Physmatics

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The Fields Institute, June 2, 2005

Mathematics and Physics

- ~400 BC, Aristotle
- 1972, Freeman Dyson
- 2001, Edward Witten

Questions

A: What happened?

B: What is happening?

Answers

A: Mathematical Physics (mostly) ran its course

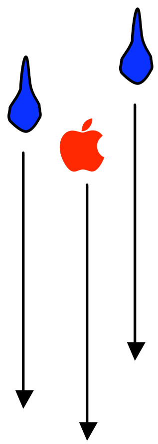
B: ***Physmatics***

I will review some concepts in physics and mathematics, showing how these disciplines relate. I will then use these concepts to explain duality, an example of ***physmatics***.

Mathematics in our mist

1. Droplets form
2. Droplets fall

- Zillions of drops, each with a jillion molecules. Too much! Consider droplet formation by making predictions about average process: statistics. See?
- To describe fall to Earth, model droplet as a single point accelerating to Earth (Newton).

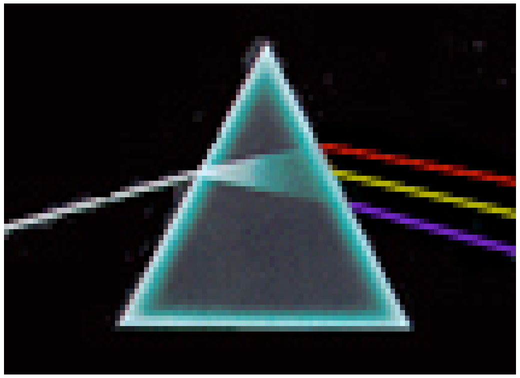


We're almost **forced** to use mathematics. For gravity, we need a space described by some number of coordinates, with notions of length and time: **geometry**.

Einstein: Length and Time are **not** universal. Need mathematics without preferred coordinates: ***differential* geometry**!

Quantum Sun

White light contains a spectrum



"Pure" light behaves this way. Why?

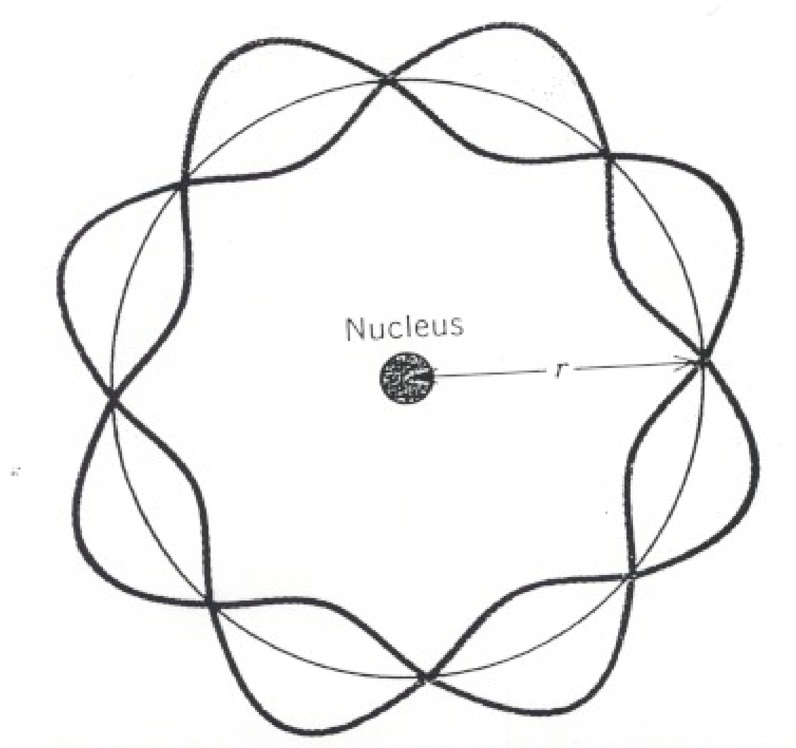
1. There are atoms, with a nucleus and electrons.
2. Bohr: electrons orbit in discrete energy bands (Schrödinger equation).
3. When electrons jump bands, the difference in energy is emitted as light.
4. If there are enough molecules and bands, the discrete rays look like a full spectrum.

Newton and Hooke were both right!

Welcome to the dark side of the moon!

