Monte Carlo Tests of Conformal Invariance and SLE predictions for Self-Avoiding Walks

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0. Introduction - def of SAW

The self-avoiding walk is defined as follows.

Fix N, the number of steps.

Take all nearest neighbor walks starting at 0

with no self-intersections.

$$\omega(i) \in \mathbf{Z}^2, \quad i = 0, 1, 2, \cdots N$$

$$\omega(0) = 0$$

$$|\omega(i) - \omega(i+1)| = 1$$

$$\omega(i) \neq \omega(j), \quad i \neq j$$

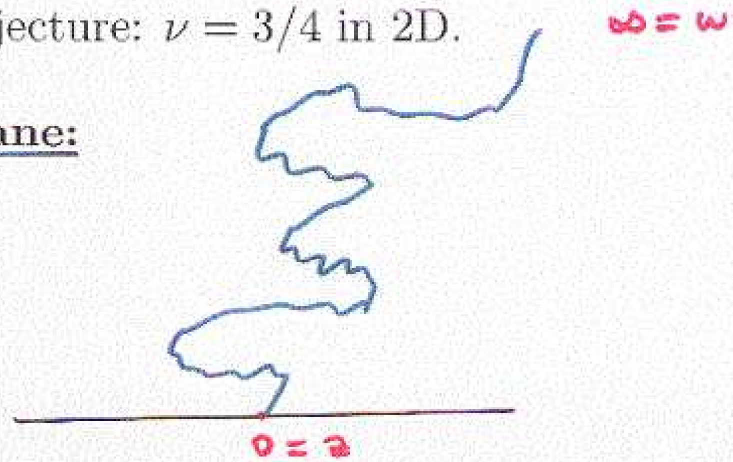
Give them equal probability.

Critical exponents:

$$E\left[|\omega(N)|^2\right] \sim N^{2\nu}$$

Conjecture: $\nu = 3/4$ in 2D.

Half-Plane:



Scaling limits:

One we do not use:

Look at $N^{-\nu}\omega$ as $N \to \infty$.

This should give a measure on curves in $\mathbf{R}^2$ that start at 0 and end at some random point.

One we do use:

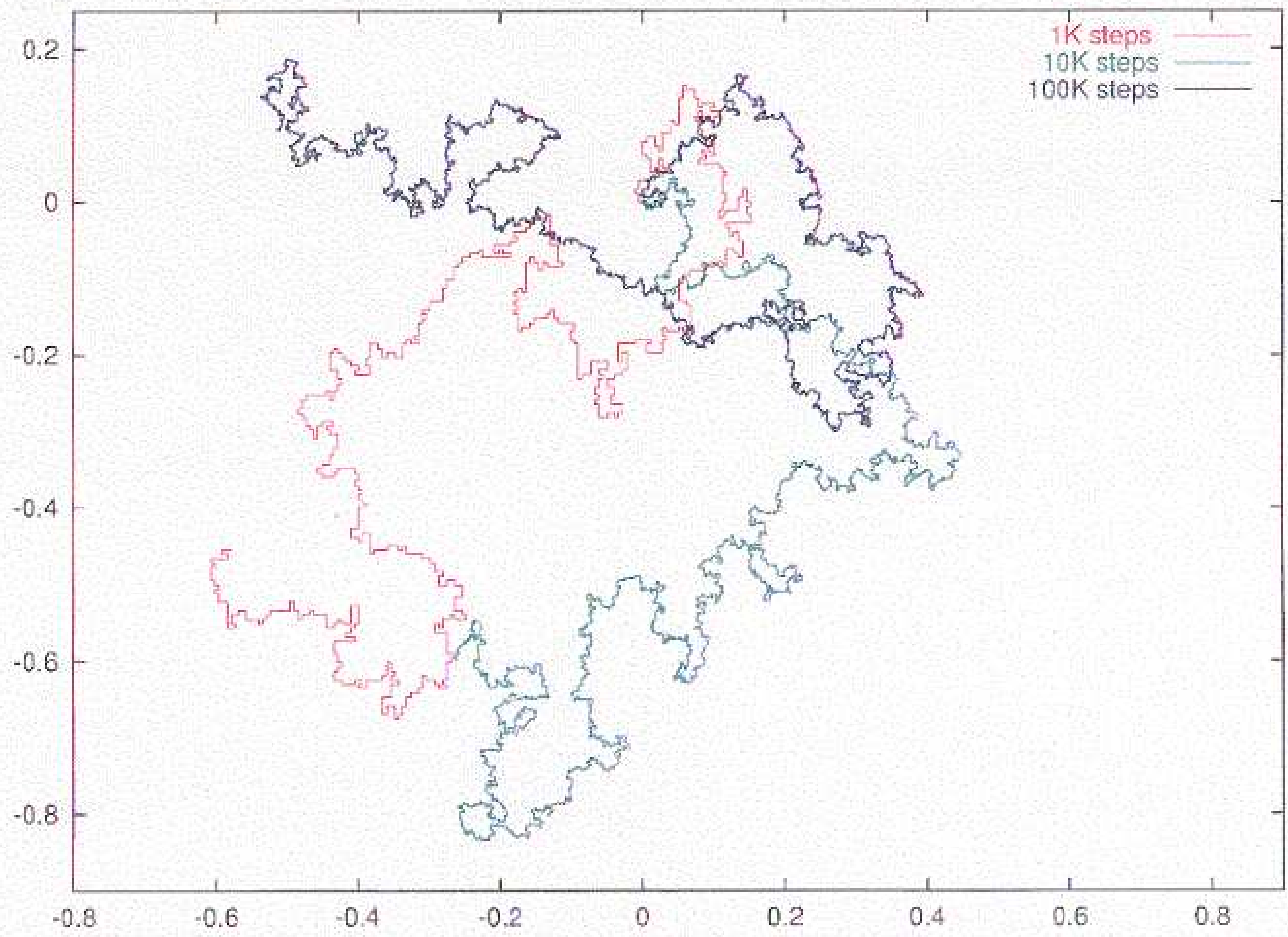
First let $N \to \infty$ to get a measure on infinite walks on $\mathbf{Z}^2$.

Then let lattice spacing go to 0.

This should give a measure on curves in $\mathbf{R}^2$ that start at 0 and go to ∞.

No simple relation between the two

Half-plane



1K steps
10K steps
100K steps
0.2
0
-0.2
-0.4
-0.6
-0.8
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8

Simulating the SAW

We use the pivot algorithm

Markov chain Monte Carlo algorithm

State space = all N-step SAW's

Find a Markov chain (transition matrix) such that its stationary distribution is the uniform distribution on SAW's with N steps.

Limitations:

Cannot fix the endpoint of the walk

Ergodic only in certain domains,

e.g., half-plane

Another method - PERM

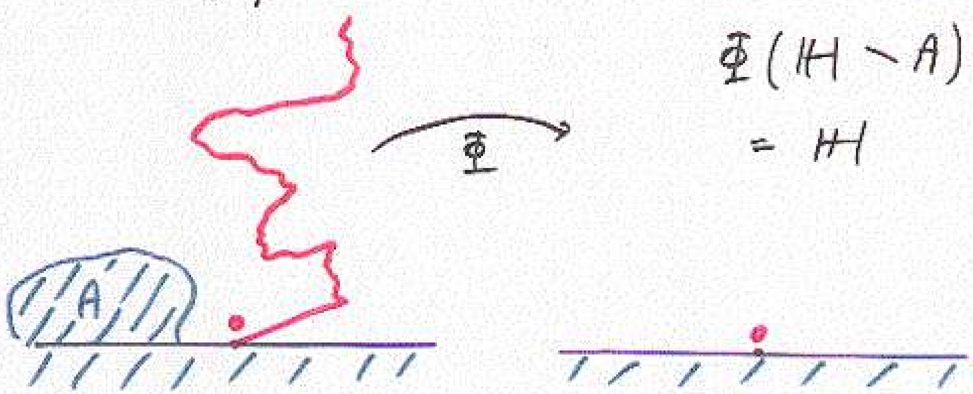
1. Tests of SLE predictions for the 2D SAW

For each $\kappa < 4$, chordal SLE gives a probability measure on curves in the upper half plane that go from 0 to ∞.

