

10 June 2005

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Equivariant Cohomology and GKM for flag varieties

Mantra : "when in doubt, localize".

(and: "when in doubt, look at the (normal map) picture")

References: Tolman, Woodward

On the cohom ring of "Flag" Toric varieties

Karlsson, Knutson, Tao, P. Klei

et al. give, via toric flag varieties

• H. Henriques, Holm

at the moment, 47 references

+ appears in Adv. in Math.

Plan: - Localization and the equiv 1-skeleton

- GKM theory and graphs

? ↗ - Application/Example: Schubert classes

→ [- More general results (joint w/ Henriques, Holm)]

GOAL: Give a (combinatorial) description of the equiv cohom. of flag varieties $H_T^*(G/P)$, suited for ring computations.

1) Localization and equiv 1-skeleton.

Idea: reduce the computation to a point.

• Suppose $T \subset \text{Spt}$, trivially. $H_T^*(X) := H^*(X \times_T ET)$, so

"
 $(S^1)^n$

$$H_T^*(\text{pt}) = H^*(\text{pt} \times_T ET)$$

$$= H^*(BT)$$

$$= H^*((\mathbb{C}P^\infty)^n)$$

= polynomials in n variables

$$\cong \text{Sym}(\mathbb{C}^*)$$

MORAL: Equiv cohom of a point carries a lot of information

(unlike ordinary coh^{gm} case)!

• In a flag variety $G/P = \coprod_{[w] \in W_G/W_P} B \tilde{w} P/P$ Bruhat cell decomposition

W_G = weyl gp of G

W_P = " - " of P

\tilde{w} = representative of

$[w] \in W_G/W_P$

we have $|W_G/W_P|$ many such T -fixed pts, where T acts trivially
as above.

$$\text{so } H_T^*((G/P)^T) = \bigoplus_{P \in (G/P)^T} H_T^*(P) = \bigoplus_{P \in (G/P)^T} \underbrace{\mathbb{C}[x_1, \dots, x_n]}_{\text{Sym}(\mathbb{C}^*)}$$

just a sum of polynomial rings!

Since for any $T \subset M$, $M^T \hookrightarrow M$, have a map $H_T^*(M; \mathbb{C}) \xrightarrow{i^*} H_T^*(M^T; \mathbb{C})$.

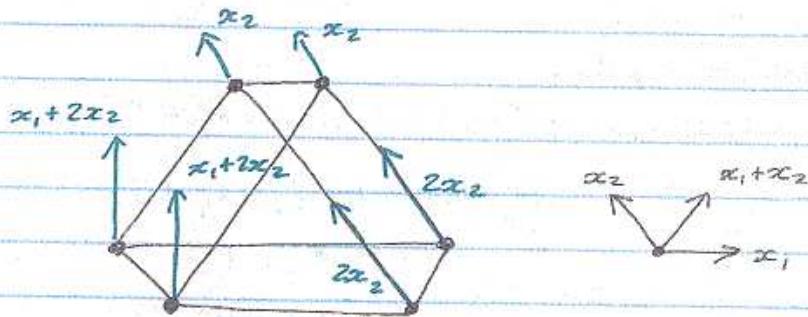
"MAGIC": for $T \subset G/P$, this map is an injection!

— so can describe a cohomology class in $H_T^*(M; \mathbb{C})$ as a list of polynomials attached to each $p \in (G/P)^T$

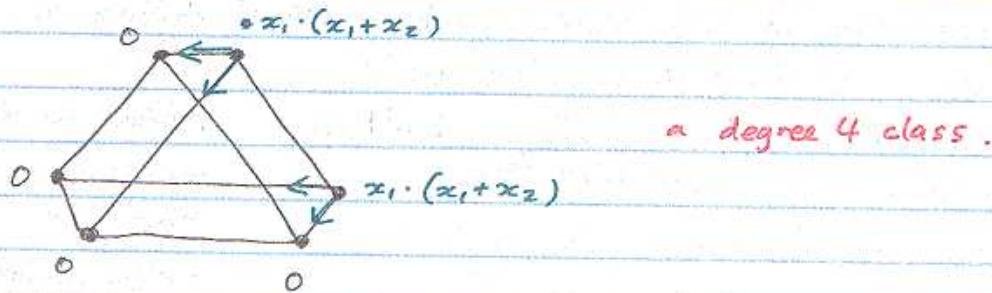
— i^* is a ring map, but $H_T^*(M^T)$ just a sum of polynomial rings, so product computations simplified.