Quantization of Momentum

- particle moving in a circle can have only certain discrete "wavelengths," as they must divide the circumference: $\lambda = 2\pi R/n$
- momentum is inversely proportional to wavelength: $p \sim 1/\lambda \sim n/R$
- if circle "large" compared to h, then momenta seem to live on a continuum



Bohr Atom

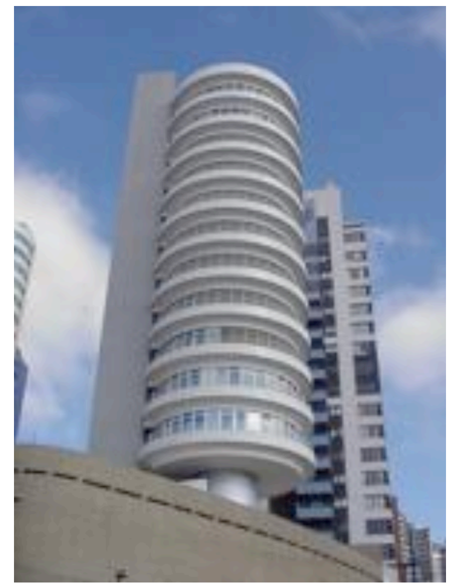
Remember this:

$p = n/R$

Gauge Symmetry

James Clerk and Katherine Maxwell, with dog

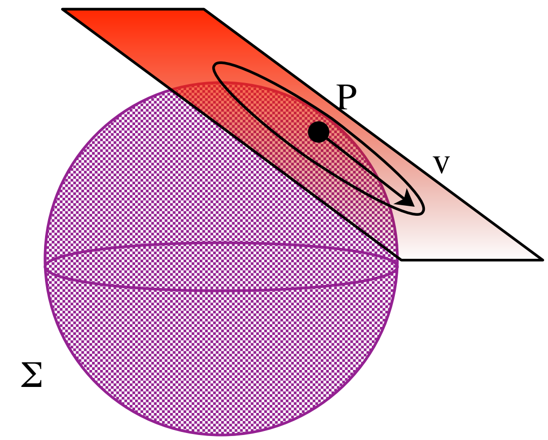
- At every point, you are allowed a "rotation" by an arbitrary angle. This rotation changes nothing, as with Suite Vollard.
- The electric/magnetic potential A has this freedom: $A \rightarrow A + df$ (here f is the angle).
- Just as b/a is unchanged if we replace b by bf and a by af, the replacement of A leaves E and B invariant.



The Suite Vollard Building, Brazil

Differential Geometry

- At each point P of a space Σ there is a flat tangent plane
- The collection of all tangent planes forms the *tangent bundle* TΣ of Σ
- A point of TΣ can be thought of as a point of Σ together with a velocity vector, v

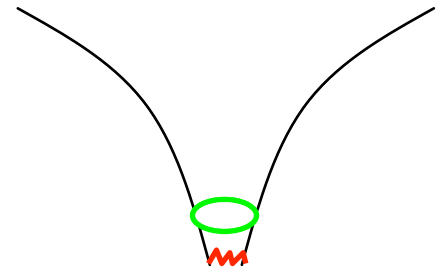


If we consider not velocities but just directions, then at each point of Σ we have a circle. This is a *circle bundle* – like the Suite-Vollard building.

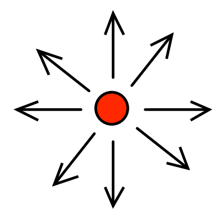
The dark side: Whereas zero velocity makes sense, zero direction does not! So constructions with circle bundles have the great freedom of starting each of the circles from an arbitrary point: gauge symmetry!

Objects in physics

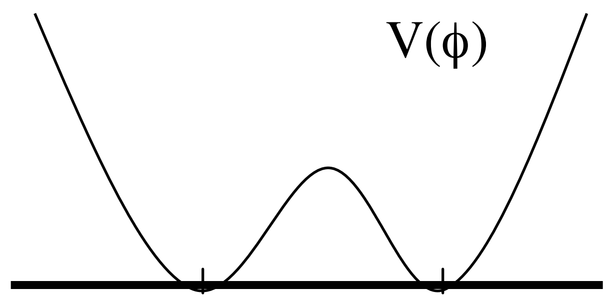
- Black holes

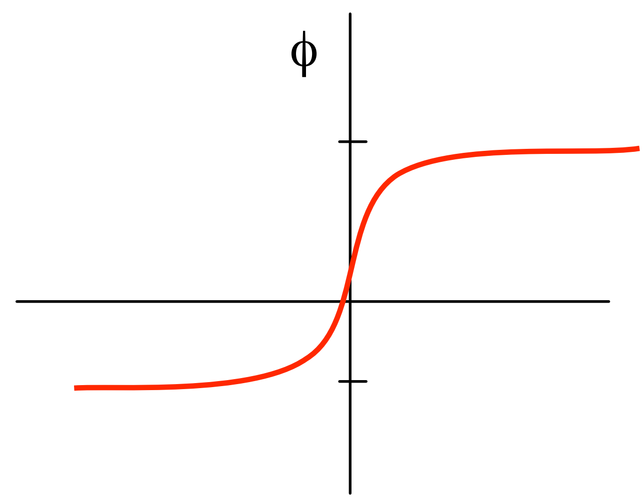


- Magnetic charges (monopoles)



