

# **Self-Avoiding Walk in Four Dimensions**

David Brydges

University of British Columbia

# Conjecture

(Brezin, Le Guillou, Zinn-Justin, 1973)

$N$  step self-avoiding walk in four dimensions:

- End-to-end distance  $\sim N^{1/2} \log^{1/8} N$

# Conjecture

(Brezin, Le Guillou, Zinn-Justin, 1973)

$N$  step self-avoiding walk in four dimensions:

- End-to-end distance  $\sim N^{1/2} \log^{1/8} N$
- Critical Green's function decays as  $|x - y|^{-2}$

# Conjecture

(Brezin, Le Guillou, Zinn-Justin, 1973)

$N$  step self-avoiding walk in four dimensions:

- End-to-end distance  $\sim N^{1/2} \log^{1/8} N$
- Critical Green's function decays as  $|x - y|^{-2}$

Easier Problem

# Conjecture

(Brezin, Le Guillou, Zinn-Justin, 1973)

$N$  step self-avoiding walk in four dimensions:

- End-to-end distance  $\sim N^{1/2} \log^{1/8} N$
- Critical Green's function decays as  $|x - y|^{-2}$

Easier Problem

- hierarchical lattice

# Conjecture

(Brezin, Le Guillou, Zinn-Justin, 1973)

$N$  step self-avoiding walk in four dimensions:

- End-to-end distance  $\sim N^{1/2} \log^{1/8} N$
- Critical Green's function decays as  $|x - y|^{-2}$

Easier Problem

- hierarchical lattice
- weak repulsion

# Conjecture

(Brezin, Le Guillou, Zinn-Justin, 1973)

$N$  step self-avoiding walk in four dimensions:

- End-to-end distance  $\sim N^{1/2} \log^{1/8} N$
- Critical Green's function decays as  $|x - y|^{-2}$

Easier Problem

- hierarchical lattice
- weak repulsion
- continuous time

# Conjecture

(Brezin, Le Guillou, Zinn-Justin, 1973)

$N$  step self-avoiding walk in four dimensions:

- End-to-end distance  $\sim N^{1/2} \log^{1/8} N$
- Critical Green's function decays as  $|x - y|^{-2}$

Easier Problem

- hierarchical lattice
- weak repulsion
- continuous time

Proof: (Brydges-Evans-Imbrie 1990), (Brydges-Imbrie 2003), (PIMS Lectures, 2001)

