GEOMETRY REPRESENTATION THEORY ON FLAG VARIETIES FIELDS INSTITUTE WORKSHOP ON SCHUBERT VARIETIES AND SCHUBERT CALCULUS

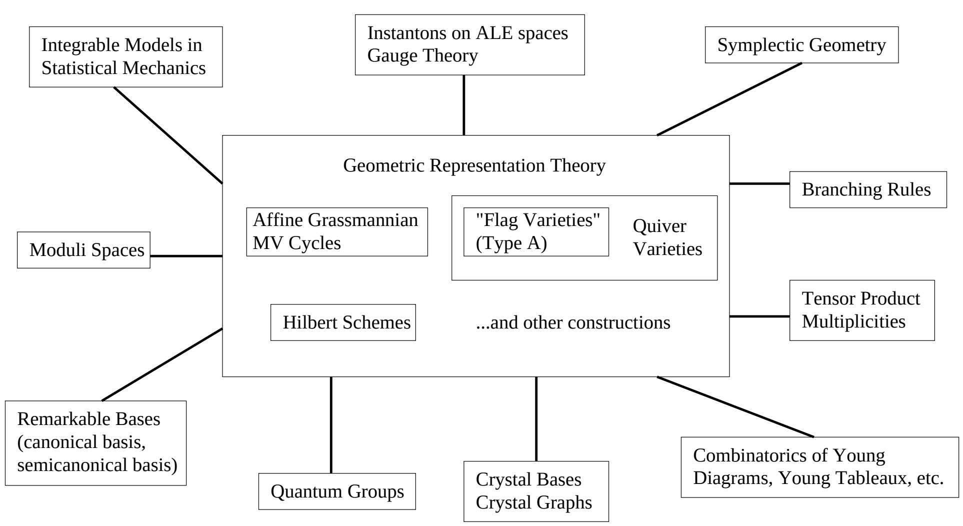
ALISTAIR SAVAGE

ABSTRACT. In this expository talk, we describe a construction due to Ginzburg which endows the homology of flag varieties and certain other, closely related, varieties with the structure of representations of $\mathfrak{sl}_n$. The action of $\mathfrak{sl}_n$ is described in terms of natural geometric operations and the construction yields bases in the representations with nice properties.

1. Introduction

Geometric representation theory is a term that encompasses many different ideas. In this talk we focuss on a small area of this field and describe in some detail one particular construction in geometric representation theory. In keeping with the topic of this workshop, we choose a construction that involves varieties consisting of flags.

Before we begin, we first mention some of the connections between geometric representation theory and other fields in mathematics. The following diagram is a summary of just a handful of these connections. There are certainly others not included here.



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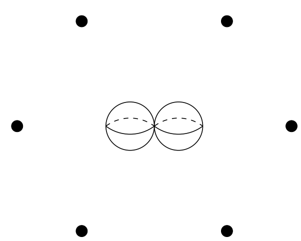
Our goal in this talk will be to describe a method of realizing in a geometric way representations of the Lie algebra $\mathfrak{sl}_n$. The main idea is as follows. For a given representation, we want to define an algebraic variety whose irreducible components enumerate a basis for this representation. More precisely, we will have one such variety for each weight space of the representation. We then consider the (top-dimensional, Borel-Moore) homology of this variety. This will become the underlying vector space of our representation. We define natural geometric operators on this space corresponding to the generators of our Lie algebra, thus constructing the representation in question. The classes in homology of the irreducible components of our varieties then provide us with a natural geometric basis in these representations. These bases turn out to have very nice properties.

For example, for the adjoint representation of $\mathfrak{sl}_3$, the varieties will look as follows.



The varieties corresponding to the six one-dimensional weight spaces are points and thus have one irreducible component, corresponding to the fact that the weight space is one-dimensional. On the other hand, the variety corresponding to the two-dimensional zero weight space (the Cartan subalgebra) is an union of two copies of $\mathbb{C}P^1$ (two irreducible components). We have organized these varieties to correspond to the usual depiction of the weight diagram of the adjoint representation of $\mathfrak{sl}_3$.