- Solitons

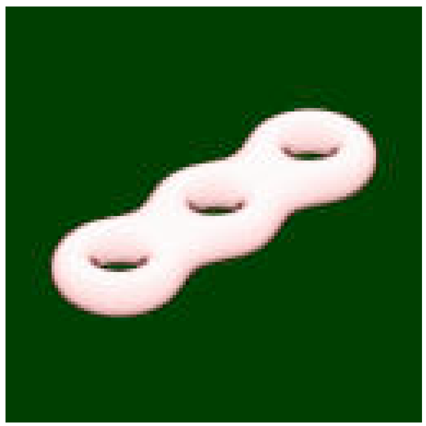


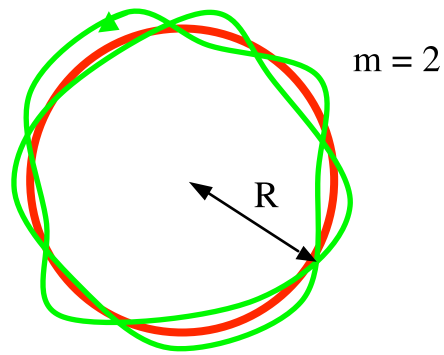


These can all be described as what we'll call branes.

Differential Topology

- A closed loop of string on a circle may wrap m times. Minimum length ~ mR.





- We know it has three holes (h=3), but what computation could a computer make to see this?
- Gauss and Bonnet: $\int K/2\pi = 2 - 2h$. There are generalizations of h to arbitrary spaces *and* bundles.

Remember:

1. Integers capture global information that is insensitive to details.
2. A minimal m-winding string on a circle has length mR.

Summarizing…

In describing physics to this stage, researchers developed the relevant mathematics along the way.

Physics	Math
Newtonian physics	Calculus
Gravity (Einstein)	Differential Geometry
Quantum theory	Analysis of functions, Diff Eq's, Representation theory
Electromagnetism and gauge theories	Geometry and Topology of bundles

After this, physics and math grew apart, leading to Dyson's "divorce."

(Also, recall p = n/R, length = mR, and objects described as "branes.")

Physmatics and Duality

- ***Physmatics*** is a reconciliation. Spouses *inter*dependent.
- String theory has led to and been influenced by new mathematics. Ideal source of ***physmatics***.
- Best example is duality symmetry, where two physical theories give *exactly* the same predictions – i.e., they're really the same!
- Mathematics ***is*** the study of when things are the same! That is, when A = B.
- The mathematics of duality is new and novel.

Duality in TV

Ralph	Fred
Alice	Wilma
Norton	Barney
Trixie	Betty

Not only must the objects correspond -- the relationships must correspond!

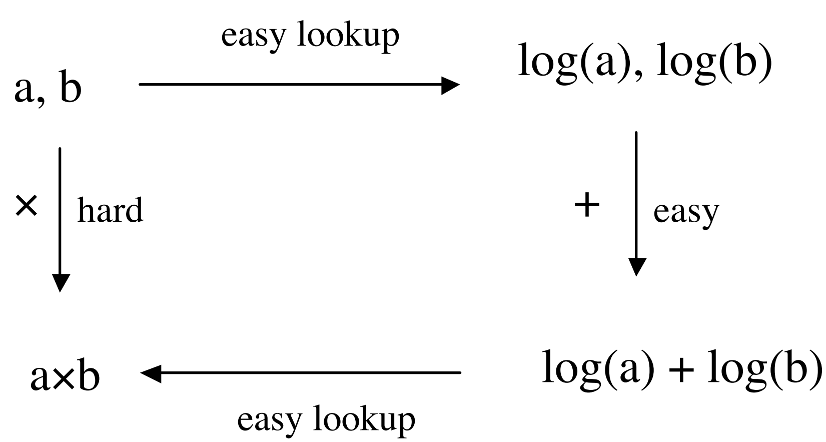
Ralph badgers Alice with good-natured bluster; Alice tolerates Ralph.

Fred badgers Wilma with good-natured bluster; Wilma tolerates Fred.

Duality in arithmetic: × = + !

$$10^2 \times 10^3 = 10^5 \overset{\text{“dual to”}}{\longleftrightarrow} 2 + 3 = 5$$

In the days of slide rules, we used this duality because addition is easier than multiplication. Map the problem to something easier, then map your answer back.