Conjecture (Lawler, Schramm, Werner)
The scaling limit of the SAW in a half-plane is chordal SLE with $\kappa = 8/3$.
They show that if the scaling limit exists and is conformally invariant, then it is $\mathrm{SLE}_{8/3}$.

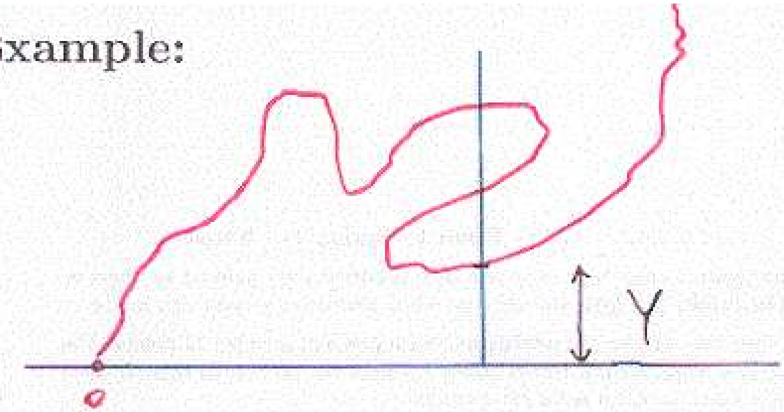
Thm (LSW) For $\mathrm{SLE}_{8/3}$ in the half-plane

$$P(\underset{= \phi}{\gamma \cap A}) = \Phi'(0)^{5/8}$$



$\Phi(0) = 0, \quad \Phi(\infty) = \infty, \quad \Phi'(\infty) = 1$

Example:

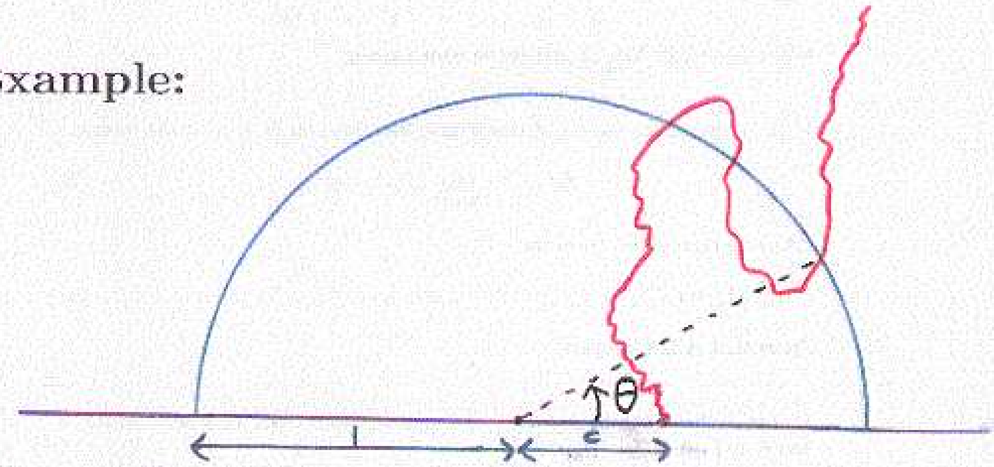


Y = lowest intersection with vert. line

$$P(Y \leq t) = P(\gamma \quad hits$$  $)$

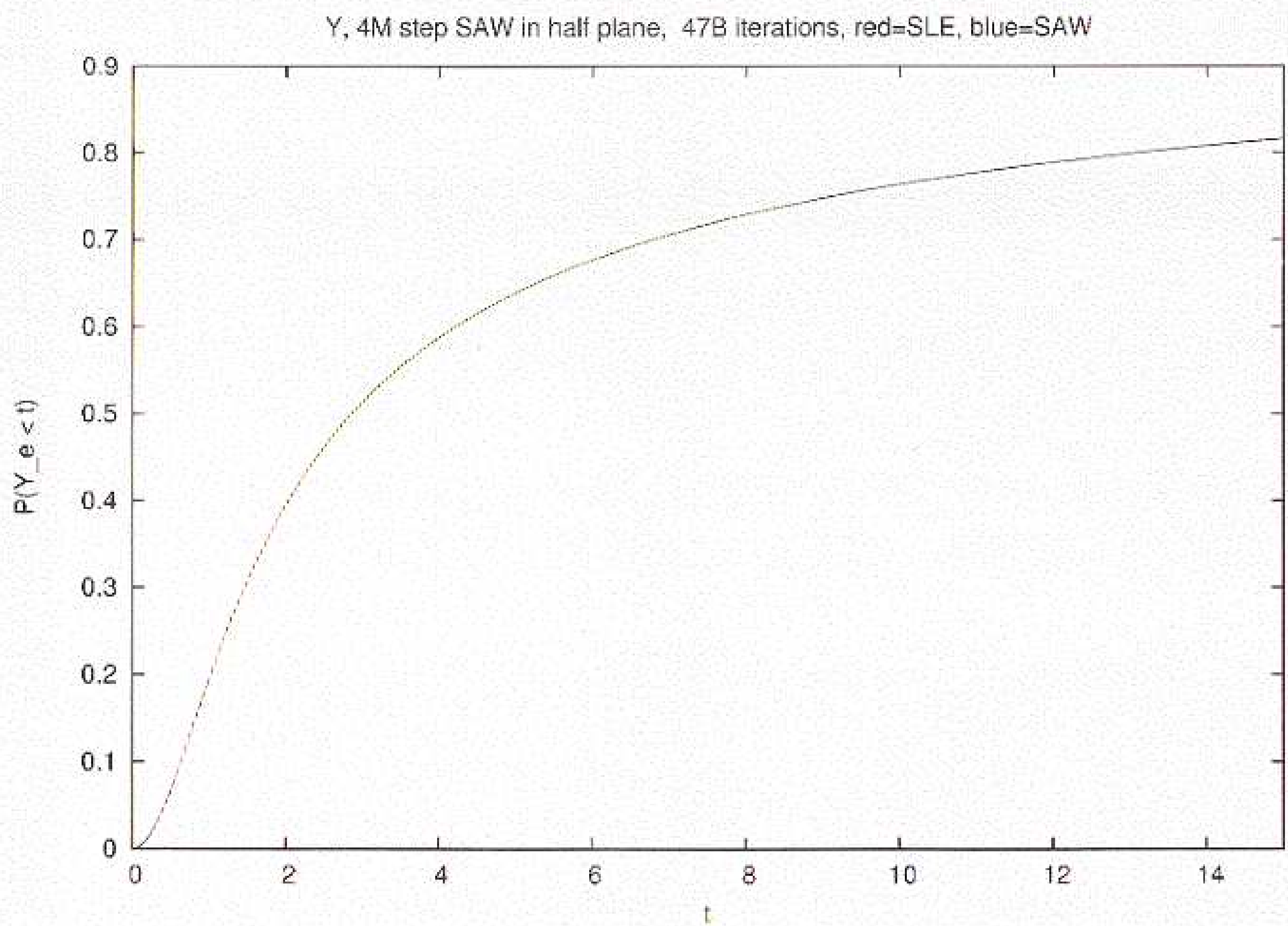
$$= 1 - {\Phi'}_{A_t}(0)^{5/8} = 1 - (1+t^2)^{-5/16}$$