Those familiar with the representation theory of Kac-Moody Lie algebras should keep in mind that the construction we describe here is just the type A case of a more general construction involving quiver varieties. Thus, the results generalize to arbitrary symmetric Kac-Moody Lie algebras (see [8] for details on the relation between the two constructions).

Our presentation will be fairly informal. We will at times sacrifice precision in the name of expository clarity. Those who are interested in the complete details should consult the references cited below.

2. Preliminaries

We first introduce the algebraic objects we are interested in. Let $\mathfrak{sl}_n$ be the Lie algebra of traceless $n \times n$ matrices. Let $e_{i,j}$ denote the matrix with entry 1 in position (i,j) and zeroes in all other positions. Then $\{e_k = e_{k,k+1}, f_k = e_{k+1,k}\}_{k=1}^{n-1}$ are called the *Chevalley generators* of $\mathfrak{g}$. The subalgebra consisting of traceless diagonal matrices is called the *Cartan subalgebra*. It is denoted by $\mathfrak{h}$ and is spanned by the matrices

$$h_k = e_{k,k} - e_{k+1,k+1},\ 1 \le k \le n-1.$$

Thus the dual space $\mathfrak{h}^*$ is spanned by the simple roots

$$\alpha_k = \epsilon_k - \epsilon_{k+1},\ 1 \le k \le n-1,$$

where $\epsilon_k(e_{l,l}) = \delta_{kl}$ and the fundamental weights are given by

$$\Lambda_k = \epsilon_1 + \cdots + \epsilon_k,\ 1 \le k \le n-1.$$

A *dominant integral weight* is an linear combination of the fundamental weights with nonnegative coefficients. Consider a dominant integral weight $\mathbf{w} = w_1\Lambda_1 + \cdots + w_{n-1}\Lambda_{n-1}$. Then

$$\mathbf{w} = \lambda_1\epsilon_1 + \cdots + \lambda_{n-1}\epsilon_{n-1}$$

where $\lambda_k = w_k + \cdots + w_{n-1}$ and so $\mathbf{w}$ corresponds to a partition $\lambda(\mathbf{w}) = (\lambda_1 \ge \cdots \ge \lambda_{n-1})$. We say that a highest weight $\mathbf{w}$ is a partition of d if $|\lambda(\mathbf{w})| = \lambda_1 + \cdots + \lambda_{n-1} = d$ or, equivalently, if $\sum_{k=1}^n kw_k = d$.

The *universal enveloping algebra* $U(\mathfrak{sl}_n)$ of $\mathfrak{sl}_n$ is the tensor algebra of the underlying vector space, modded out by the commutation relations of $\mathfrak{sl}_n$. Representations of $U(\mathfrak{sl}_n)$ are the same as representations of $\mathfrak{sl}_n$. The isomorphism classes of finite-dimensional irreducible representations of $\mathfrak{sl}_n$ are in one-to-one correspondence with integral dominant weights. Our goal is to construct these representations geometrically.

3. Convolution algebra in homology

We now give a brief overview of one of the main tools in the construction: the convolution algebra in homology. Those interested in further details should consult [1].

In this talk $H_*(Z)$ will denote the Borel-Moore homology with $\mathbb{C}$-coefficients of a locally-compact space Z. We use Borel-Moore homology rather than ordinary homology because it is better suited to the spaces we will be dealing with. There are many equivalent definitions of Borel-Moore homology. One definition mirrors the usual definition of singular homology but one allows infinite chains that are locally finite (any compact set intersects only a finite number of the complexes appearing in any chain). From this definition, one can see that for compact spaces, Borel-Moore homology and ordinary homology coincide. As an example of a non-compact space where the two differ, consider $\mathbb{R}^n$. We have the following:

$$H_i(\mathbb{R}^n) = \begin{cases} \mathbb{C} & \text{if } i = n \\ 0 & \text{if } i \neq n \end{cases}, \quad H_i^{\mathrm{ord}}(\mathbb{R}^n) = \begin{cases} \mathbb{C} & \text{if } i = 0 \\ 0 & \text{if } i \neq 0 \end{cases}.$$

Here $H_i^{\mathrm{ord}}(\mathbb{R}^n)$ denotes ordinary homology. The degree zero ordinary homology is generated by the class of a point. Since a point is not the boundary of any finite chain, this class in non-zero. However, in Borel-Moore homology, one can have a chain whose "tail" goes off to infinity and thus a point is the boundary of a locally finite chain. Also, one can have a locally finite chain covering all of $\mathbb{R}^n$. This chain has no boundary and this is why $H_n(\mathbb{R}^n) = \mathbb{C}$.

