

An inadequate introduction to the singularities of Schubert varieties. - A. Woo (1)

Reference: Billey-Lakshminarayanan, Sing. Loci of Schub. Var.
(2000)

(i.e. 5 yrs out of date,

1 Which Schubert varieties are smooth?

A d-dim. variety X is smooth if every pt $p \in X$ has a nbhd diffeomorphic (holomorphic) to \mathbb{C}^d , ~~for some~~. Equivalently (assuming X connected), the tangent space at each pt p is of dimension d .

Algebraically, we can compute the tangent space as

$$\overline{\mathfrak{m}_p/m_p^2}^*$$

What is $\overline{\mathfrak{m}_p/m_p^2}^*$?

At a point p , we have a local ring \mathfrak{o} of germs of functions at p :

$$\mathfrak{o}_{X,p} = \left\{ \text{rational functions } \frac{f}{g} \text{ on an affine nbhd of } p \text{ st. } g(p) \neq 0 \right\}$$

$$\mathfrak{m}_p = \text{ideal of } \mathfrak{o}_{X,p} \text{ of functions } \frac{f}{g}(p) = 0.$$

Thm: (Lakshmi bai - Sandhya '90 (previous less combinatorial answers by Ryan, Wolper, others))

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X_w is smooth iff w avoids 4231 and 3412
i.e. $\nexists i < j < k < l$, s.t. $w(i) > w(k) > w(j) > w(l)$
and $\nexists i < j < k < l$, s.t. $w(j) > w(i) > w(l) > w(k)$

Proof by reducing $\dim T_{P_{id}}(X_w)$ ($P_v = T$ -fixed pt $\in \overline{B}$)
to combinatorics, and doing combinatorics.

Why does calculating ~~not~~ $T_{P_{id}}(X_w)$ only suffice?

Principle of semi-continuity: Singularities get worse on closed sets. Since the B -action gives isom between pts on X_v^0 , if X_w is singular at P_v , it is singular on X_v , and therefore P_{id} .

(Types B + C, 27 patterns, Type D, 49 patterns)
(Billey, alternatively, singular ~~not~~ X_w for A_3, B_3, C_3, D_4 give all necessary patterns (Billey-Postnikov '03).

2 Where is X_w singular?

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(i.e. Find v maximal in Bruhat order s.t.

X_w is singular at ~~P_v~~ P_v)

Billey-Warrington, Kassel-Lascoux-Reutenauer,
Manivel, Cortez

Ist 3: use known reduction of calculating $\dim T_{P_v}$ to combinatorics, solve combinatorics

Cortez: (partial) resolutions of X_w

Resolution of singularities: Y smooth

$\downarrow \pi$ projective, isom on
 \times a dense open set

Try to study X by understanding Y .

Pf idea:

1) Explicit calculation that X_w is singular at certain P_v .

2) For 3412-avoiding w , use Zelevinsky resolution,

$$Z_w = \mathcal{E}(U_0, V_0)$$

show that it is an isomorphism ~~on~~ on X_w^0 for $u \neq v; v_i$

3) In general, construct partial resolutions π_i , and show that at X_u° , $u \neq v$, π_i is an isom on X_u° for at least one π_i , and $\pi_i^{-1}(X_u^\circ)$ is nonsingular by induction. (The partial resolutions are fiber bundles over Schubert varieties.) \square

This problem is open for other types

3 How singular is X_w ?

Measures of singularity:

- a) multiplicity (and Hilb series)
- b) local intersection cohomology
- c) type

A Multiplicity:

Given a local ring $(\mathcal{O}_{X,p}, m_{X,p})$, the associated graded ring

$$\text{gr } \mathcal{O}_{X,p} = \frac{\mathcal{O}_{X,p}}{m_{X,p}} \oplus \frac{m_{X,p}}{m_{X,p}^2} \oplus \frac{m_{X,p}^2}{m_{X,p}^3} \oplus \dots$$

