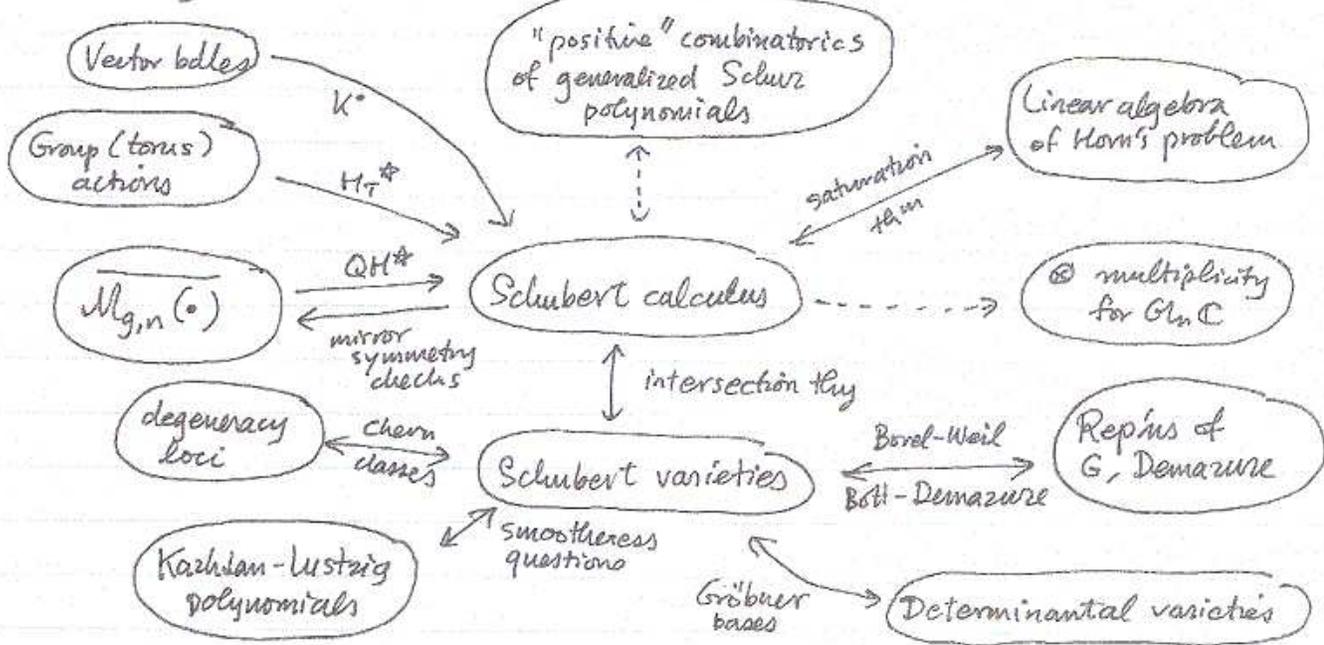


Alex Yong: Introduction. (notes by Megumi Harada)



- algebraic geometry
- representation theory
- combinatorics
- symplectic geometry
- algebra

Classical Enumerative Geometry (intersection theory)

$[Q]:$  # pts in intersection of  $n$  polynomials in  $n$  variables  $\in \mathbb{R}^n$   
homog.

$\leftrightarrow H_1, \dots, H_n$  hypersurfaces

$[A]:$  [Bezout]  $\leq d_1 \dots d_n$  "Suppose" want equality, not  $\leq$ .

Rephrase:

- $\mathbb{R} \rightsquigarrow \mathbb{C}$  (non-real sol'n)
- $\mathbb{C}^n \rightsquigarrow \mathbb{CP}^n$  (asymptotic sol'n)
- multiplicity of sol'n

Let  $P \in H_1 \cap \dots \cap H_n$

$$i(P) = i(P, H_1 \cap \dots \cap H_n) = \begin{cases} 1 & \text{transverse} \\ \geq 1 & \text{otherwise} \end{cases}$$

↑  
intersection multiplicity

Idea: (Principle of Continuity) vary  $H_i(t) \xrightarrow{t \rightarrow 0} H_i$

$t=1$ :  $H_1(1) \cap \dots \cap H_n(1)$  transverse

$$P_i(t) \rightarrow P_i$$

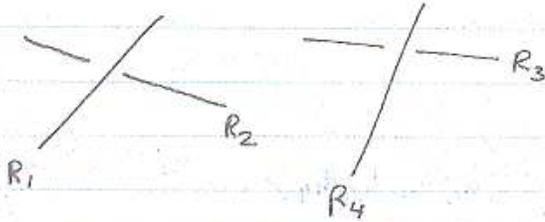
$$i(P) = \# P_i \text{ 's.}$$

Principle: • Want compact space  $\mathbb{C}P^n$   
• "moving lemma"

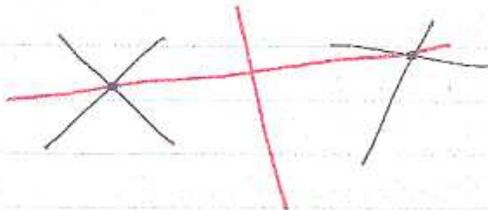
"basic"

• Classic Schubert problem: "4-line problem" [basic for what?]  
- what is the problem in other Schubert settings?

Q How many lines intersect 4 generic (random) lines in  $\mathbb{C}P^3$



answer: move  $R_1, R_2$  to intersect, with  $R_3, R_4$



A: 2

but why shouldn't it change?