Another definition of Borel-Moore homology, and the one we will use, is the following. If Z is a closed subset of a smooth, oriented manifold M, then

$$H_k(Z) := H^{\dim_{\mathbb{R}} M - k}(M, M \backslash Z).$$

Thus, we define Borel-Moore homology by Poincaré duality. Note that if X itself is smooth, we may take $X = M$ and then Borel-Moore homology is dual to usual cohomology (recall that ordinary homology is dual to cohomology with compact support). We use the above definition of Borel-Moore homology because it makes it natural to define an intersection pairing. If Z and Z' are closed subsets of a smooth variety M, we have a $\cup$-product map

$$H^k(M, M\backslash Z) \times H^l(M, M\backslash Z') \to H^{k+l}(M, M\backslash(Z \cap Z')).$$

This is just the relative version of the usual $\cup$-product in cohomology. Then, by duality, we construct the intersection pairing in Borel-Moore homology

$$\cap : H_k(Z) \times H_l(Z') \to H_{k+l-d}(Z \cap Z'), \quad d = \dim_{\mathbb{R}} M.$$

Note that this pairing depends on the particular embedding of Z and Z' in M.

Now we describe the so-called *convolution product* in Borel-Moore homology. Let M_1, M_2 and M_3 be smooth, oriented manifolds and $p_{kl} : M_1 \times M_2 \times M_3 \to M_k \times M_l$ be the obvious projections. Let $Z \subset M_1 \times M_2$ and $Z' \subset M_2 \times M_3$ be closed subvarieties and assume that the map

$$p_{13} : p_{12}^{-1}(Z) \cap p_{23}^{-1}(Z') \to M_1 \times M_3$$

is proper (the preimage of any compact set is compact) and denote its image by $Z \circ Z'$. The operation of convolution

$$\star : H_k(Z) \times H_l(Z') \to H_{k+l-d}(Z \circ Z'), \quad d = \dim_{\mathbb{R}} M_2,$$

is defined by

$$c \star c' = (p_{13})_*(p_{12}^* c \cap p_{23}^* c'),$$

where $p_{12}^* c$ means $c \boxtimes [M_3]$ (we have a Knünneth formula in Borel-Moore homology), etc.

Now, let M be a smooth manifold and $\mu : M \to N$ be a proper morphism. Let

$$Z = M \times_N M = \{(m_1, m_2) \in M \times M \mid \mu(m_1) = \mu(m_2)\} \subset M \times M.$$

Then $Z \circ Z = Z$ and so convolution makes $H_*(Z)$ a finite-dimensional associative $\mathbb{C}$-algebra with unit.

For $x \in N$, let $M_x = \mu^{-1}(x)$. We also identify M_x with the variety $M_x \times \mathrm{pt}$. Then setting $M_1 = M_2 = M$ and $M_3 = \mathrm{pt}$, we have $Z \circ M_x = M_x$ and convolution makes $H_*(M_x)$ a $H_*(Z)$-module.

There are a few key properties of Borel-Moore homology that we shall use. First, any closed subvariety of a variety Z determines a class in the Borel-Moore homology. Furthermore, if Z is of complex dimension d then the top degree part of Borel-Moore homology $H_{2d}(Z)$ has as a basis the classes of the irreducible components of dimension d.

The above convolution product construction can be repeated with any theory that has operations of "pullback" for smooth morphisms, "pushforward" for proper morphisms, and "intersection" (e.g., the linear space of constructible functions, equivariant K-theory, etc.).

Our next goal is to define varieties so that the convolution product on these varieties realizes (a quotient of) the universal enveloping algebra $U(\mathfrak{sl}_n)$ and its representations.

4. Ginzburg's construction

We now describe Ginzburg's construction of the enveloping algebra $U(\mathfrak{sl}_n)$ and its irreducible highest weight representations via convolution in homology. This construction was inspired by the work of Beilinson, Lusztig and MacPherson. Proofs omitted here can be found in [2] or [1].

