

EQUIVARIANT QUANTUM SCHUBERT CALCULUS

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REFERENCES: * EQUIVARIANT QUANTUM SCHUBERT CALCULUS? - TO APPEAR
IN ADV. OF MATH, ALSO ON ARXIV.

"POSITIVITY IN EQUIVARIANT QUANTUM SCHUBERT CALCULUS"
TO APPEAR IN ANN. OF MATH, ALSO ON ARXIV.

* EQUIVARIANT QUANTUM COHOMOLOGY OF HOMOGENEOUS SPACES
- AVAILABLE ON ARXIV

NOTATIONS:

$X = G/P$ where G - complex, connected, semisimple Lie grp.

$P \subset G$ parabolic subgroup

$B \subset P \subset G$

$\overset{P}{\sim}$
Borel subgroup.

Usual notations: $\Delta = \{\alpha_1, \dots, \alpha_r\}$ - simple positive roots.

$\Delta_P \subset \Delta$ subset determining P .

$W = \langle s_{\alpha_1}, \dots, s_{\alpha_r} \rangle$ - Weyl group

$W_P = \langle s_{\alpha_i} : \alpha_i \in \Delta_P \rangle$

for each $w \in W_P \in W/W_P \exists$ a minimal length representative
 $\tilde{w} \in W$ s.t. $wW_P = \tilde{w}W_P$.

$W^P = \{ \text{minimal length representatives of } w/W_P \}$

$H^*(X)$

Schubert variety $X(w) = \overline{B \cdot w P / P}$, $w \in W^P$

Schubert class $T_w \in H^2(X)$, $T_w = [X(w)]$.

and $c(w) = \dim X(w) \sim l(w)$.

$H^*(X)$ is a graded \mathbb{Z} -algebra with a \mathbb{Z} -basis
 $\{T_w\}_{w \in W^P}$.

$$\text{Multiplication: } \tau_u * \tau_v = \sum c_{uv} \tau_w$$

$c_{uv} \in \mathbb{Z}_+$, # of pts in the intersection of
 • ~~general translates of~~ general translates of
 $X(u), X(v), X(w)$,
 where $w \in W^P$ is the m.l.s. of uv , w
 (w - longest el. in W).

$QH^*(X)$ (Quantum cohomology). *lives of*

A degree d is a nonnegative combination of Schubert curves (dim 1),
 $d \in H_2(X; \mathbb{Z})$ ~~identifies~~ $d = (d_{\alpha_i})_{\alpha_i \in \Delta \setminus \Delta_P}$, $d_{\alpha_i} \in \mathbb{Z}_+$

let $g = (g_{\alpha_i})_{\alpha_i \in \Delta \setminus \Delta_P}$, $\deg g_i = \dots$ (but ≥ 2)

$$g^d := \prod_{\alpha_i \in \Delta \setminus \Delta_P} g_{\alpha_i}^{d_i}$$

$QH^*(X)$ - $\mathbb{Z}[q]_{\geq 3}$ -algebra, with $\mathbb{Z}[q]_{\geq 3}$ -basis

$\{\tau_w\}_{w \in W^P}$. Multiplication:

$$\tau_u * \tau_v = \sum_{d, w} q^d c_{uv}^{w, d} \tau_w$$

$c_{uv}^{w, d}$ = # of rational curves of degree d which pass through
 general translates of $X(u), X(v)$ and $X(w)$.

GOAL: Compute $c_{uv}^{w, d}$!

- for Grassmannians (including some Grassmannians in
 other types): many algorithms (Perrin + Giambelli, min. root
 $\geq \text{st} \alpha$ -flags etc.); \exists a positive conjecture (Bud-Kreuzl-
 - Tamvakis).

