Orbifold Vertex Algebras and EALA Representations

(work in progress with Y. Billig)

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Goal: Use vertex algebras to construct interesting representations for EALAs.

Outline:

1. Review Affine Theory
   - Loop Realization of Affine L.A.s
   - Vertex Algebras & Vertex L.A.s
2. Vertex Algebras for EALAs
   - EALAs
   - Toroidal Example
   - Multiloop Realization
   - Orbifold Vertex Algebras
3. Representations for EALAs
   - Thin Coverings
   - Irreducible Modules

(We will work over $\mathbb{C}$.)

Loop Algebras

$\dot{\mathfrak{g}}$ finite dim. simple L.A.

$\sigma : \dot{\mathfrak{g}} \to \dot{\mathfrak{g}}$ auto of order m

Fix ξ prim. mth root of 1.

Write $k \mapsto \overline{k}$ in $\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$

$$\dot{\mathfrak{g}}_{\overline{k}} := \{x \in \dot{\mathfrak{g}} \mid \sigma x = \xi^k x\}$$

σ extends to $\dot{\mathfrak{g}} \otimes \mathbb{C}[t, t^{-1}]$:

$$\sigma(x \otimes t^n) = \xi^{-n} \sigma(x) \otimes t^n$$

Def: The <u>loop algebra</u>

$$L(\dot{\mathfrak{g}}, \sigma) := \sum_{k \in \mathbb{Z}} \dot{\mathfrak{g}}_{\overline{k}} \otimes t^k = \{\dot{\mathfrak{g}} \otimes \mathbb{C}[t, t^{-1}]\}^\sigma$$

Thm [V. Kac, 1969]: Every affine L.A. is a loop algebra extended by a central element (and a derivation).

Vertex Algebras

Def: A vertex algebra consists of the following data:

- (space of states) a vector space V
- (vacuum vector) $\mathbb{1} \in V$
- (vertex operators)
  $$\begin{aligned} Y(\cdot, z):\ V &\longrightarrow \operatorname{End} V[[z^{\pm}]] \\ A &\mapsto Y(A,z) = \sum_{n\in\mathbb{Z}} A_{(n)} z^{-n-1} \end{aligned}$$

Satisfying 2 Axioms (Vacuum & Locality)

Vertex Lie Algebras

A L.A. $\mathfrak{g}$ is a vertex Lie algebra if there is a family of formal Laurent series (called *fields*) in $\mathfrak{g}[[z^{\pm}]]$, whose commutators encode the multiplication in $\mathfrak{g}$ and satisfy certain vertex operator-like conditions.

Example: The affine Lie algebra

$\widehat{\mathfrak{g}} := \dot{\mathfrak{g}} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}\mathbf{c}$ is a vertex L.A.

Fields: $x(z) := \sum_{n \in \mathbb{Z}} (x \otimes t^n) z^{-n-1}$

$$\mathbf{c}(z) = \mathbf{c} z^0$$

$$[x(z), y(w)] = [x, y](w) z^{-1} \delta\left(\frac{w}{z}\right) + (x|y)\mathbf{c} z^{-1} \frac{\partial}{\partial w} \delta\left(\frac{w}{z}\right)$$

Remark: If $\mathfrak{g}$ is a vertex L.A.,
then $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_+$ (as vector spaces)
where $\mathfrak{g}_\pm = \{\text{coeffs of } z^n \mid n \in \mathbb{Z}_\mp\}$ are
Lie subalgebras.($\mathbb{Z}_- = \mathbb{Z}_{<0}$, $\mathbb{Z}_+ = \mathbb{Z}_{\geq 0}$)

Thm [e.g. Dong-Li-Mason, 2001]:
If $\mathfrak{g}$ is a vertex Lie algebra, then the $\mathfrak{g}$-module $\mathrm{Ind}_{\mathfrak{g}_+}^{\mathfrak{g}}(\mathbb{C}\mathbb{1}) \cong \mathcal{U}(\mathfrak{g}_-) \otimes \mathbb{C}\mathbb{1}$ has a natural vertex algebra structure (called the <u>enveloping vertex algebra</u>).

Remark: $\left\{\text{modules of V.A. } \mathrm{Ind}_{\widehat{\mathfrak{g}}_+}^{\widehat{\mathfrak{g}}}(\mathbb{C}\mathbb{1})\right\}$

$$= \left\{\text{smooth modules of L.A. } \widehat{\mathfrak{g}}\right\}$$

(M is *smooth* if for each $m \in M$ and $x \in \dot{\mathfrak{g}}$, $(x \otimes t^N).m = 0$ for $N \gg 0$.)