The arrow notation, ie. "the snapshot":

Example: $\mathrm{Fl}(\mathbb{C}^3)$



- each arrow is to be thought of as a linear polynomial.
- each variable has degree 2, so this is a deg 2 class.



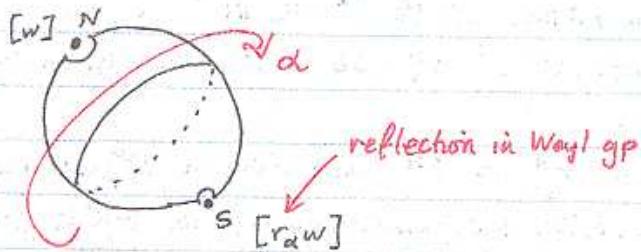
--- "bouquet of arrows" notation.

What is image of i^* ?

Fields : Eqvt Coh and
GKM (2)

More data : $\left(\begin{array}{l} T\text{-fixed pts} = \text{pts with stabilizer equal to } T \\ \text{"stabilizer as big as possible"} \\ = *T\text{-eqvt } \mathbb{C}\text{-skeleton}* \end{array} \right)$

now consider $*T\text{-eqvt 1-skeleton}^* := \text{pts with stabilizer codim 1 in } T$
 $= \text{eqvt 1-cells (copies of } \mathbb{C}^*) \text{, with 2}$
 $\text{fixed pts at } N \text{ and } S \text{ poles}$



The residual T -action on \mathbb{C}^* is α .
 (sign ambiguity here, but
 doesn't matter)

Another way to think about this: look at isotropy repn $T \mathbb{C}^* T_{[w]} G/P$

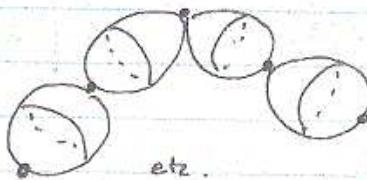
(so we're saying these \mathbb{C}_α 's close up to
 \mathbb{P}^1 's in G/P)

$$\cong \bigoplus_\alpha \mathbb{C}_\alpha$$

as a T -rep

\uparrow
 T -weights
 \downarrow
 T -wt spaces

so the eqvt 1-skeleton is a "balloon animal," with spheres connecting
 some of the fixed pts :



2) GKM theory and graphs.

(... getting to an answer to the question...)

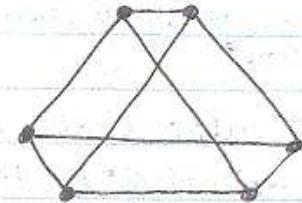
"GKM graph" = [Goresky-Kottwitz-MacPherson]

vertices = $(G/P)^T$

edges = when \exists a \mathbb{P}^1 connecting the vertices [for G/P : when related by simple reflection

PLUS extra data: each edge e decorated with weight $\alpha_e \in \mathbb{Z}^\times$.

Ex: $\mathrm{Fl}(\mathbb{C}^3)$



Observations: - Think of the picture in $(\mathbb{C}^2)^* \cong \mathbb{R}^2$

- Given action $W \subset \mathbb{C}^* \wr$, the vertices are given by a Weyl orbit.

- The weights $t_{\mathbb{Z}}^* \subseteq t^*$ also live in \mathbb{C}^* , so are also vectors. With this embedding of the GKM graph (given by the T-moment map), the weight data $\{\alpha_e\}$ encoded by the direction of the edge [up to sign, again irrelevant].