Fix an integer $d \geq 1$. Let

$$\mathcal{F} = \{0 = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n = \mathbb{C}^d\}$$

be the set of all n-step partial flags in $\mathbb{C}^d$. The space $\mathcal{F}$ is a smooth compact manifold with connected components parameterized by partitions

$$\mathbf{d} = (d_1 + d_2 + \cdots + d_n = d), \quad d_i \in \mathbb{Z}_{\geq 0}.$$

The connected component of $\mathcal{F}$ corresponding to $\mathbf{d}$ is

$$\mathcal{F}_{\mathbf{d}} = \{F = (0 = F_0 \subset \cdots \subset F_n = \mathbb{C}^d) \mid \dim F_i/F_{i-1} = d_i\}.$$

Let

$$N = \{x \in \mathrm{End}(\mathbb{C}^d) \mid x^n = 0\}.$$

Then

$$T^*\mathcal{F} \cong M = \{(x, F) \in N \times \mathcal{F} \mid x(F_i) \subset F_{i-1},\ 1 \leq i \leq n\}.$$

The above decomposition of $\mathcal{F}$ yields a decomposition of M given by $M = \bigsqcup_{\mathbf{d}} M_{\mathbf{d}}$ where $M_{\mathbf{d}} = T^*\mathcal{F}_{\mathbf{d}}$ for an n-step partition $\mathbf{d}$ of d.

The natural projections give rise to the diagram

$$N \xleftarrow{\mu} M \xrightarrow{\pi} \mathcal{F}.$$

We have a natural action of $GL_d(\mathbb{C})$ on $\mathcal{F}$, N and M by conjugation and the projections commute with this action.

For $x \in N$, let $\mathcal{F}_x = \mu^{-1}(x)$. It has connected components $\mathcal{F}_{\mathbf{d},x}$ given by $\mathcal{F}_{\mathbf{d},x} = \mathcal{F}_{\mathbf{d}} \cap \mathcal{F}_x$. Define

$$Z = M \times_N M = \{(m_1, m_2) \in M \times M \mid \mu(m_1) = \mu(m_2)\} \subset M \times M.$$

Proposition 4.1. *The variety Z is the union of the conormal bundles to the $GL_d(\mathbb{C})$-orbits in $\mathcal{F} \times \mathcal{F}$. The closures of these conormal bundles are precisely the irreducible components of Z.*

We now have exactly the setup of Section 3. Thus we have the following proposition.

Proposition 4.2. *We have $Z \circ Z = Z$. Thus $H_*(Z)$ is an associative algebra with unit and $H_*(\mathcal{F}_x)$ is an $H_*(Z)$-module for any $x \in N$.*

Proposition 4.3. *All irreducible components of Z contained in $M_{\mathbf{d}^1} \times M_{\mathbf{d}^2}$ are half dimensional. In particular, they all have the same dimension.*

Let $H_{\mathrm{top}}(Z)$ be the vector subspace of $H_*(Z)$ spanned by the fundamental classes of the irreducible components of Z and let $H_{\mathrm{top}}(\mathcal{F}_x)$ be the vector subspace of $H_*(\mathcal{F}_x)$ spanned by the fundamental classes of the irreducible components of $\mathcal{F}_x$.

Proposition 4.4. *The homology group $H_{top}(Z)$ is a subalgebra of $H_*(Z)$ and $H_{top}(\mathcal{F}_x)$ is an $H_{top}(Z)$-stable subspace of $H_*(\mathcal{F}_x)$.*

Now, for a partition $\mathbf{d}$ we have the diagonal subvariety $\Delta \subset \mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}}$ which is a $GL_d(\mathbb{C})$-orbit. We define

$$H_k = \sum_{\mathbf{d}} (d_k - d_{k+1})[T^*_{\Delta}(\mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}})],$$

where $T^*_\Delta(\mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}})$ denotes the conormal bundle to the $GL_d(\mathbb{C})$-orbit Δ. The conormal bundle $T^*_\Delta(\mathcal{F}_{\mathbf{d}} \times \mathcal{F}_{\mathbf{d}})$ is simply the diagonal in $T^*\mathcal{F}_{\mathbf{d}} \times T^*\mathcal{F}_{\mathbf{d}}$.