Extended Affine Lie Algebras

- EALAs are multivariable generalizations of affine L.A.s
- have a nondeg. symm. bilinear form
- have a notion of type, corresponding to finite irred. root systems
- can be written in the form [e.g. Allison-Benkart-Gao, 2000]

$$(\dot{\mathfrak{g}} \otimes A) \oplus (V \otimes B) \oplus \mathcal{C} \oplus \mathcal{D}$$

where

$\dot{\mathfrak{g}}$ fin. dim. simple Lie algebra

V (little adjoint) $\dot{\mathfrak{g}}$-module

$A \oplus B$ algebra

$\mathcal{C}$ elts commuting with all but D

$\mathcal{D}$ some derivations of $A \oplus B$

Example: Toroidal Lie Algs

$\dot{\mathfrak{g}}$ fin. dim. simple L.A.

$A = \mathbb{C}[t_0^{\pm 1}, \ldots, t_N^{\pm 1}]$

Then $\dot{\mathfrak{g}} \otimes A$ is an EALA

More interesting rep theory with a central extension:

$$\mathrm{uce}(\dot{\mathfrak{g}} \otimes A) = (\dot{\mathfrak{g}} \otimes A) \oplus \Omega_A^1 / dA$$

$$[x \otimes f(\mathbf{t}),\, y \otimes g(\mathbf{t})] = \\ [x, y] \otimes f(\mathbf{t}) g(\mathbf{t}) + (x|y) \overline{g(\mathbf{t}) df(\mathbf{t})}$$

EALAs require nondeg. bilin. form $\rightsquigarrow$ must add $\mathcal{D} = \{$skew-centr. ders of $A\}$
$= \{$div. 0 vector fields on $\mathbb{T}^{N+1}\}$
(cf. Neher, 2004)

Toroidal EALA: $(\dot{\mathfrak{g}} \otimes A) \oplus \Omega^1_A/dA \oplus \mathcal{D}$

We want a vertex L.A., so we add <u>all</u> (polyn.) vector fields on $\mathbb{T}^{N+1}$.

Also twist mult. by a 2-cocycle τ, generalizing the Virasoro cocycle:

$$\tau : \ \mathrm{Vect}(\mathbb{T}^{N+1}) \times \mathrm{Vect}(\mathbb{T}^{N+1}) \to \Omega^1_A/dA$$

Full Toroidal L.A. $\mathfrak{g}^{\tau}_{\mathrm{tor}} =$

$$(\dot{\mathfrak{g}} \otimes A) \oplus \Omega^1_A/dA \oplus_{\tau} \mathrm{Vect}(\mathbb{T}^{N+1})$$

No symm. invar. bilin. form.

Let $\mathfrak{g}(\mu\tau) :=$ toroidal EALA with $\mu\tau$, for $\mu \in \mathbb{C}$

Simple quotient of enveloping VA of $\mathfrak{g}_{\text{tor}}^{\mu\tau}$ is tensor product of several well-known VAs. This lets one find modules for $\mathfrak{g}_{\text{tor}}^{\mu\tau}$.

Restricting to $\mathfrak{g}(\mu\tau)$ gives:

Thm [Billig, 2005]: The tensor product

$$L_c(\dot{\mathfrak{g}}) \otimes \mathrm{Hyp}_N^+ \otimes L_{c_1}(\mathfrak{sl}_N) \otimes Vir(c_2)$$

is an irreducible module for $\mathfrak{g}(\mu\tau)$ where $L_c(\dot{\mathfrak{g}})$ is irred. hi wt $\widehat{\mathfrak{g}}$-module of level c (resp. for $L_{c_1}(\mathfrak{sl}_N)$ and $Vir(c_2)$), Hyp_N^+ is a certain sub-VA of a lattice VA, and c_1, c_2 depend on $c, \mu \in \mathbb{C}$.

Multiloop Algebras

$\dot{\mathfrak{g}}$ fin. dim. simple L.A.

$\sigma_0, \ldots, \sigma_N : \ \dot{\mathfrak{g}} \to \dot{\mathfrak{g}}$ commuting autos

of order $m_0, \ldots, m_n$, resp.

$\mathbb{Z}^N \to G := \bigoplus_{i=0}^{N} (\mathbb{Z}/m_i\mathbb{Z})$ canon. map.

Decompose $\dot{\mathfrak{g}} = \bigoplus_{\mathbf{k} \in G} \dot{\mathfrak{g}}_{\overline{\mathbf{k}}}$ according to common eigenspaces for the σ_i.

