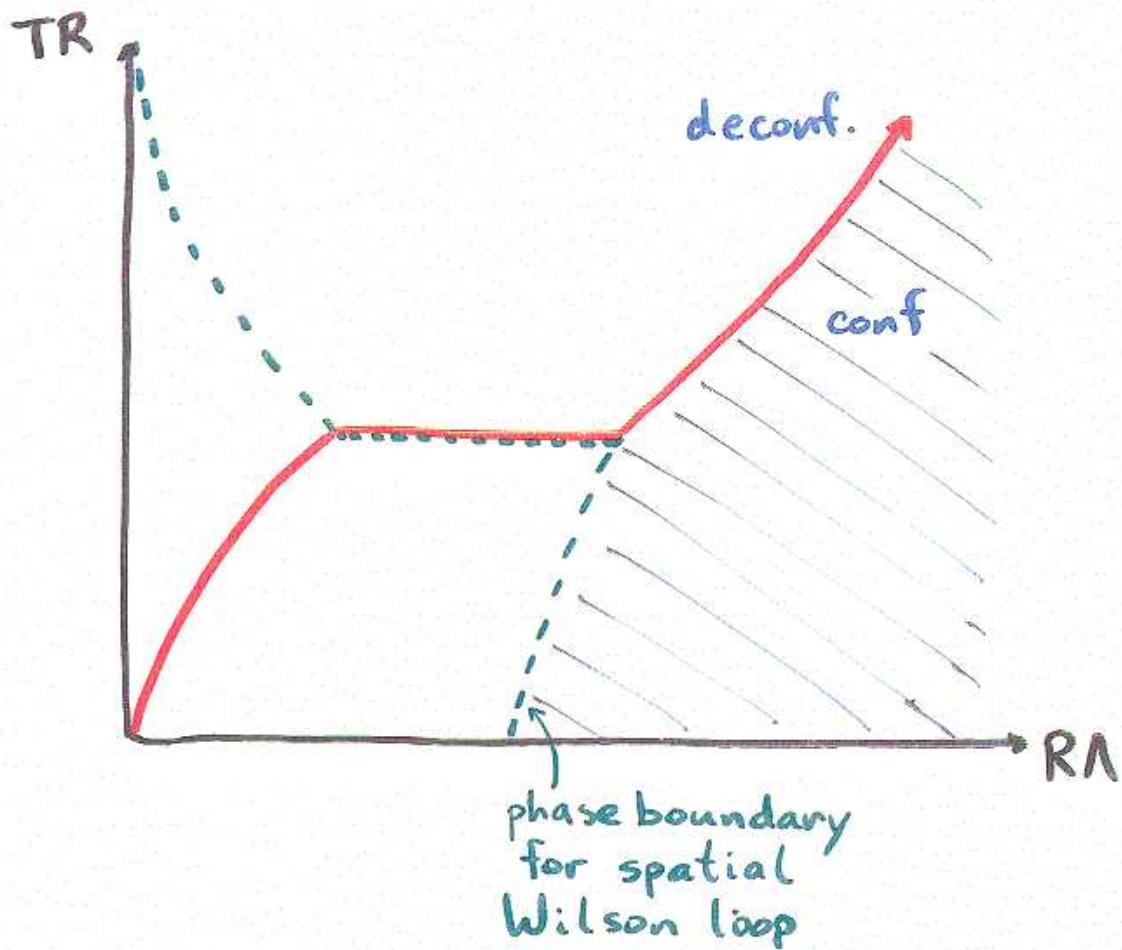
$$Z = \int dU dV e^{-S_{\text{YM}}(U, V) - f(U) - g(V)}$$

$$e^{-S_{\text{YM}}(U, V)} = \underset{\text{large } N}{\text{const}} + e^{-\sum c_n(r_i r_j) \text{tr}(U^n) \text{tr}(U^{n'})} + e^{-\sum c_n(r_i r_j) \text{tr}(V^n) \text{tr}(V^{n'})}$$