SLIDE: Examples of GKM graphs.

Theorem: [G-K-M]

$$\text{Let } (x(v))_{v \in W \backslash G / W_p} \in H_T^*((G/p)^T) \cong \bigoplus_{v \in (G/p)^T} H_T^*(v).$$

$$(x(v))_{v \in W \backslash G / W_p} \in \text{im}(i^*) \iff \exists | x(v) - x(r_\alpha v) \\ \forall v, \alpha.$$

... so identifies $H_T^*(G/p)$ as a subring of $H_T^*((G/p)^T)$ in terms

of combinatorial data in the GKM graph. Now can do

computations. NOTE: result is actually much more general...

3) **Q** So, for example, what do Schubert classes look like?

SLIDE: • Example: $\mathrm{Fl}(\mathbb{C}^3)$. Can see some Morse theory here: looking at euler classes of negative normal bundles.

SLIDE: • Example: Schubert-type computation.

- It mainly proves "equivariant generalizations" aspects of the Goresky-May theorem
 - e.g. for K_T^* of Kac-Moody groups, May's theorem would be:
even doesn't make sense, but in different language (equivariant)
 - after Linda Chen has work on Schubert varieties for GKM theory, can use ΩG instead of G
- Fields: Equiv Coh and
GKM (3)

4) More general results (joint w/ Henriques, Holm)

arXiv: math.AT/0409305

- more general equivariant cohomology theories: T -equivariant complex oriented cohomology theory w/ some assumptions "have Euler classes"
- eg: $H_T^*(-, \mathbb{Z}), K_T^*, MU_T^* \leftarrow$ equivariant complex cobordism
- $\underbrace{\qquad\qquad\qquad}_{\text{all examples we can actually compute are here}}$
- more general spaces: an equivariant stratified space, possibly ∞ -dimensional (so covers flag varieties of Kac-Moody groups, e.g. $\Omega G = LG/G$), w/ some conditions on attaching maps.
- explicit examples of module generators for some Kac-Moody flag varieties.
(not obvious! Schubert varieties singular...?)

SLIDE: $H_T^*(\Omega SU(2))$

SLIDE: $K_T^*(\Omega SU(2))$.

SLIDE: Our 3 main theorems.

Some rank 2 examples :

$$\begin{array}{c} \underline{G} \\ \underline{P} \\ SL(3, \mathbb{C}) \end{array} \quad \left(\begin{matrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{matrix} \right) \quad \begin{array}{c} \underline{G/P} \\ \mathbb{C}\mathbb{P}^2 \end{array}$$

$$\Gamma \subset \mathfrak{t}^* \cong \mathbb{R}^2$$



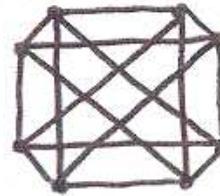
$$SL(3, \mathbb{C}) \quad \left(\begin{matrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{matrix} \right) \quad Fl(\mathbb{C}^3)$$



$$Spin(5)_{\mathbb{C}} \quad \text{Isotropic lines in } \mathbb{C}^5$$



$$Spin(5)_{\mathbb{C}} \quad \text{Isotropic flags in } \mathbb{C}^5$$



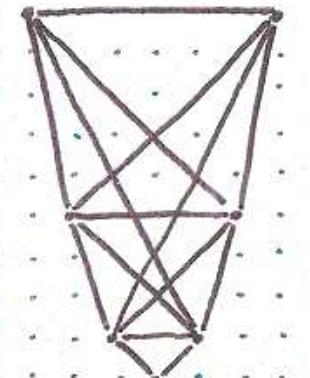
$$G_2 \quad P_{\text{long}}$$



$$LSL(2, \mathbb{C}) \quad \text{loops extending over } O \in \mathbb{C}^\times \quad \cap SU(2)$$

" $L^+SL(2, \mathbb{C})$ "

:



Main results: ($E_T^* = H_T^*$ or K_T^*)

Theorem 1: $E_T^*(G/P) \xrightarrow{i^*} E_T^*((G/P)^T) \cong \prod_{w_G/w_P} E_T^*(pt)$
is injective.