Def: The multiloop algebra

$$L(\dot{\mathfrak{g}}, \sigma_0, \ldots, \sigma_N) := \bigoplus_{\mathbf{k} \in \mathbb{Z}^N} \dot{\mathfrak{g}}_{\overline{\mathbf{k}}} \otimes \mathbf{t}^{\mathbf{k}}.$$

Thm [Allison-Berman-Faulkner -Pianzola, 2005 with Neher, 2004]: All EALAs (except for those of Type A over certain quantum tori) are obtained by adjoining skew-centroidal derivations and central elts to multiloop algebras.

Note that $\dot{\mathfrak{g}}_{\overline{0}} \otimes \mathbb{C}[t_0^{\pm m_0}, \ldots, t_N^{\pm m_N}]$
$\subseteq L(\dot{\mathfrak{g}}, \sigma_0, \ldots, \sigma_N) \subseteq \dot{\mathfrak{g}} \otimes \mathbb{C}[t_0^{\pm 1}, \ldots, t_N^{\pm 1}]$

Thm [Benkart-Neher, 2005]: The uce of $L(\dot{\mathfrak{g}}, \sigma_0, \ldots, \sigma_N)$ can be determined from its centroid $C = \mathbb{C}[t_0^{\pm m_0}, \ldots, t_N^{\pm m_N}]$.

Adding appropriate derivations to the uce of the multiloop algebra, we define the twisted toroidal EALA

$$\mathfrak{g}_{\mu\tau}(\sigma_0, \ldots, \sigma_N) := L(\dot{\mathfrak{g}}, \sigma_0, \ldots, \sigma_N) \oplus \Omega^1_C / dC \oplus SCDer(C).$$

The autos σ_i extend to $\mathfrak{g}^{\mu\tau}_{\mathrm{tor}}$ and its enveloping VA V. The fixed point set in $\mathfrak{g}(\mu\tau) \subseteq \mathfrak{g}^{\mu\tau}_{\mathrm{tor}}$ under $G = \langle \sigma_i \mid 0 \leq i \leq N \rangle$ is $\mathfrak{g}(\mu\tau)^G = \mathfrak{g}_{\mu\tau}(\sigma_0, \ldots, \sigma_N)$, whose representation theory we might expect to be related to the sub-vertex algebra V^G (an <u>orbifold vertex algebra</u>).

Modules for $\mathfrak{g}_{\mu\tau}(\sigma_0, \ldots, \sigma_N)$

Since $\mathfrak{g}_{\mu\tau}(\sigma_0, \ldots, \sigma_N) \subseteq \mathfrak{g}^{\mu\tau}_{\text{tor}}$, the module $L_c(\dot{\mathfrak{g}}) \otimes \text{Hyp}^+_N \otimes L_{c_1}(\mathfrak{sl}_N) \otimes Vir(c_2)$ is a module for $\mathfrak{g}_{\mu\tau}(\sigma_0, \ldots, \sigma_N)$, but it need not be irreducible.

To describe irreducible modules, we define "thin coverings" of a module M for a Lie algebra $\mathfrak{g}$ graded by G.

Def: A set of subspaces $\{M_k \mid k \in G\}$ is a <u>covering</u> of M if $\sum_{k \in G} M_k = M$ and $\mathfrak{g}_k \cdot M_\ell \subseteq M_{k+\ell}$ for all $k, \ell \in G$. The covering is <u>thin</u> if there is no covering $\{N_k\} \neq \{M_k\}$ with $N_k \subseteq M_k$.

Irreducible Modules

Let $\widehat{\mathfrak{g}}(\sigma_0) := \left(\bigoplus_{r\in\mathbb{Z}} \dot{\mathfrak{g}}_{\overline{r}} \otimes t_0^r \right) \oplus \mathbb{C}\mathbf{c}$.

Let L be irred hi wt module for $\widehat{\mathfrak{g}}(\sigma_0)$ of level c, with thin covering $\{L_{\overline{k}}\}$ relative to $H = \langle \sigma_1, \dots, \sigma_N \rangle$. Motivated by vertex algebra ideas, we obtain

Thm [Billig-Lau, 2005]: The space

$$\left(\bigoplus_{\mathbf{k}\in\mathbb{Z}^N} L_{\overline{\mathbf{k}}} \otimes \mathbf{t}^{\mathbf{k}} \right)$$
$$\otimes \mathbb{C}[u_{pj}, v_{pj} \mid 1 \le p \le N,\, j \in \mathbb{Z}]$$
$$\otimes L_{c_1}(\mathfrak{sl}_N) \otimes Vir(c_2)$$

is an irreducible module for the twisted toroidal EALA $\mathfrak{g}_{\mu\tau}(\sigma_0, \dots, \sigma_N)$.