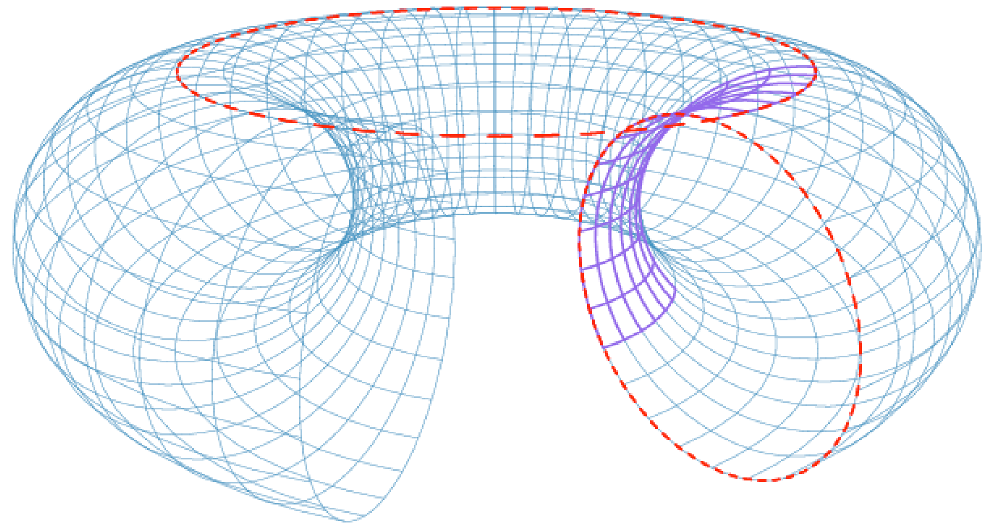


Duality in physics: string on a circle

- String can have winding (m) and momentum (n)
- An (m,n) string on a circle of radius R has energy
  $E = (n/R)^2 + (mR)^2 = [m/(1/R)]^2 + [n(1/R)]^2$
- Same as an (n,m) string on a circle of radius 1/R !
- Big and small are **equivalent**

If the two string theories A and B are "dual," then…
EVERY object of A has a "mirror" object of B.
EVERY calculation of A has an equal mirror calculation of B.

Bagel



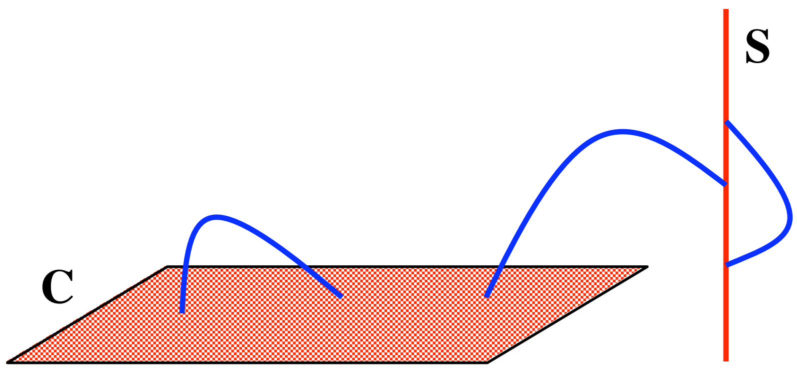
A bagel is the product of two circles of radii R_1, R_2.

In string theory, an equivalent bagel is the product circles of radii $1/R_1$, R_2.

Remarkably, there are dual string theories for pairs of spaces which differ in **topology** as well as **size**, i.e. spaces as different as bagel and bialy!

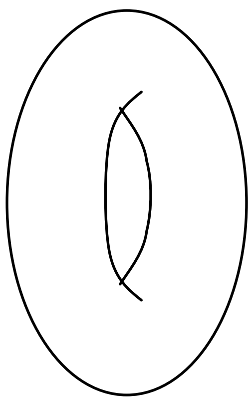
Reaping the math harvest

- First: identify different objects in the two theories.
- Objects: "branes." Types: C(omplex), S(ymplectic)
- Branes are (charged) locations where strings can end.



- Under the equivalence, the types of branes are interchanged: C-branes in A ↔ S-branes in B.
- Identify correspondence, then check relationships.