Modern approach: intersection theory

Need a compact space to work on: space of lines in  $\mathbb{C}P^3$   
 $= \text{Gr}(2 \text{ planes in } \mathbb{C}^4)$

- $X_i =$  subvariety of lines that meet  $R_i$ .
- $\text{Gr}(2, \mathbb{C}^4)$  compact, so  $X_i \xrightarrow{\text{Poincaré}} [X_i] \in H^*(\text{Gr}(2, \mathbb{C}^4))$ .

• By conts deformation

$$X_i \longrightarrow X_j$$

$$[X_1] = [X_2] = [X_3] = [X_4] \in H^*(\text{Gr}(2, \mathbb{C}^4))$$

Answer:  $\int_{\text{Gr}(2, \mathbb{C}^4)} [X_1 \cap \dots \cap X_4] = \int_{\text{Gr}(2, \mathbb{C}^4)} [X_1] \cdot [X_2] \cdot \dots \cdot [X_4]$

$$= \int_{\text{Gr}(2, \mathbb{C}^4)} [X_1]^4$$

becomes a question of a cohom ring computation.

Conclude: • Get a ring presentation of  $H^*(\text{Gr} \dots)$   
 • computer algebra

Example: Computation in question:

$$(x_1 + x_2)^4 \text{ in } \mathbb{Z}[x_1, \dots, x_4] / \langle h_1, h_2, h_3, h_4 \rangle \quad h_i = \sum \text{deg } i \text{ monomials in } x_1, \dots, x_4$$

... compute a Gröbner basis...

$$\begin{cases} X_4 = -(x_1 + x_2 + x_3) \\ X_3^2 = -(x_1^2 + x_1 x_2 + x_1 x_3 + x_2^2 + x_2 x_3) \\ X_2^3 = -(x_1^3 + x_1^2 x_2 + x_1 x_2^2) \\ X_1^4 = 0 \end{cases}$$

$$\text{so: } = x_1^4 + 4x_1^3 x_2 + 6x_1^2 x_2^2 + 4x_1 x_2^3 + x_2^4$$

⋮

$$= 2 x_1^2 x_2^2$$

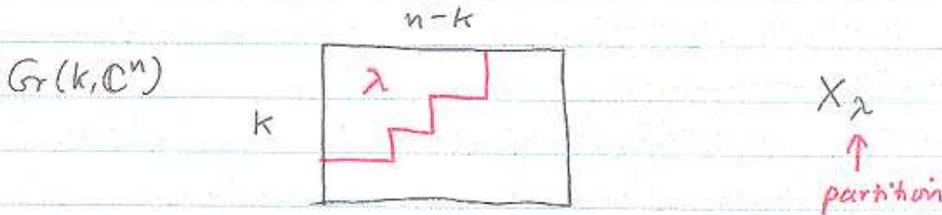
↳ this is the 2 from above!

BUT: can't get a positive formula. Lots of minus signs.

WANT: positive, combinatorial answer to how to compute these.

These #'s show up everywhere... want to understand them as well as possible.

GOAL: "manifestly positive" = (combinatorial) sol'ns for these numbers.



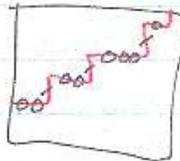
$$\sigma_\lambda = [X_\lambda] \in H^*(Gr)$$

$$\sigma_\mu \cdot \sigma_\lambda = \sum_{\nu \subseteq k \times (n-k) \text{ box}} c_{\mu\lambda}^\nu \sigma_\nu$$

Littlewood-Richardson coeff,  $\geq 0$ .

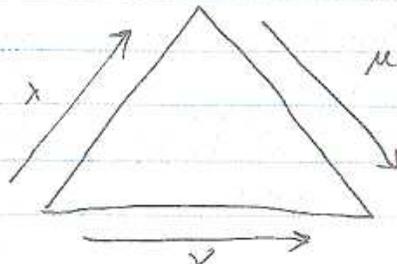
Nice solution for  $c_{\mu\lambda}^\nu$ : [Knutson-Tao]

Think of  $\lambda$  as a lattice path, get sequence of 0's and 1's  
(0 for horiz, 1 for vertical)



Do this for  $\mu, \lambda, \nu$ .

Now put on a  $\Delta$



fill in by: fill in by:

don't flip,  
but OK to rotate

$c_{\lambda\mu}^{\nu}$  = # puzzle fillings of this triangle.

Fix  $Gr(k, \mathbb{C}^n)$ .

Problem: Characterize pairs  $(\lambda, \mu)$  such that  $c_{\lambda\mu}^{\nu} \in \{0, 1\} \forall \nu$ .

multiplicity-free expansions.

Why we care: similarities with Saturation Thm.

Recently: 1) Horn's problem. Let  $A, B, C \in \mathcal{H}$

$$A + B = C$$

$\alpha, \beta, \gamma$  e-values.

Characterize  $(\alpha, \beta, \gamma)$  that can appear?

Horn: conjectured list of relations that characterize.

(reformulation)  $\updownarrow$  Klyachko, Zelevinsky, Fulton

$$c_{\lambda\mu}^{\nu} \neq 0 \iff c_{N\lambda, N\mu}^{N\nu} \neq 0 \quad \text{Saturation conj.}$$

$\uparrow$  . YES: Knutson-Tao.

. Knutson-Tao-Woodward  $c_{\lambda\mu}^{\nu} = 1 \iff$  facets.

2) Reality: replace back  $\mathbb{C} \rightarrow \mathbb{R}$

How many of those expected cx solutions can you achieve?

Sottile:  $Gr(2, \mathbb{C}^n)$  can get all.

Vakil:  $Gr(k, \mathbb{C}^n)$  same.

...  $\rightarrow$  Q To what extent do these generalize?

Hilbert's 15<sup>th</sup> problem: Justify Schubert!

General "policy": enumerative problems  $\longleftrightarrow$  better understanding of moduli spaces.