Let

$$\begin{aligned} Y_{k,+} &= \{(x', F', F) \in Z \mid F_l = F'_l \ \forall\ l \neq k,\ F_k \subset F'_k, \dim(F'_k/F_k) = 1\}, \\ Y_{k,-} &= \{(x', F', F) \in Z \mid F_l = F'_l \ \forall\ l \neq k,\ F'_k \subset F_k, \dim(F_k/F'_k) = 1\}. \end{aligned}$$

Let

$$E_k = [Y_{k,+}], \quad F_k = \pm[Y_{k,-}].$$

We should say a few words about the above definitions. First of all, the sets $Y_{k,\pm}$ are not connected. They are a disjoint union of irreducible components of Z. Their homology class is defined to be the sum of the homology classes of these components. Secondly, the '$\pm$' in the definition of F_k indicates that we need to put some signs in front of the classes of each of these components. We omit these details here.

Theorem 4.5 ([2]). *The map*

$$e_k \mapsto E_k,\ f_k \mapsto F_k,\ h_k \mapsto H_k,$$

extends to a surjective algebra homomorphism $U(\mathfrak{sl}_n) \twoheadrightarrow H_{top}(Z)$. Under this homomorphism, $H_{top}(\mathcal{F}_x)$ is the irreducible highest weight module of highest weight $w_1\Lambda_1 + \cdots + w_{n-1}\Lambda_{n-1}$ where ω_i are the fundamental weights and w_i is the number of $(i \times i)$-Jordan blocks in the Jordan normal form of x.

Thus, we have realized the representations of $\mathfrak{sl}_n$ in the homology of certain varieties consisting of flags. The highest weight of the representation is encoded in the Jordan normal form of $x \in N$. One can think of the action of e_k (resp. f_k) as increasing (resp. decreasing) the dimension of the kth step of the flag. Recall that the homology $H_{\mathrm{top}}(\mathcal{F}_x)$ has as a basis the classes of its irreducible components. Thus we obtain a natural geometric basis in each representation. These bases have very nice compatibility and integrality properties that follow naturally from the geometry.

We have also given a geometric realization of the universal enveloping algebra $U(\mathfrak{sl}_n)$ in the homology $H_{\mathrm{top}}(Z)$. More precisely, we have realized a quotient of this algebra. To describe this quotient, let I_d be the annihilator of $(\mathbb{C}^n)^{\otimes d}$, a two-sided ideal of finite codimension in the enveloping algebra $U(\mathfrak{sl}_n)$. Here $\mathbb{C}^n$ is the natural $\mathfrak{sl}_n$-module.

Theorem 4.6 ([1, Proposition 4.2.5]). *The homomorphism of Theorem 4.5 yields an algebra isomorphism*

$$U(\mathfrak{sl}_n)/I_d \cong H_{top}(Z).$$

Using a stabilization technique and taking $d \to \infty$, one can recover the entire universal enveloping algebra. It is known that the simple $\mathfrak{sl}_n$ modules that occur with non-zero multiplicity in the decomposition of $(\mathbb{C}^n)^{\otimes d}$ are precisely those modules whose highest weight is a partition of d.

5. Further Directions

What we have described here is just the tip of the geometric representation theory iceberg. As was mentioned in the introduction, one can generalize the above to arbitrary symmetric Kac-Moody algebras using the theory of quiver varieties (see [4, 5]). Demazure modules can also be realized geometrically using quiver varieties (see [9]), as can crystal graphs of universal enveloping algebras and representations (see [3, 6]). Crystal graphs can be thought of as the $q \to 0$ limit of quantum groups. In this limit, certain problems in representation theory such as computing weight and tensor product multiplicities are reduced to combinatorics. There are also realizations of crystal graphs in terms of classical combinatorial objects such as Young diagrams and Young tableaux. One can see a direct connection between quiver varieties and these combinatorial constructions, thus giving a geometric realization of these classical objects (see [7]). And there is much more...

References

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Fields Institute and University of Toronto, Toronto, Ontario, Canada

E-mail address: `alistair.savage@aya.yale.edu`