Mathematical description of C-branes

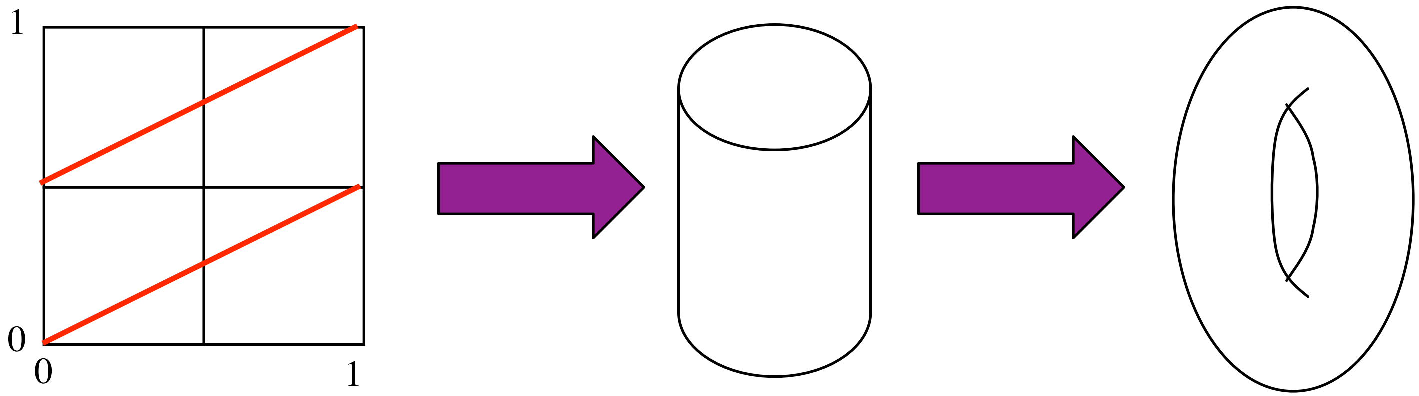


- Consider even-dimensional space, Σ (D=2 here). C-branes filling Σ are described as bundles.
- So the C-branes can be labeled with the same integers (charges) that label bundles.
- The relevant labels are 1) the dimension, or "rank," r, of the fiber space, and 2) the "degree" d, which generalizes the number of holes h from Gauss-Bonnet theorem and describes twistiness.

C-branes on a bagel are described by a pair of integers (r, d)

Mathematical description of S-branes

S-branes are minimal subspaces of dimension D/2. (D = 2 here.)



Closed paths on (unwrapped) bagel described by straight lines of slope d/r. The line $(r, d) = (2, 1)$ is shown.

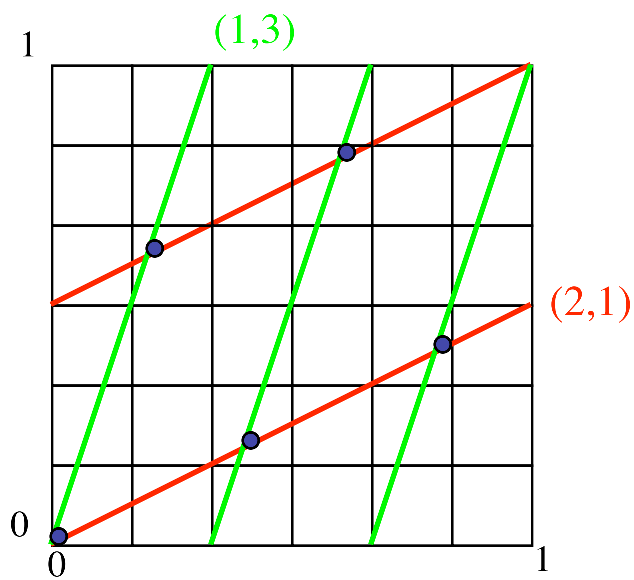
So S-branes on dual bagel also described by integers (r, d). Objects correspond!

Checking relationships I: connecting strings

- We have shown a correspondence among objects: C-brane *bundle* (r,d) on one bagel corresponds to S-brane *line* (r,d) on its mirror.
- Now we have to show that the *relationships* correspond.
- A famous extension of Gauss-Bonnet calculates the degree "between" bundles (r,d) and (r',d') as $|rd' - dr'|$. Counts strings between C-branes.
- What is the *dual* calculation for S-branes?

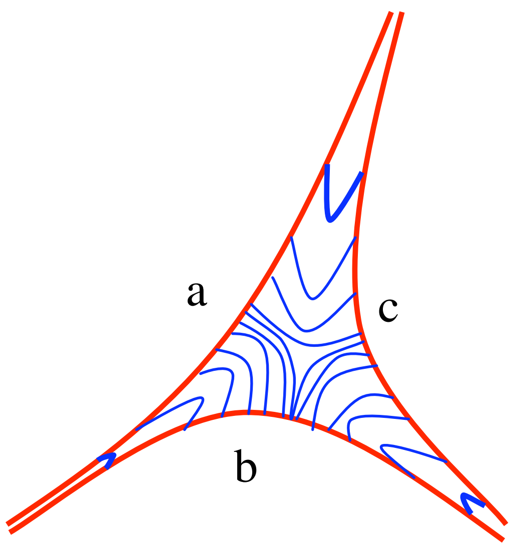
Testing duality

Dual Calcultion: Count minimal open strings between (2,1) S-brane and (1,3) S-brane.