Theorem 2: [generalizes G-K-M]

$$(x(v))_{v \in W_G/W_P} \in \text{Im}(i^*) \text{ iff } e(\alpha) | x(v) - x(r_\alpha v) \quad \forall v, \alpha.$$

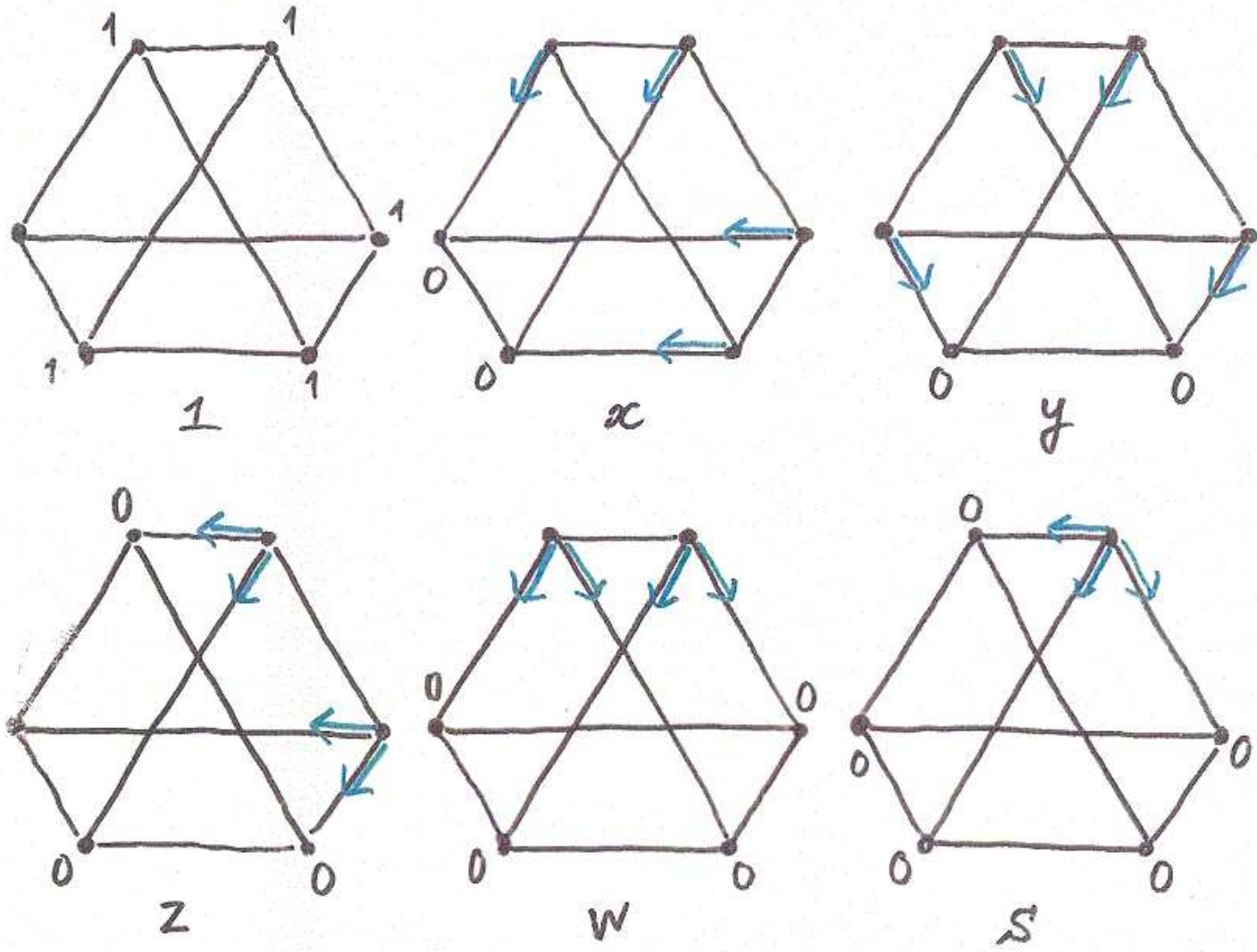
Theorem 3: If a collection $\{x_v\}$ $x_v \in E_T^*(G/P)$,
satisfies one for each vertex