The Hilbert series $H_{X,p}(t) = \sum_{d \geq 0} \dim_F \left(\frac{m_{X,p}^d}{m_{X,p}^{d+1}} \right) t^d$

$$\stackrel{\text{thm}}{=} \frac{C(t)}{(1-t)^{\dim(X_p)}}$$

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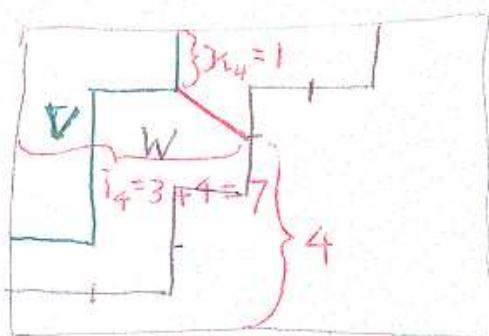
The multiplicity $\mu_{x,p} = C(1)$.

Multiplicity is 1 for smooth variety ($C(t)=1$). In general, it is the degree of $\text{Proj}(\text{gr } \mathcal{O}_{X,p})$ "projective tangent cone"; measures "how many branches at p ". Semi-continuity holds "Known" for (minuscule) Grassmannians

Recursive formula (Lakshmibai-Weyman '90)

For $G(d, \mathbb{C}^n)$: (Krattenthaler, Rosenthal-Zelevinsky, Kreiman-Lakshmibai)

$\mu_{p_v}(x_w) = \# \{ \text{family of paths from } (1,d), \dots, (d,d) \text{ to } (d-\chi_e, \chi_e + i_e) \}$



Hilbert series (Kodiyalam-Raghavan)

$$C(t) = \sum_{\text{family of paths}} t^{\#(\text{path})}$$

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Pf: Std monomial theory and combinatorics.

Charts: Standard nbhd of X_w around P_v allowing calculation of $\theta_{X,P}$.

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 1 \\ 1 & 2 & 3 & 4 & 9 \\ 0 & 2 & 1 & 3 & 8 \\ 0 & 1 & 2 & 3 & 7 \\ 0 & 0 & 1 & 2 & 6 \end{bmatrix}$$

$$w = 45231$$

~~$w = 45231$~~

$$v = 24351$$

At v , a nbhd $\cong \mathbb{C}[a_{i,j}] / \langle c=0, at+bd=0 \rangle$

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rks! 2×2 matrices

For Grassmannian w , standard monomial theory gives a Gröbner basis on all charts, allowing combinatorial calculation of mult/Hilb series

B Kazhdan-Lusztig polynomials

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$$P_{v,w}(q) = \sum_i \dim (\mathcal{H}^i(X_w))_{P_v} q^i \in \mathbb{Q}[q]$$

Contribution of P_v to intersection homology

Measures singularity ~~at v~~ of w at v .

1 at (rat) smooth point
coeff of q^i is ~~is~~ semi-continuous

~~Originally~~ These polynomials give a change of basis
in Hecke algebra ~~&~~ btw canonical and naive basis.
~~multiplicity~~ # times irrep appears in comp series of
Verma,

Link to Schubert varieties only way to prove
positivity in general.

Combinatorial rules exist only in case of
small resolution of singularities:

$\pi: Z \rightarrow Y$ is small if

$$\text{codim}_X \{x \in X \mid \dim \pi^{-1}(x) \geq i\} > 2 \quad \forall i > 0.$$

$$\text{Then } \dim (\mathcal{H}^i(Y))_p = \dim H^i(\pi^{-1}(p)).$$

- 1) 321 Hexagon-avoiding ($321 + 4 S_8$ patterns),
Bott-Samelson (Mathieu yesterday) is small
(Billey-Warrington, Deodhar)
- 2) 3412 avoiding - Zelevinsky resolution, which is
small (Lascoux, Lascoux-Schützenberger)

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\subseteq "type"
 $\Rightarrow \text{rk}(\omega_{X_w P_v})$
 $=$ ~~(can be calculated from free resolution of nbhd of X_w around P_v)~~
 Rank of leftmost term of minima

| for smooth and Gorenstein points
 def.

Semicontinuity

Thm (-, Yong): ~~X_w~~ Gorenstein iff it is Gorenstein at generic singularities (i.e. at maximal v s.t. X_w sing. at P_v).)