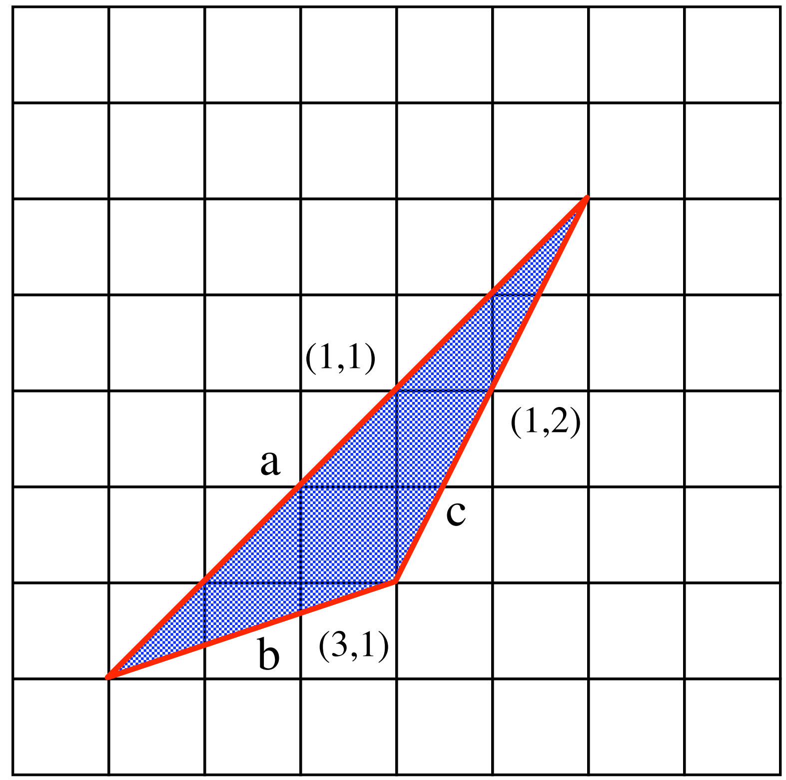


According to prediction, there should be
$rd' - dr' = 2\times3 - 1\times1 = 5$ points of intersection!

Checking relationships II: composition



An a-b and a b-c open string can form an a-c string in a physical scattering process.



Mathematically, count "triangles." Answer depends on area of triangles and box R_1R_2. (For C-branes, depends on shape R_1/R_2.)