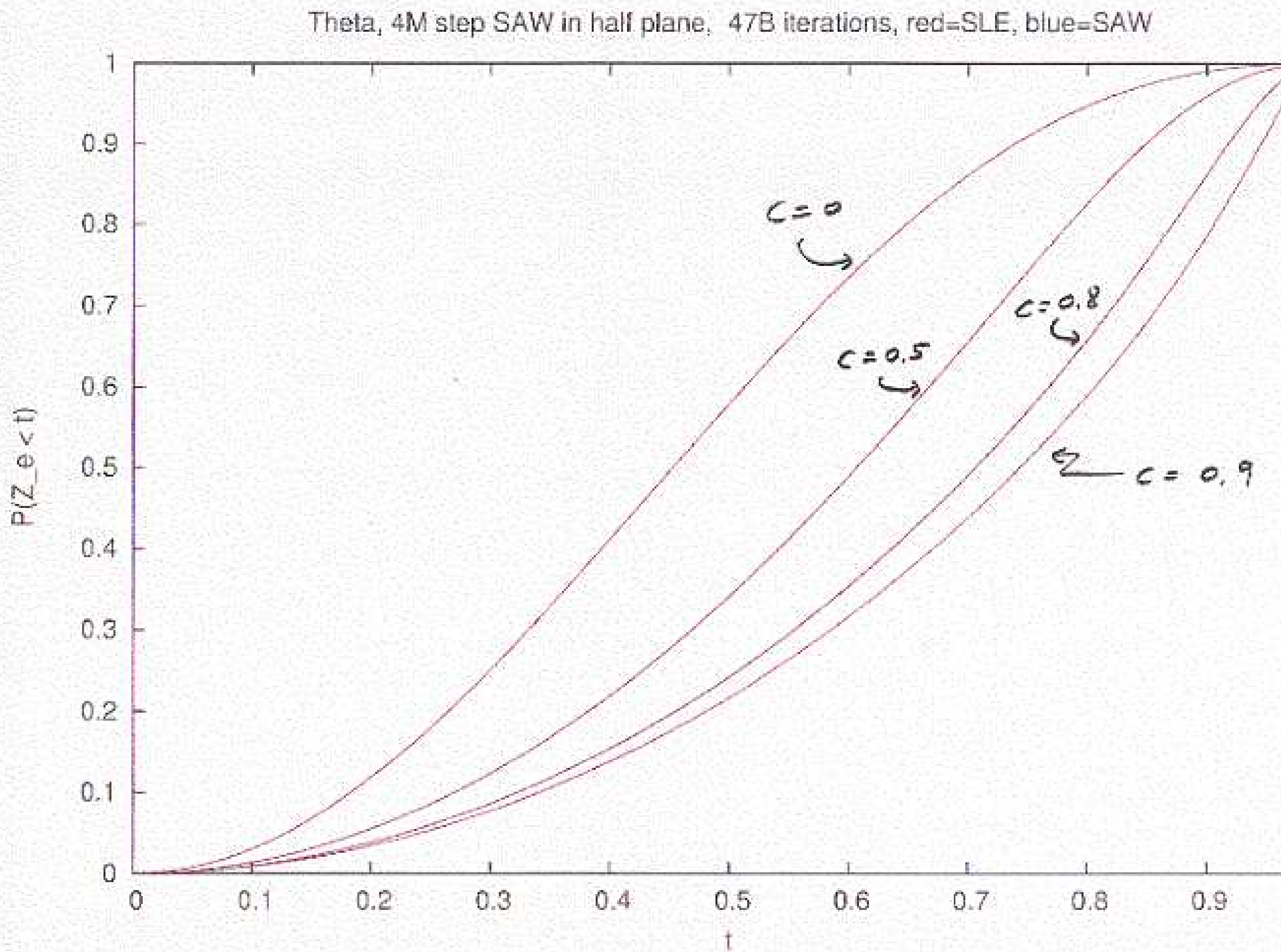
Example:



Θ = smallest polar angle of intersects

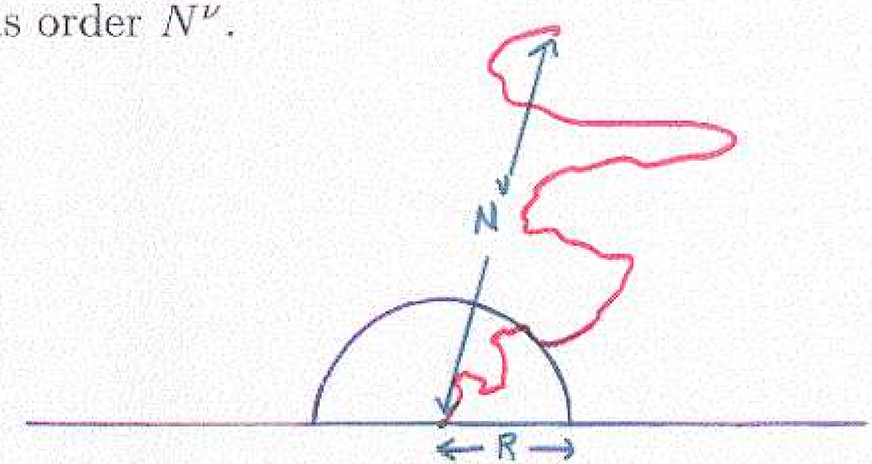
$$P(\Theta \leq t) = 1 - \left(\frac{1+\cos t}{2}\right)^{5/4}, \; c = 0$$


Y, 4M step SAW in half plane, 47B iterations, red=SLE, blue=SAW
P(Y_e < t)
t
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
2
4
6
8
10
12
14


Theta, 4M step SAW in half plane, 47B iterations, red=SLE, blue=SAW
P(Z_e < t)
t
c = 0
c = 0.5
c = 0.8
c = 0.9
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Scaling Limit

With lattice spacing=1, size of typical walk is order N^ν.



Scaling limit: $N \to \infty$ then $R \to \infty$.

Simulations:

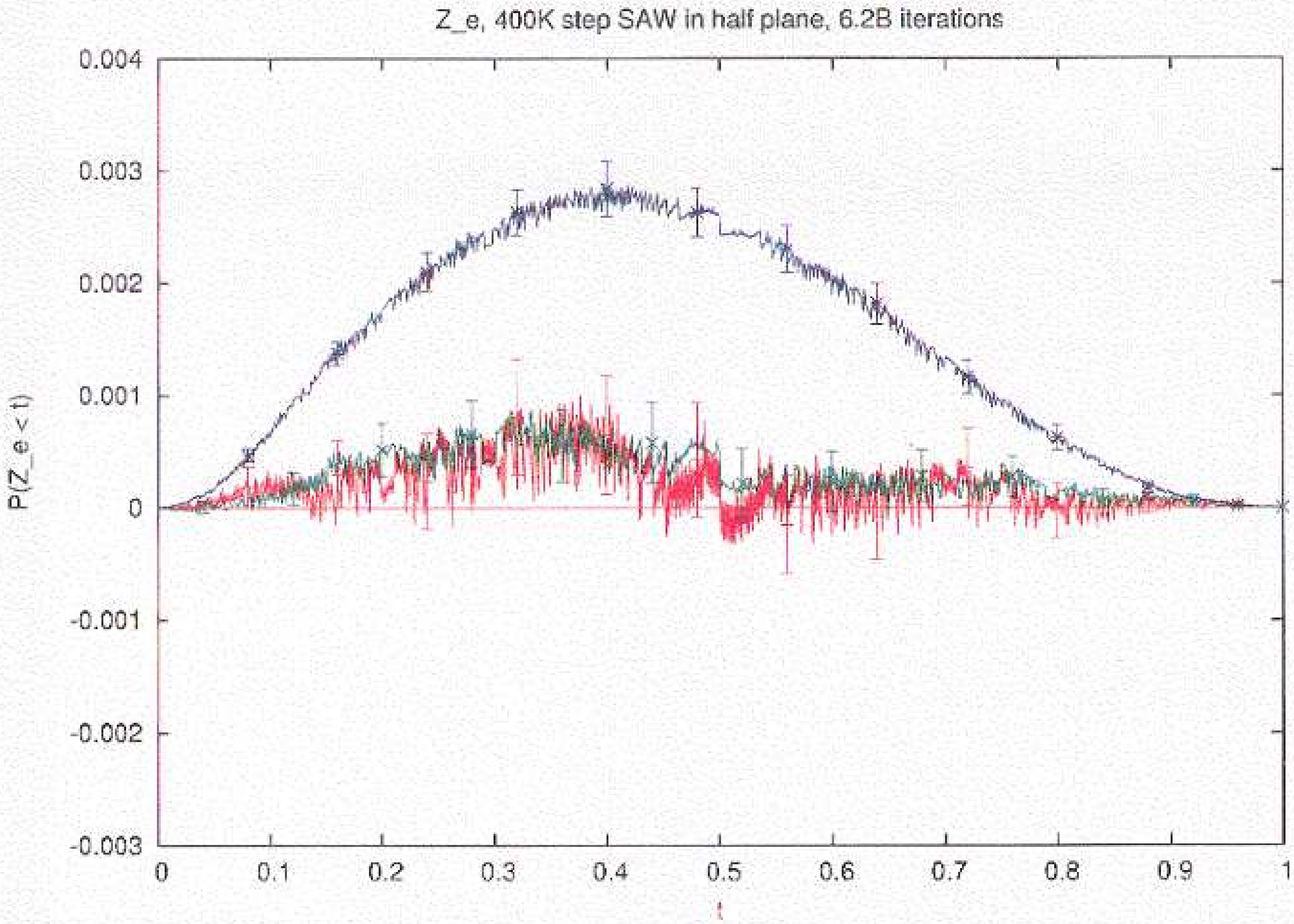
$l = R/N^\nu$

l is RV scale over walk scale

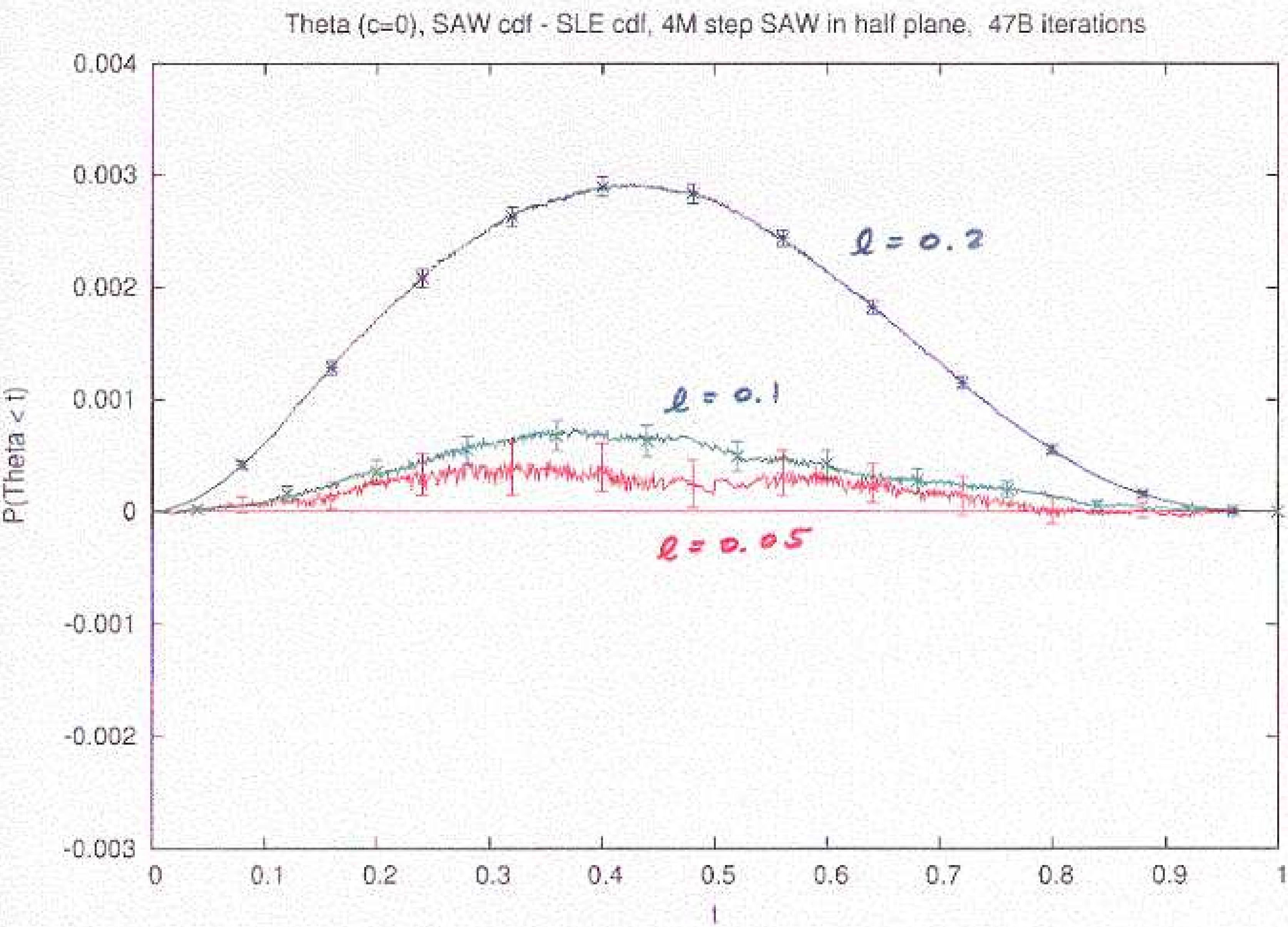
N fixed, several l's

l too small $\Rightarrow$ see lattice effects

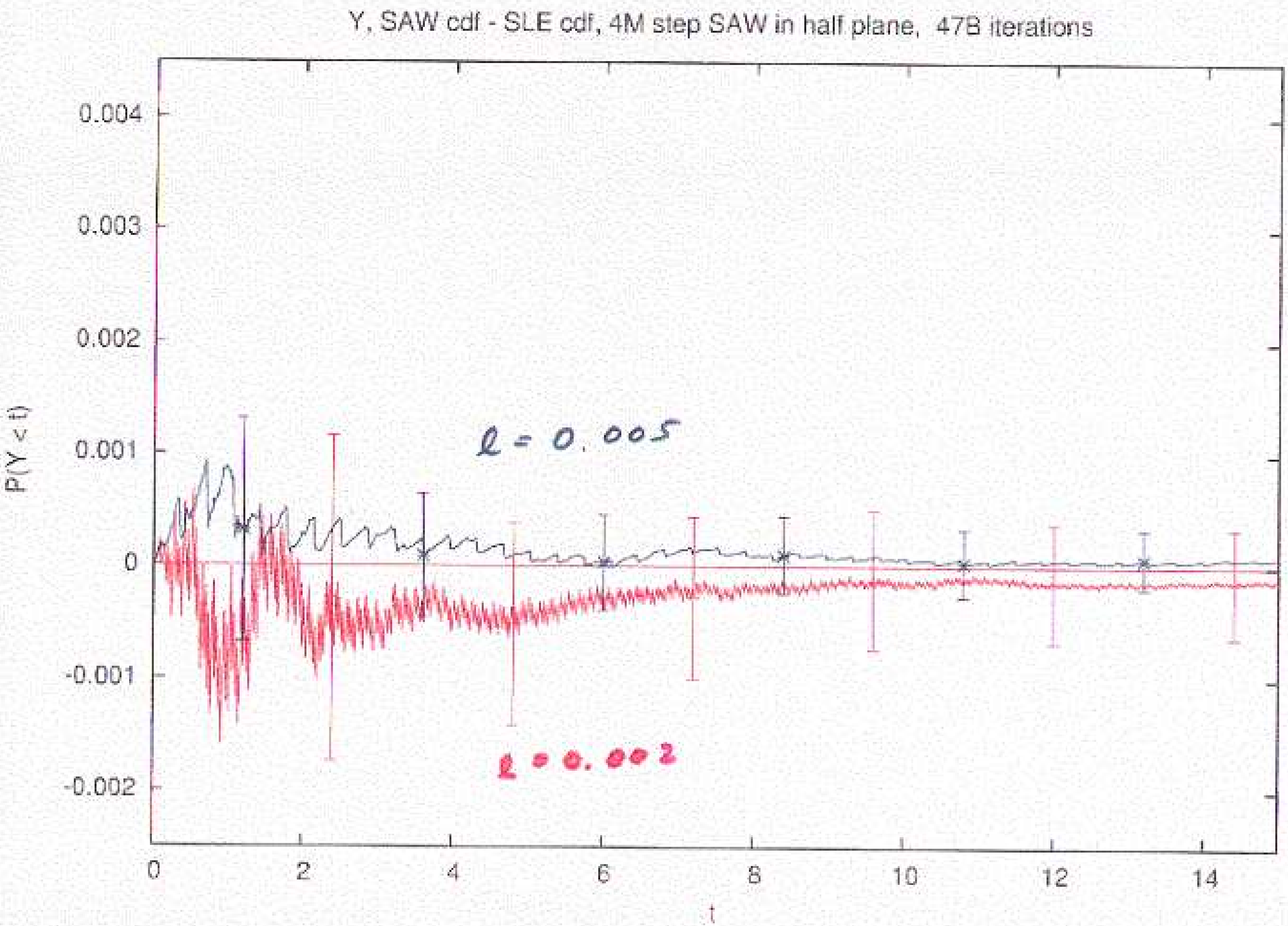
l too large $\Rightarrow$ see finite length effects