- $x_v(w) :=$ class x_v restricted to w (as above)
 $= 0$ when $w \not\succeq v$ in Bruhat order
- $x_v(v) = \prod_{\alpha: r_\alpha(v) < v} e(\alpha)$

then $\{x_v\}$ form a basis of $E_T^*(G/P)$

as a $E_T^*(pt)$ -module.

Example: $E_T^*(pt)$ -module generators for $E_T^*(\mathrm{Fl}(\mathbb{C}^3))$
 (Again $E_T^* = H_T^*$ or K_T^*)

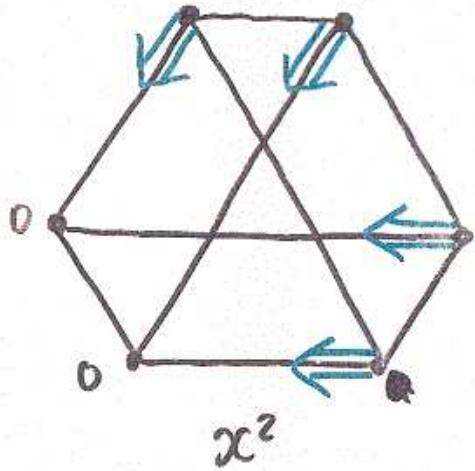


A "bouquet" of arrows represents the product of the Euler classes of the corresponding characters of T .

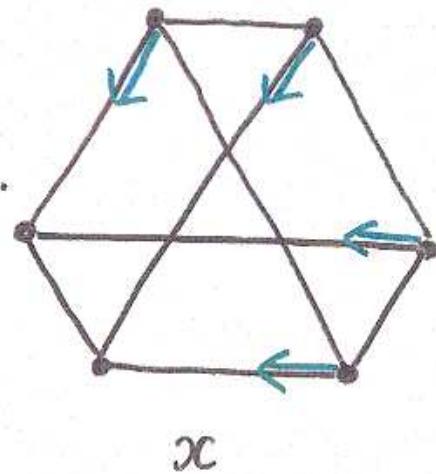
(In H_T , $e(\alpha) = \alpha \in \Lambda \subset \{\star \cong H_T^2(pt);$
 in K_T , $e(\alpha) = 1 - e^\alpha \in K_T(pt) \cong R(T)\})$

Example: a Schubert-type computation

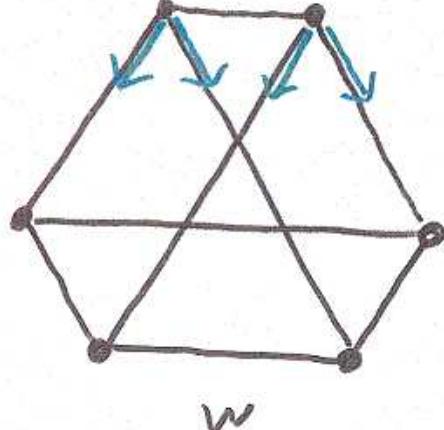
Let $G/P = \text{Fl}(\mathbb{C}^3)$.