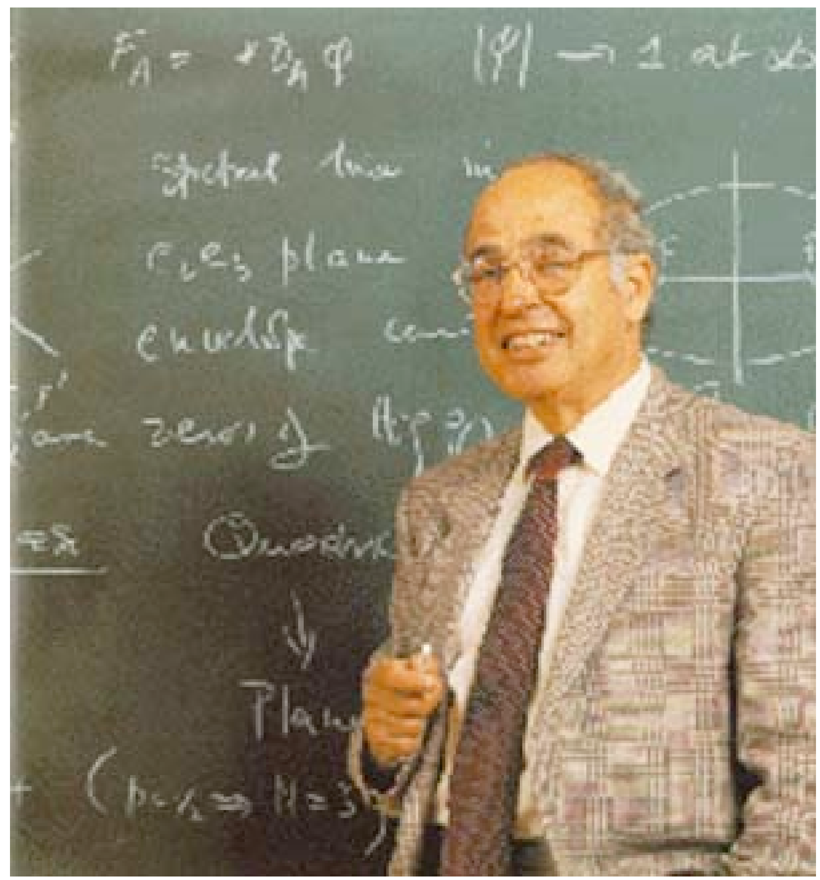
Conclusions

- This and all other tests of the duality between bagels have been proven conclusively. For example, C-brane calculation predicts exactly: $\sum_N e^{-5N^2 R_1 R_2}$
- Closer comparison of C-branes to S-branes actually requires the inversion of R to 1/R.
- Radically different spaces have been shown to be "mirror" pairs.
- This new math has been indispensible to the advancement of new physics.
- The ideas of physics are fundamental to pure, abstract math. This is ***physmatics***.

Now I feel as if I should succeed in doing something in mathematics, although I cannot see why it is so very important…
The knowledge doesn't make life any sweeter or happier, does it?

— Helen Keller (USA, 1880–1968)

A picture is worth a thousand retorts:



Sir Michael Atiyah

The End

Aristotle

The so-called Pythagoreans, who were the
first to take up mathematics,
not only advanced the subject, but saturated
with it, they fancied that the principles of
mathematics were the principles of all things.

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Freeman Dyson

I am acutely aware that the marriage between mathematics and physics, which was so enormously fruitful in the past centuries, has recently ended in divorce.

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Edward Witten

If you went back to the nineteenth century or earlier, mathematicians and physicists tended to be the same people. But in the twentieth century, mathematics became much broader and in many ways much more abstract. What has happened in the last 20 years or so is that some areas of mathematics that seemed to be so abstract that they were no longer connected with physics instead turn out to be related to the new quantum physics, the quantum gauge theories, and especially the super-symmetric theories and string theories that physicists are developing now.

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droplet formation

VOLUME 90, NUMBER 1 PHYSICAL REVIEW LETTERS week ending 10 JANUARY 2003

Kinetic Potential and Barrier Crossing: A Model for Warm Cloud Drizzle Formation

Robert McGraw and Yangang Liu

Environmental Sciences Department, Atmospheric Sciences Division, Brookhaven National Laboratory, Upton, New York 11973

(Received 21 June 2002; published 9 January 2003)

The kinetic potential of nucleation theory is used to describe droplet growth processes in a cloud. Drizzle formation is identified as a statistical barrier-crossing phenomenon that transforms cloud droplets to drizzle size with a rate dependent on turbulent diffusion, droplet collection, and size distribution. Steady-state and transient drizzle rates are calculated for typical cloud conditions. We find drizzle more likely under transient conditions. The model quantifies an important indirect effect of aerosols on climate-drizzle suppression in clouds of higher droplet concentration.

DOI: 10.1103/PhysRevLett.90.018501 PACS numbers: 92.60.Nv, 47.55.Dz, 82.60.Nh

Clouds and precipitation play crucial roles in regulating Earth's energy balance and water cycle [1]. Although it has been well established that three basic physical processes (nucleation, condensation, and collection) are involved in the formation of warm rain where the ice phase plays no role, many issues regarding the initiation of warm rain remain unsolved [2–5]. One of the funda- the equilibrium:

$$A_g + A_1 = A_{g+1}, \quad (1)$$

where A_1 represents the water vapor monomer and A_g a drop of size g. Under conditions of stable or (constrained) metastable equilibrium, detailed balance gives