Z_e, 400K step SAW in half plane, 6.2B iterations
P(Z_e < t)
t
0.004
0.003
0.002
0.001
0
-0.001
-0.002
-0.003
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1



Theta (c=0), SAW cdf - SLE cdf, 4M step SAW in half plane. 47B iterations
0.004
0.003
0.002
0.001
0
-0.001
-0.002
-0.003
P(Theta < t)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
t
l = 0.2
l = 0.1
l = 0.05


Y, SAW cdf - SLE cdf, 4M step SAW in half plane, 47B iterations
P(Y < t)
t
l = 0.005
l = 0.002
0.004
0.003
0.002
0.001
0
-0.001
-0.002
0
2
4
6
8
10
12
14

2. Tests of SLE predictions - Weakly 2D SAW

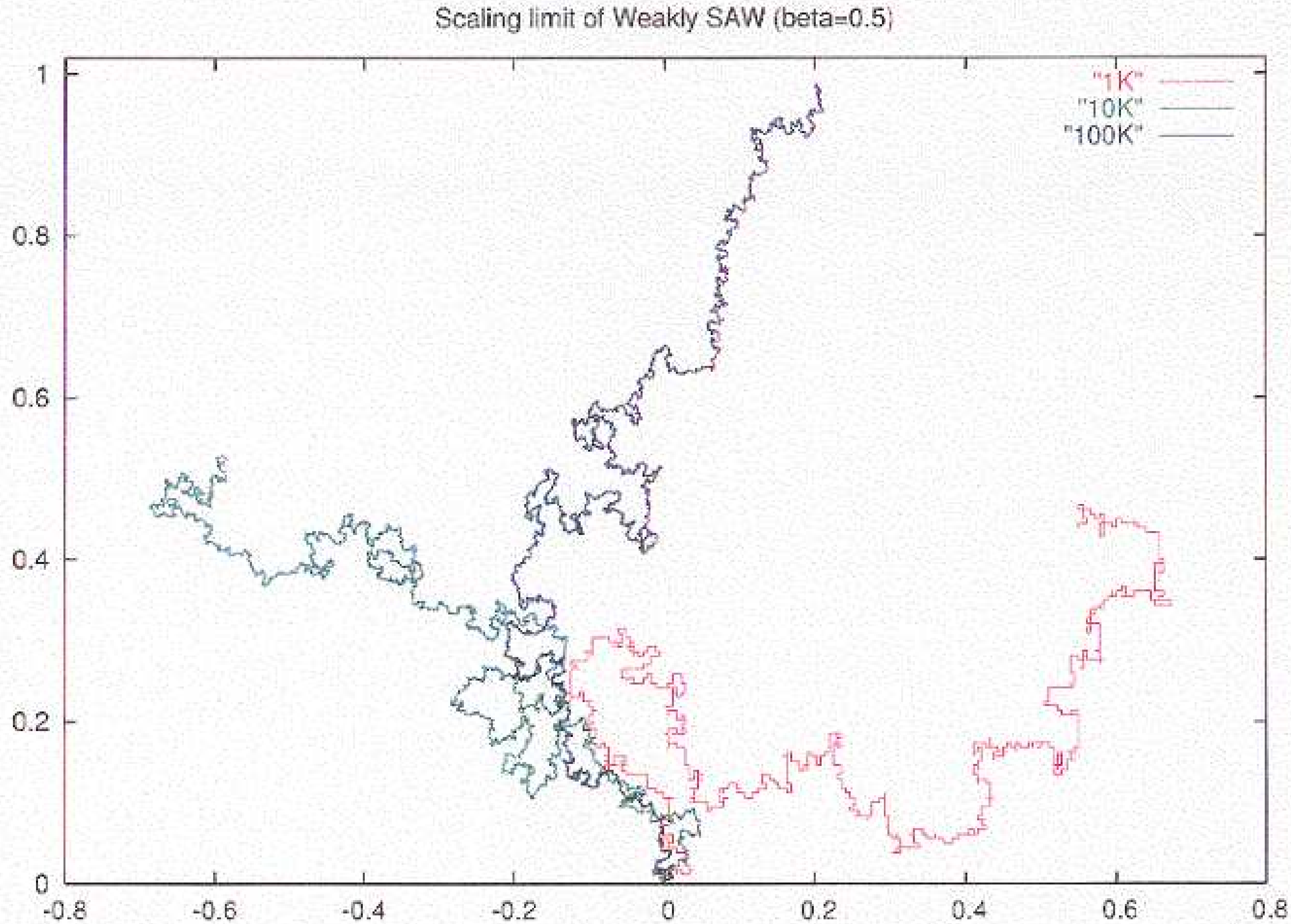
Weakly self-avoiding walk:

parameter β

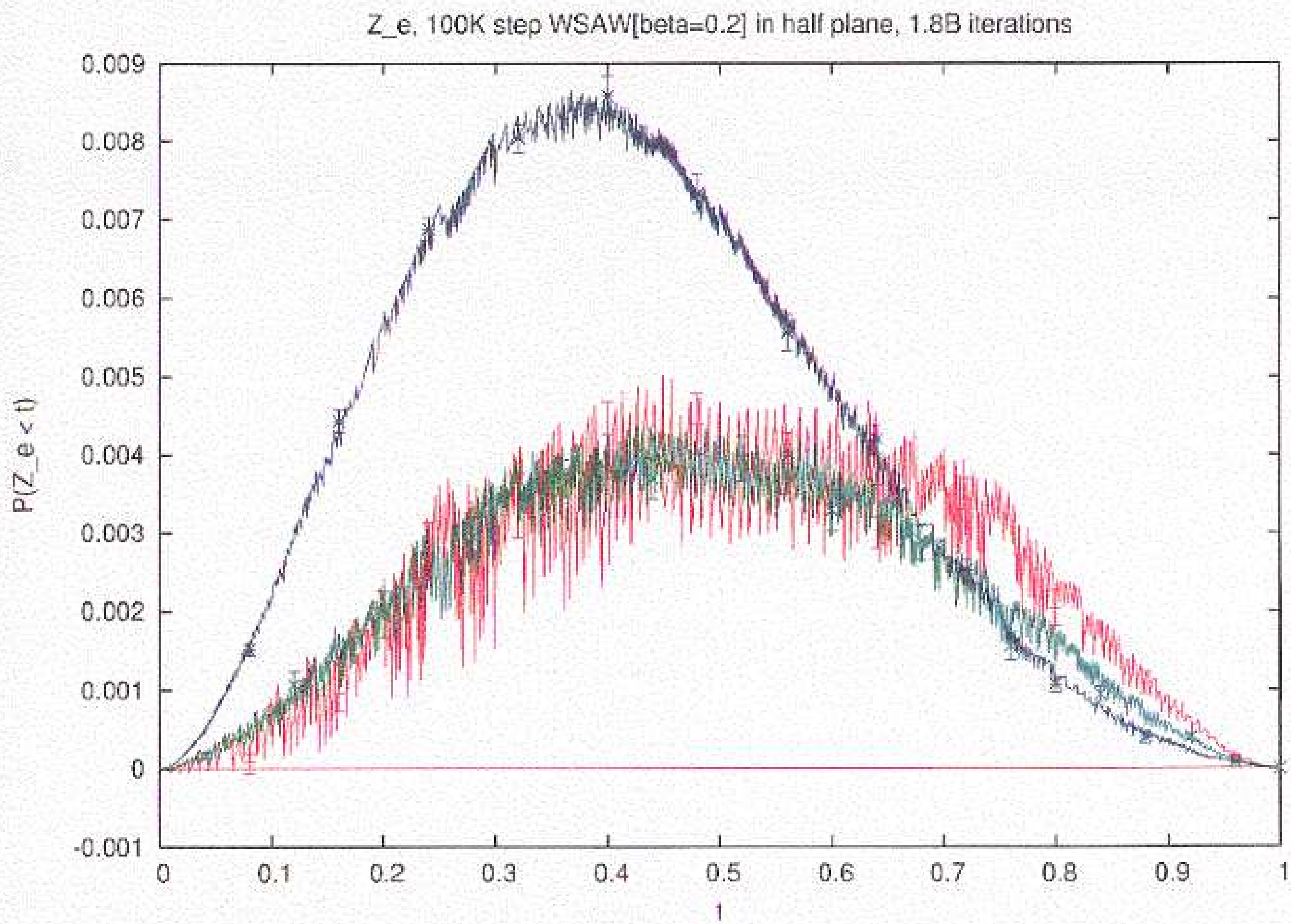
Take all N step walks starting at origin, allowed to self-intersect, but weighted by

$$\exp[-\beta \sum_{i<j} 1(\omega(i) = \omega(j))]$$

Believed that for all $\beta > 0$ this has same exponents and same scaling limit as SAW



Scaling limit of Weakly SAW (beta=0.5)
"1K"
"10K"
"100K"
1
0.8
0.6
0.4
0.2
0
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8



Z_e, 100K step WSAW[beta=0.2] in half plane, 1.8B iterations
P(Z_e < t)
t
0.009
0.008
0.007
0.006
0.005
0.004
0.003
0.002
0.001
0
-0.001
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

3. Wrong conformal invariance - 2D SAW

What conformal invariance does not mean:

Let D be simply connected, $0 \in D$.

Let U be unit disc.

Let $\phi : D \to U$, $\phi(0) = 0$. Full plane

Consider infinite SAW's w starting at 0.

t is time of first intersection of walk and ∂D.

$\phi(\omega[0,t])$ is a path from 0 to ∂U.

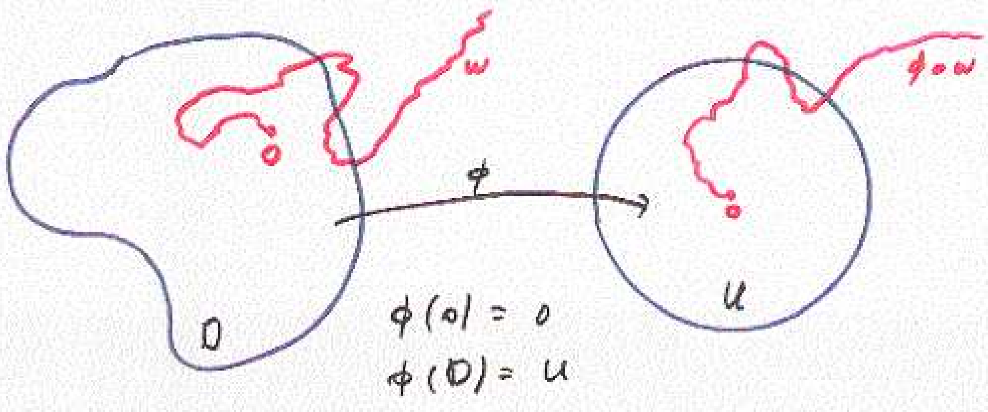
If you do this with Brownian motion,

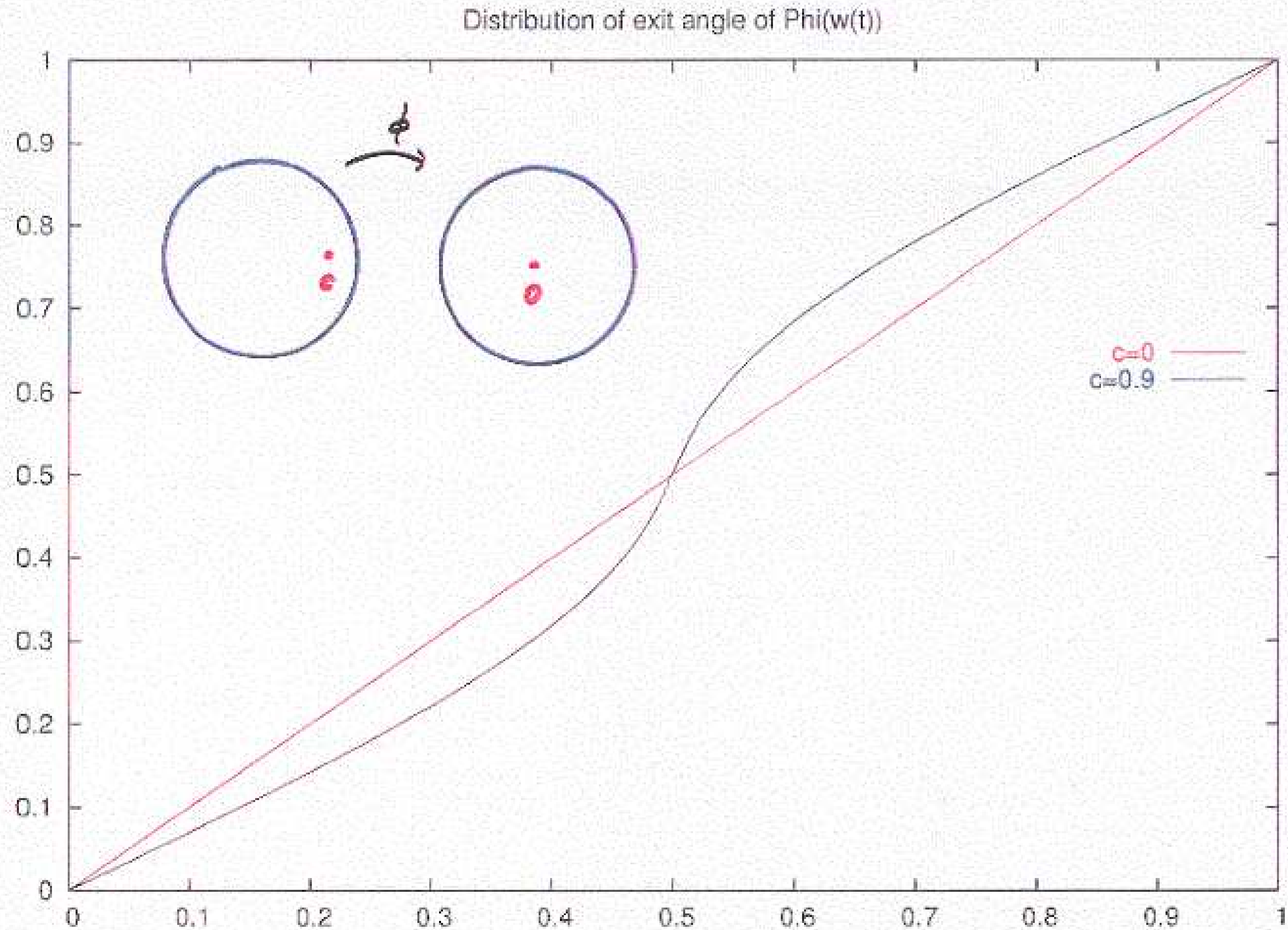
$\phi(\omega[0,t])$ has the same distribution as a BM

from 0 until it exits U.