$$f(U) = -\mu(r_1, r_2, m) \text{tr}(U) \text{tr}(U^\dagger)$$

$$g(V) = -\mu(r_1, r_2, m) \text{tr}(V) \text{tr}(V^\dagger)$$

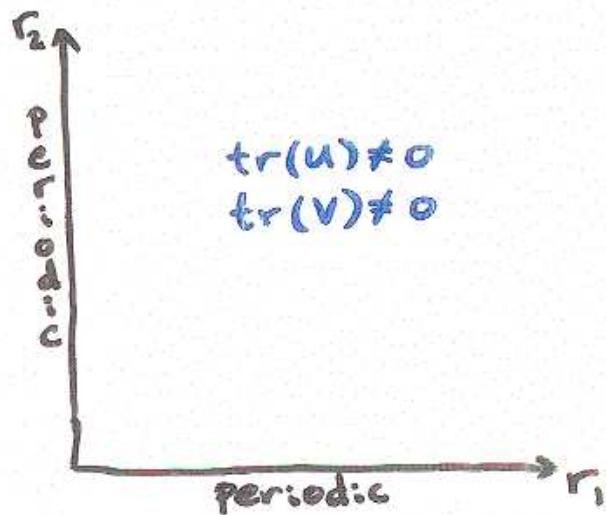
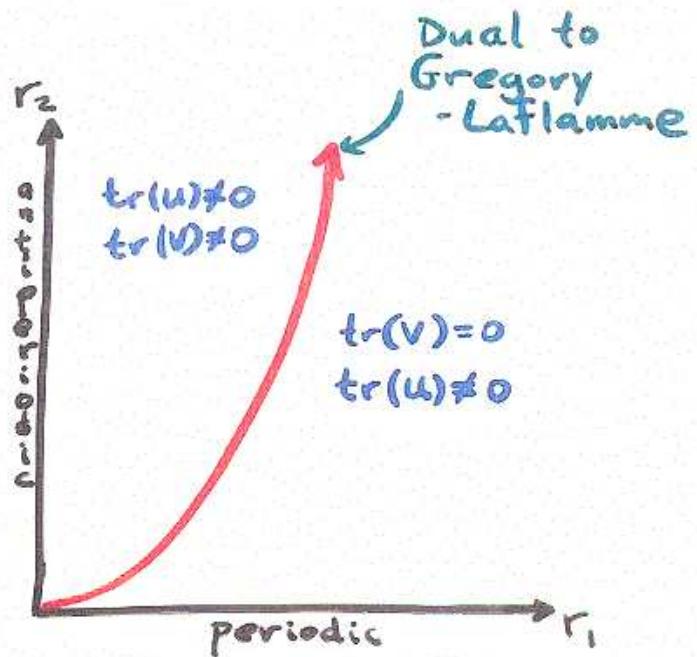
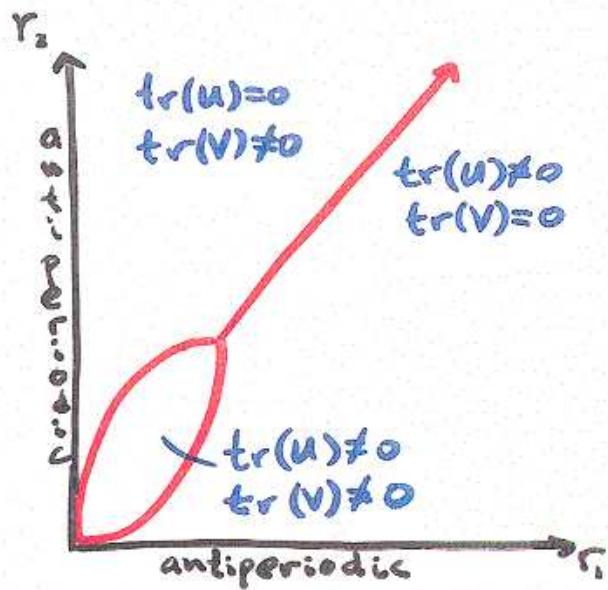
# Phase diagram for large $N$ 1+1 d $U(N)$ YM with adjoint scalars on $S^1$



$F = \Theta(1)$  in shaded region

# MAXIMALLY SUSY I+I Y.M.

→ Use AdS/CFT for strong coupling



# CONCLUSIONS

Finite volume + Large  $N$

↳ allows tractable analytic analysis  
of phase transitions

Interesting to interpret phase diagrams  
in dual gravity/string theory

New phases of  
gravitational  
theories?

Properties of black  
holes from gauge  
theories?