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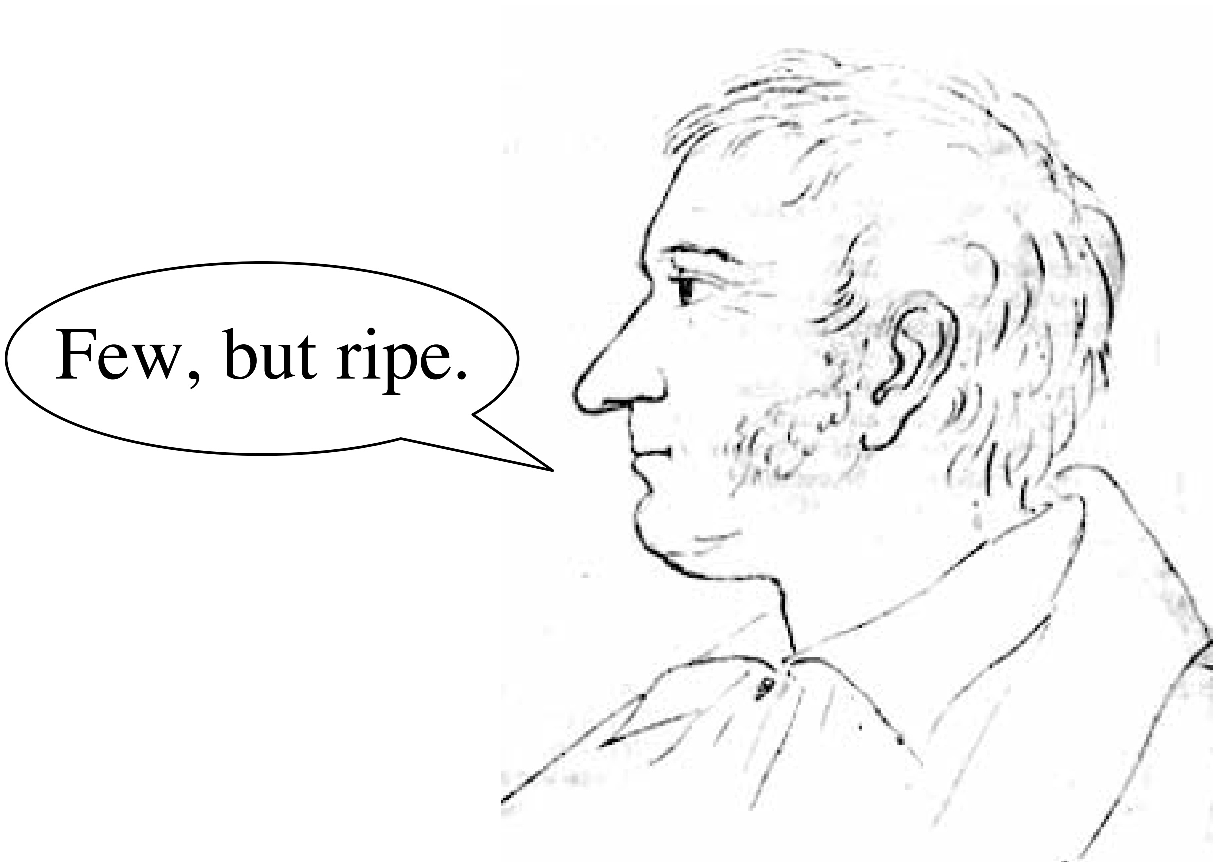
Karl Friedrich Gauß

(Germany, 1777-1855)



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Albert Einstein

(Germany, 1879–1955)

"Put your hand on a hot stove for a minute, and it seems like an hour. Sit with a pretty girl for an hour, and it seems like a minute. That's relativity."

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Relationships

- No one really understood music unless he was a scientist, her father had declared, and not just a scientist, either, oh, no, only the real ones, the theoreticians, whose language mathematics. She had not understood mathematics until he had explained to her that it was the symbolic language of relationships. "And relationships," he had told her, "contained the essential meaning of life."

  —Pearl Buck (USA, 1892-1973), *The Goddess Abides*, Pt. I, 1972.

- Mathematicians do not study objects, but relations between objects. Thus, they are free to replace some objects by others so long as the relations remain unchanged. Content to them is irrelevant: they are interested in form only.

  – Jules Henri Poincaré (France, 1854-1912)

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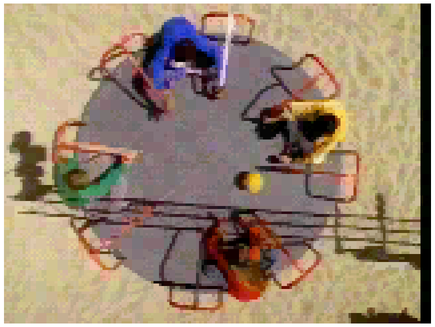
Examples of *Physmatics*

- Mirror symmetry
- String dualities (heterotic/type-II)
- Conifold transition (GW=CS=DT open/closed)
- Noncommutative geometry
- Geometrization with dilaton action
- Formality conjecture and Feynman diags
- Langlands/CFT
- Topology/Verlinde formula
- Chern-Simons theory and knot invariants

Coriolis “Force”

Newtonian physics looks different in spinning coordinates.



Einstein’s formulation is valid in *any* set of coordinates.

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(Movie taken from http://ww2010.atmos.uiuc.edu/(Gh)/guides/mtr/fw/crls.rxml)