(Ignore parametrization)

$\phi(\omega(t))$ is uniform on unit circle for BM.





Distribution of exit angle of Phi(w(t))
ϕ
c=0
c=0.9
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

4. Conformal invariance - 2D SAW

Conformal invariance $\Rightarrow$ SLE

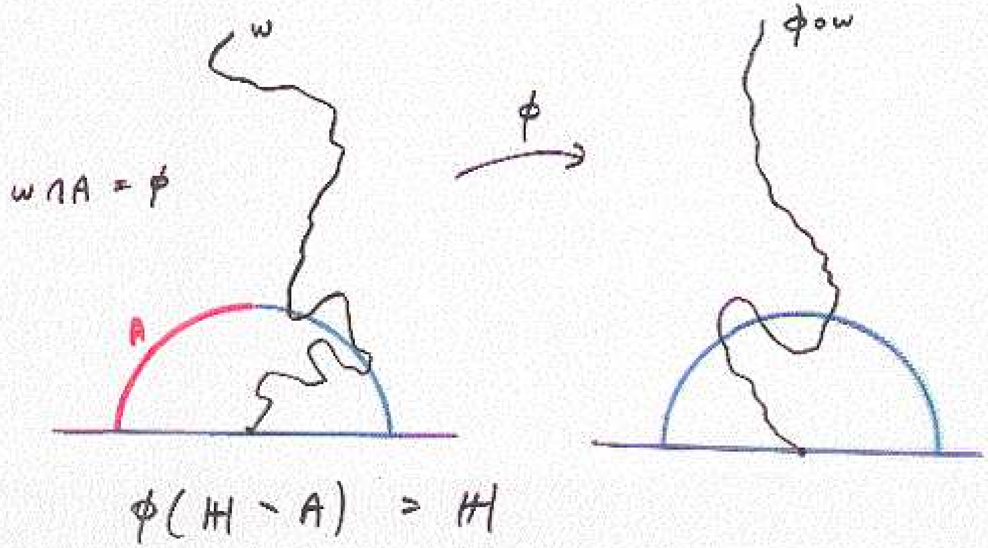
Seen MC tests of SLE predictions

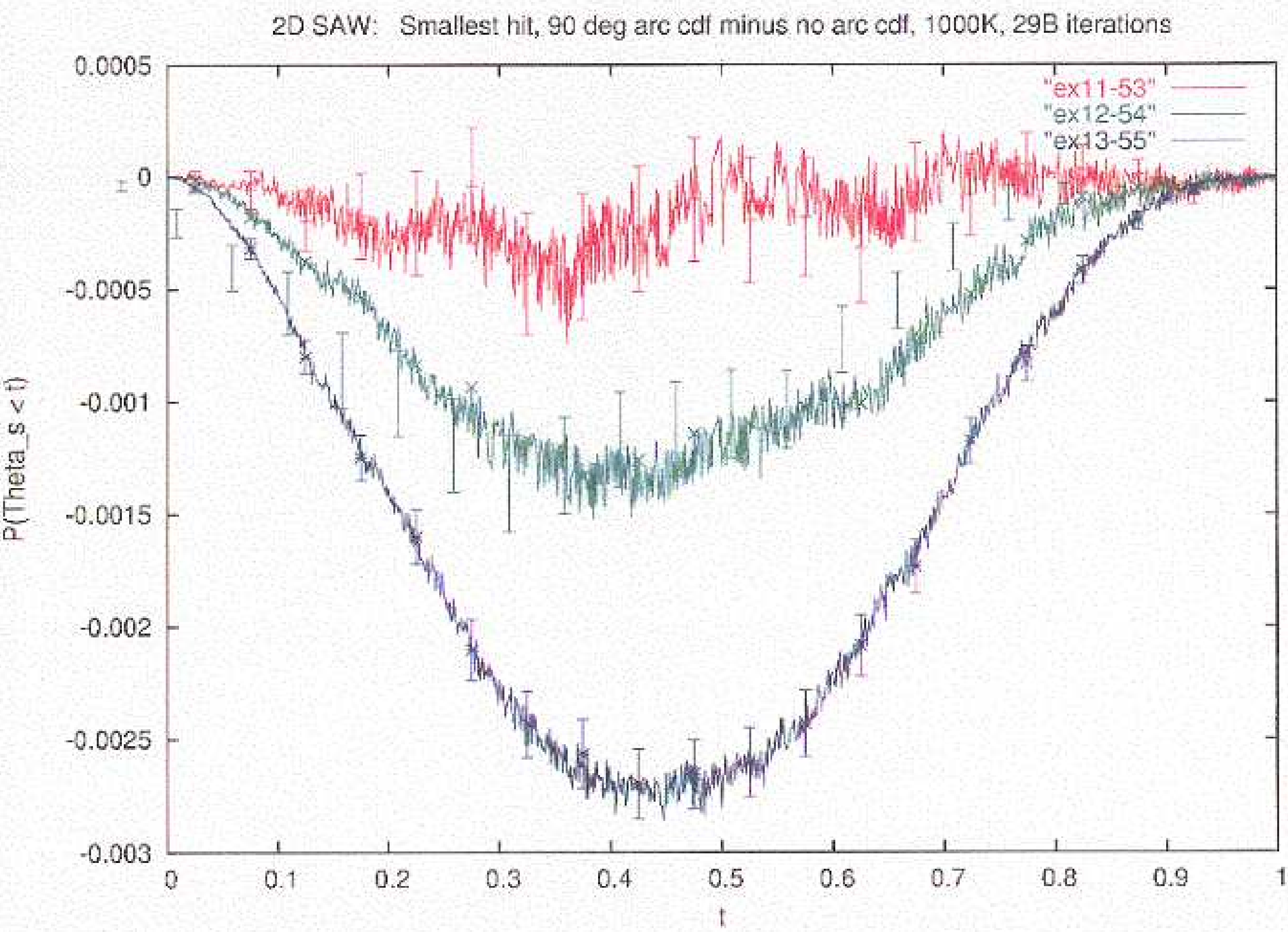
Can test conformal invariance directly

Simulate the SAW in half-plane minus something, conformally map this "perturbed" half-plane to the half-plane.

Compare resulting measure on half-plane with original SAW measure on half-plane.

Our "perturbation" is to remove an arc:





2D SAW: Smallest hit, 90 deg arc cdf minus no arc cdf, 1000K, 29B iterations
"ex11-53"
"ex12-54"
"ex13-55"
P(Theta_s < t)
t
0.0005
0
-0.0005
-0.001
-0.0015
-0.002
-0.0025
-0.003
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

5. Conformal invariance - projected 3D SAW

Take the 3D SAW in the half-space ($y > 0$) and project it onto the x-y plane.

Result is a 2D curve with self-intersections.

Is it conformally invariant?