III

- in type A (i.e. $X = \mathrm{Fl}(m_1, \dots, m_k; n)$ -
- a partial flag manifold)

Pier + Giambelli (I.C. Fontanino, Forum-Geoff
- Postnikov).
 - in general types:
 - for G/B : a Chevalley formula $\mathbb{T}_X \times \mathbb{T}_{S_{\lambda}}$
(Peterson, Fulton-Woodward)
 - Note: In this case $\{\mathbb{T}_{S_{\lambda}}\}$ generate $H^*(X)$, $QH^*(X)$
 - a Giambelli formula (algorithm for finding polynomial P s.t. $\mathbb{T}_X = P(\mathbb{T}_{S_{\lambda}} : \lambda \in \mathcal{B} \times S_p)$) obtained by Mare
 - for G/P :
 - no multiplication by gens known
 - no Giambelli
- However Peterson comparison formula (proved by C. Woodward):
- $$\begin{matrix} c_{uv}^{w,d} \\ \downarrow \\ \text{GW for } QH^*(G/P) \end{matrix} = \begin{matrix} c_{u'v'}^{w',d'} \\ \downarrow \\ \text{GW for } QH^*(G/P) \end{matrix}$$
- where w, v, w', v' are explicitly defined.
- Conclusion: Study of $QH^*(G/P)$ is hard.

New idea intuition To study $\mathrm{CH}^*(X)$, study an equivariant version of it.

PROMISE: - will be able to compute c_{equiv} avoiding Giambelli, mult. by generators (but not Chiralley), Peterson comparison formula.

First, briefly recall:
~~Equivariant cohomology~~

To make the statements simpler, in what follows we restrict to $X = G(p, m)$ - Grassmannian of p -planes in \mathbb{C}^m .

~~Almost all the statements will generalize to G/p . We point out those that do not.~~

w/w_p cross partitions $\lambda = (\lambda_1, \dots, \lambda_r) \subset \boxed{\quad}_{m-p}$

~~polytopes on divisor class~~ Fix $F_0 = \langle e_1, \dots, e_m \rangle \subset \mathbb{C}^m$ - std. flag

Schubert variety $\mathcal{R}_\lambda(F_0) = \{V \in G(p, m) : \dim V \cap F_{m-r+i-\lambda_i} \geq c_i\}$.

$T_\lambda = \mathcal{R}_\lambda(F_0) \} \in H^{2(\lambda)}(X; \mathbb{K})$, $\forall i = \lambda_1 + \dots + \lambda_p$

- one divisor class: T_\square .

Equivariant cohomology

$T \cong (\mathbb{G}^\times)^m \curvearrowright X$ (if $\lambda = H_T^*$ wt) = $\mathcal{R}\{T_1, \dots, T_m\}$

($T_i - T_{i+1}$ cross $-d_i$ in previous case)

$T_\lambda^T = \{\mathcal{R}_\lambda(F_0)\}_T \in H^{2(\lambda)}_T(X)$

$\widehat{\mathcal{H}}_T^Y(X)$ - Λ -algebra with Λ -basis $\{\mathfrak{T}_\lambda^Y\}_{\lambda \in \square_p}$.

Multiplication

$$\mathfrak{T}_\lambda^Y \cdot \mathfrak{T}_\mu^Y = \sum c_{\lambda\mu}^Y(t) \cdot \mathfrak{T}_\nu^Y$$

$c_{\lambda\mu}^Y(t) \in \Lambda$ - homogeneous, of degree $(\lambda_1 + \mu_1 - \nu_1)$.

$$\text{If } \deg(c_{\lambda\mu}^Y(t)) = 0 \Rightarrow c_{\lambda\mu}^Y(t) = c_{\lambda\mu}^Y.$$

Equivariant quantum cohomology (Givental-Kim)

$Q\mathcal{H}_T^Y(X)$ - graded $\Lambda_{\{e\}}$ -algebra, $\deg e = m$, with $\Lambda_{\{e\}}$ -basis $\{\mathfrak{T}_\lambda\}_{\lambda \in \square_p}$.

Multiplication:

$$\mathfrak{T}_\lambda \circ \mathfrak{T}_\mu = \sum_{d, \lambda} q^d \cdot c_{\lambda\mu}^{Y,d}(t) \mathfrak{T}_\nu$$

$c_{\lambda\mu}^{Y,d}(t)$ - eg. Gromov-Witten invariants (Givental-Kim).

$c_{\lambda\mu}^{Y,d}(t) \in \Lambda$ homogeneous of poly degree $(\lambda_1 + \mu_1 - \nu_1 - md = 0)$.

Proposition • $d=0$, $c_{\lambda\mu}^{Y,d}(t) = c_{\lambda\mu}^Y(t)$ - eq. coeff.