$$= (1 - e^{\leftarrow}) \cdot$$



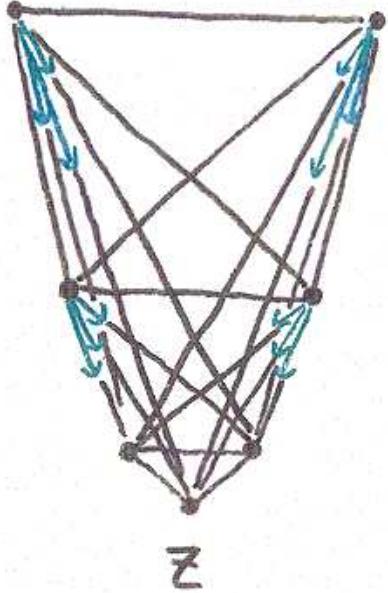
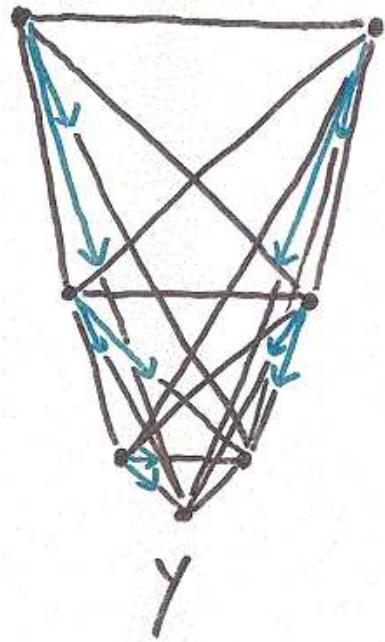
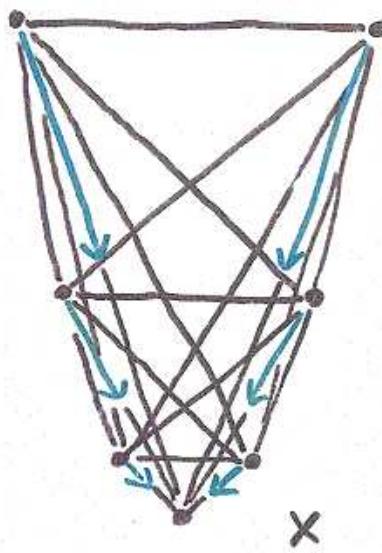
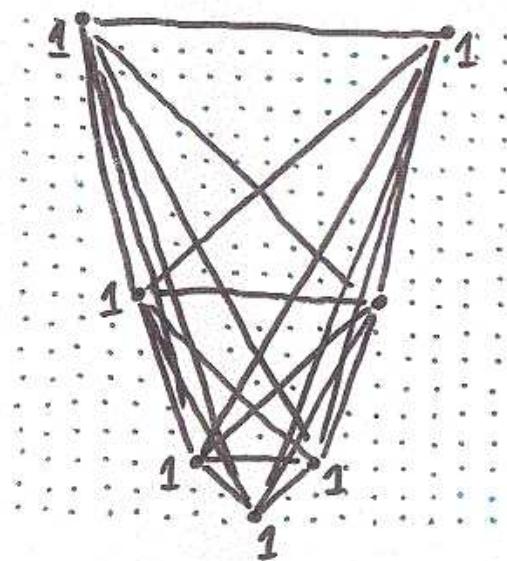
$$+ e^{\leftarrow} \cdot$$



so

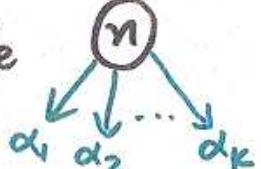
$$x^2 = (1 - e^{\leftarrow}) \cdot x + e^{\leftarrow} \cdot w$$

Example : The first few module generators
of $H_T^*(\Omega SU(2))$



Example: $K_T^*(\Omega SU(2))$

$$\text{Let } p_n(\alpha_1, \dots, \alpha_k) = \left[\prod_{i=1}^k (1 - e^{\alpha_i}) \right] \cdot \left[\sum_{\substack{a_1 + \dots + a_k = 0 \\ a_1, \dots, a_k \geq 0}} e^{a_1 \alpha_1 + \dots + a_k \alpha_k} \right]$$

and now write  for $p_n(\alpha_1, \dots, \alpha_k)$.

Here are the first few (omitting 1) $K_T^*(\text{pt})$ -module generators of $K_T^*(\Omega SU(2))$.