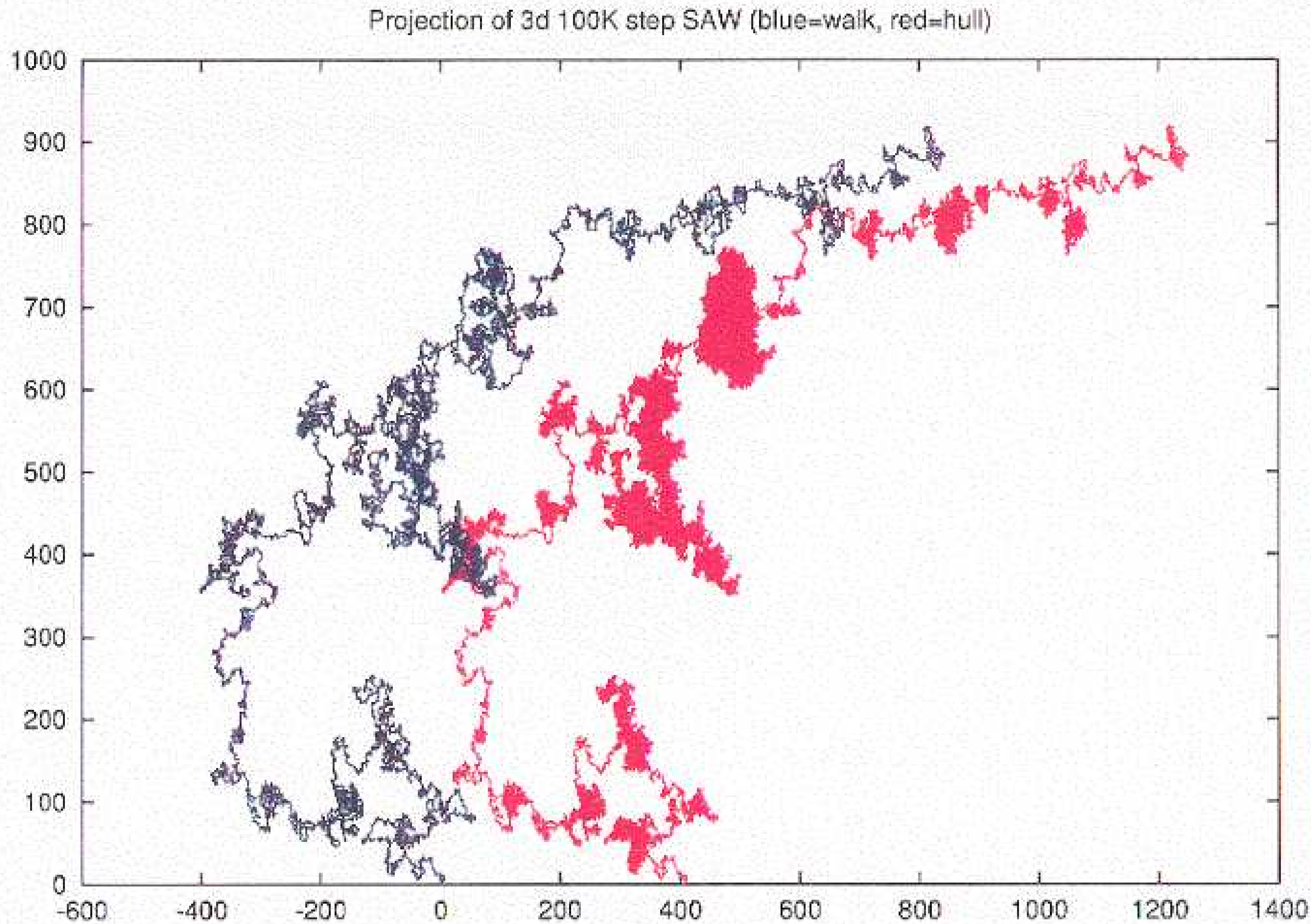
Why the *%&? should it be ?

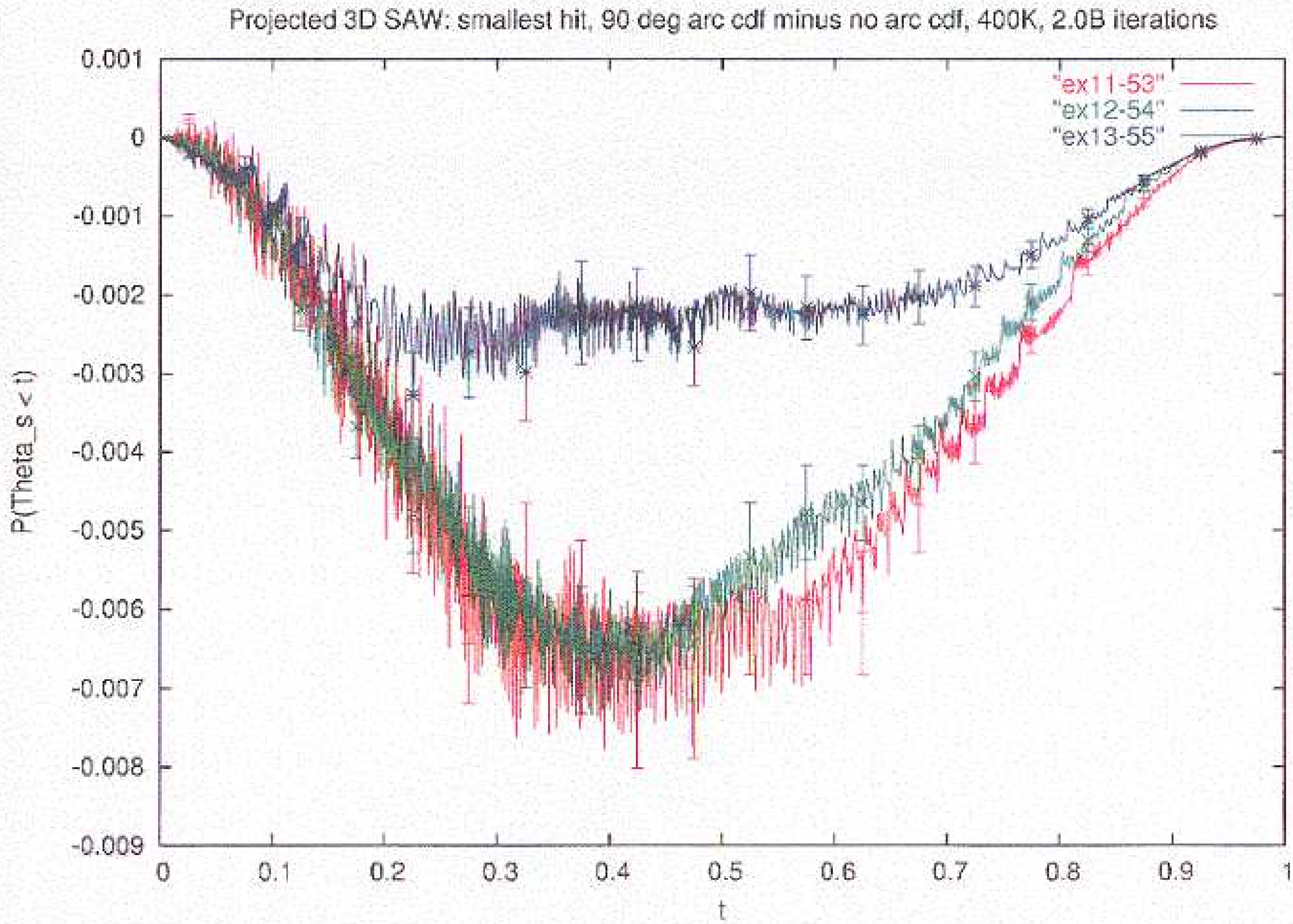
If it is, it is a LSW "conformal-restriction" measure, and we have

$$P(\Gamma \cap A) = \Phi'(0)^a$$

for some $a \geq 5/8$.

You can fit it extremely well with the CR measures with $a \approx 0.83$.


Projection of 3d 100K step SAW (blue=walk, red=hull)
1000
900
800
700
600
500
400
300
200
100
0
-600
-400
-200
0
200
400
600
800
1000
1200
1400



Projected 3D SAW: smallest hit, 90 deg arc cdf minus no arc cdf, 400K, 2.0B iterations
"ex11-53"
"ex12-54"
"ex13-55"
P(Theta_s < t)
t
0.001
0
-0.001
-0.002
-0.003
-0.004
-0.005
-0.006
-0.007
-0.008
-0.009
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1