• $\deg(c_{\lambda\mu}^{Y,d}(t)) = 0 \quad c_{\lambda\mu}^{Y,d}(t) = \hat{c}_{\lambda\mu}^{Y,d}$
 $(\lambda_1 + \mu_1 - \nu_1 - md = 0)$ ordinary GW

Example $Q\mathcal{H}_T^Y(\mathbb{P}^2)$:

$\lambda: (0), \square, \square\square, \deg e = 3$.

$$\begin{aligned} \mathfrak{T}_0 \circ \mathfrak{T}_\square &= \mathfrak{T}_\square + (T_2 - T_3) \mathfrak{T}_\square, & \mathfrak{T}_\square \circ \mathfrak{T}_\square &= (T_1 - T_2)(T_1 - T_3) \mathfrak{T}_\square + 2\mathfrak{T}_\square + \\ \mathfrak{T}_0 \circ \mathfrak{T}_\square &= (T_1 - T_3) \mathfrak{T}_\square + 9. & & + (T_1 - T_2), \end{aligned}$$

Note: In $\tau_{\square} \circ \tau_{\square}$ get mixed term.

$$\text{Thm (Eg Chvalley)} \quad \tau_{\lambda} \circ \tau_{\alpha} = \sum_{\mu \rightarrow \lambda} \tau_{\mu} + c_{\alpha \lambda}^{\lambda} (\tau_{\lambda} + \tau_{\lambda^-})$$

$\mu \rightarrow \lambda \Leftrightarrow \lambda \subset \mu$ and .

$$c_{\alpha \lambda}^{\lambda} (\tau) = \sum_{i=1}^p T_{m-p+i-\lambda_i} - \sum_{j=m-p+1}^m T_j$$

λ^- - partition obtained from λ by removing $m-1$ boxes from last row of λ .

Ex: $p=2, m=4, \text{ & } \begin{array}{|c|c|}\hline \square & \square \\ \hline \end{array} \rightsquigarrow \square$

Note: NO MIXED TERMS!

Cor: Eg Chvalley determines $Q_{\lambda}^{\alpha}(\chi)$ i.e.

let (A, \diamond) - $\underset{\text{graded}}{\mathcal{A}_{\Sigma}}$ - algebra, & commutative at

(a) have $\{t_{\lambda}\}_{\lambda \in \square(p)}$ A_{Σ} - basis

(b) $t_{\lambda} \diamond t_{\mu}$ is given by Eg Chvalley.

Then \exists sur. hom. $(A, \diamond) \cong (Q_{\lambda}^{\alpha}(\chi), \circ)$

Proof By effective algorithm.

Idea:

$$(\tau_\lambda \circ \tau_\mu) \circ \tau_\nu = \tau_\lambda \circ (\tau_\mu \circ \tau_\nu)$$

$\widehat{v\mu}$

$$\text{if } \lambda \neq \nu \quad c_{\lambda\mu}^{\nu, d}(t) = \frac{\sum_{\epsilon \rightarrow \lambda} c_{\epsilon, \mu}^{\nu, d}(t) - \sum_{\nu \rightarrow \epsilon} c_{\lambda\mu}^{\epsilon, d}(t)}{c_{\alpha\nu}^{\nu}(t) - c_{\alpha\lambda}^{\lambda}(t)} +$$

$$+ \frac{c_{\lambda\mu}^{\nu, d-1} - c_{\lambda\mu}^{\nu+, d-1}}{c_{\alpha\nu}^{\nu}(t) - c_{\alpha\lambda}^{\lambda}(t)} \quad (*)$$

$$\nu^+ \text{ is st } (\nu^+)^\sim = \nu$$

$(*)$ is a recurrence relation; by using double induction, on d , and on deg $c_{\lambda\mu}^{\nu, d}(t)$, can determine $c_{\lambda\mu}^{\nu, d}(t)$

There should be a 'nice' combinatorial formula for $c_{\lambda\mu}^{\nu, d}(t)$. This is suggested by:

Thm (Peterson - Gasharov, $d=0$; $-d > 0$)

$$c_{\lambda\mu}^{\nu, d}(t) \in \mathbb{K}_+ [T_1 - T_2, \dots, T_{m+1} - T_m].$$

In fact, I have conjecture (just for $G_r(q, n)$) for $c_{\lambda\mu}^{\nu, d}(t)$ in terms of puzzles, generalizing cons. of Burc - Klesh - Tamrak.