# Hierarchical Lattice, $L = 2$

Points in hierarchical lattice



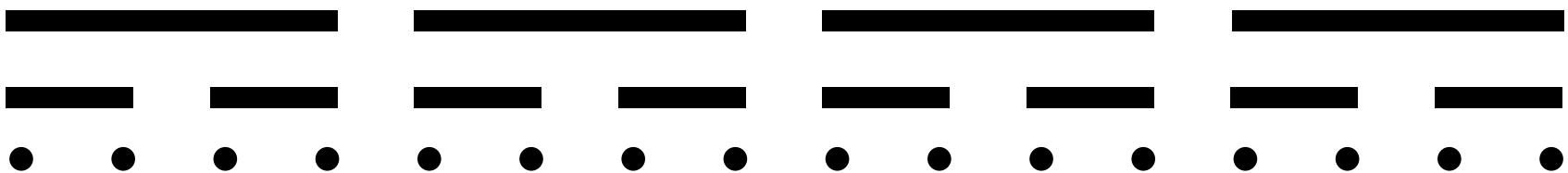
# Hierarchical Lattice, $L = 2$

Hierarchical balls, diameter  $L$



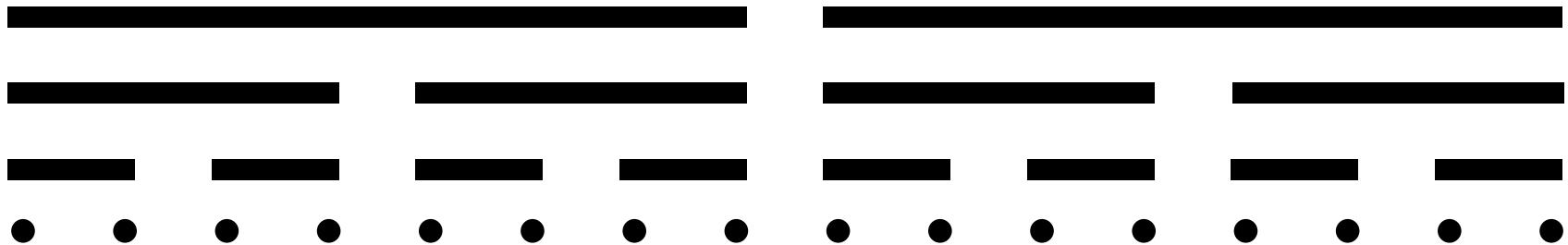
# Hierarchical Lattice, $L = 2$

Hierarchical balls, diameter  $L, L^2$



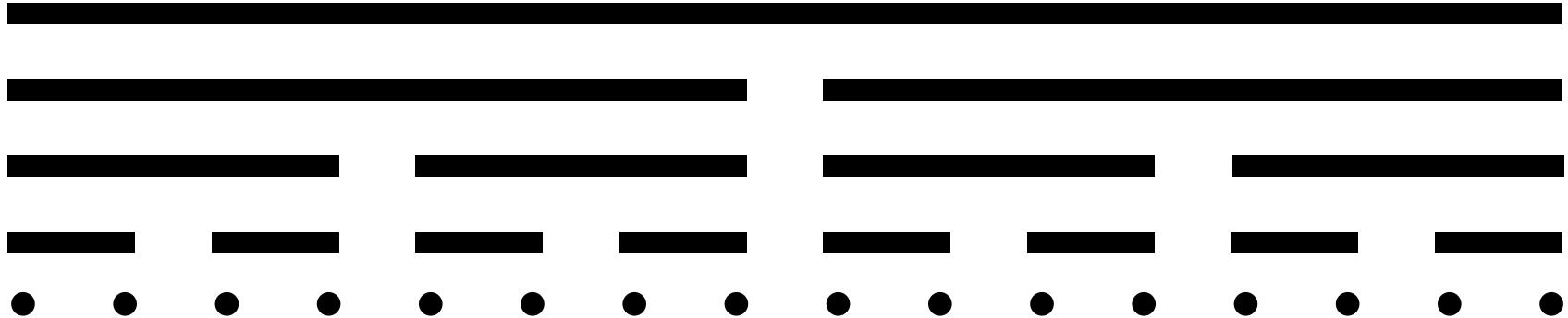
# Hierarchical Lattice, $L = 2$

Hierarchical balls, diameter  $L, L^2, L^3$

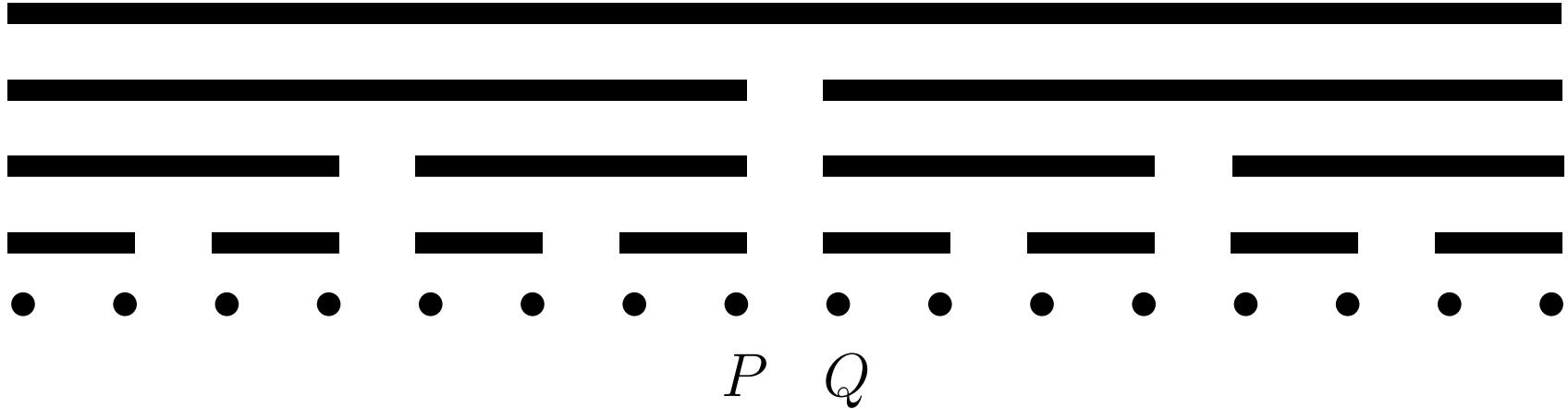


# Hierarchical Lattice, $L = 2$

Hierarchical balls, diameter  $L, L^2, L^3, L^4$



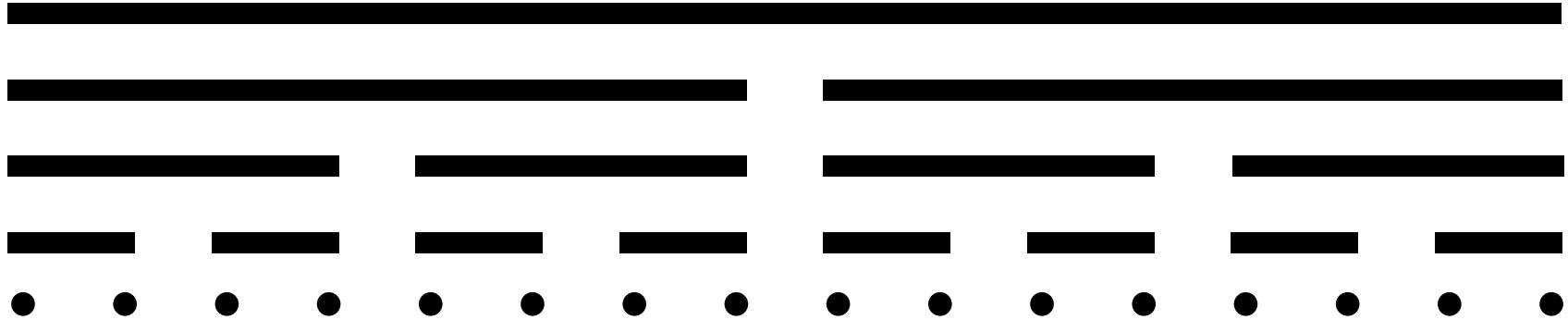
# Hierarchical Lattice, $L = 2$



$$|P - Q| = L^4.$$

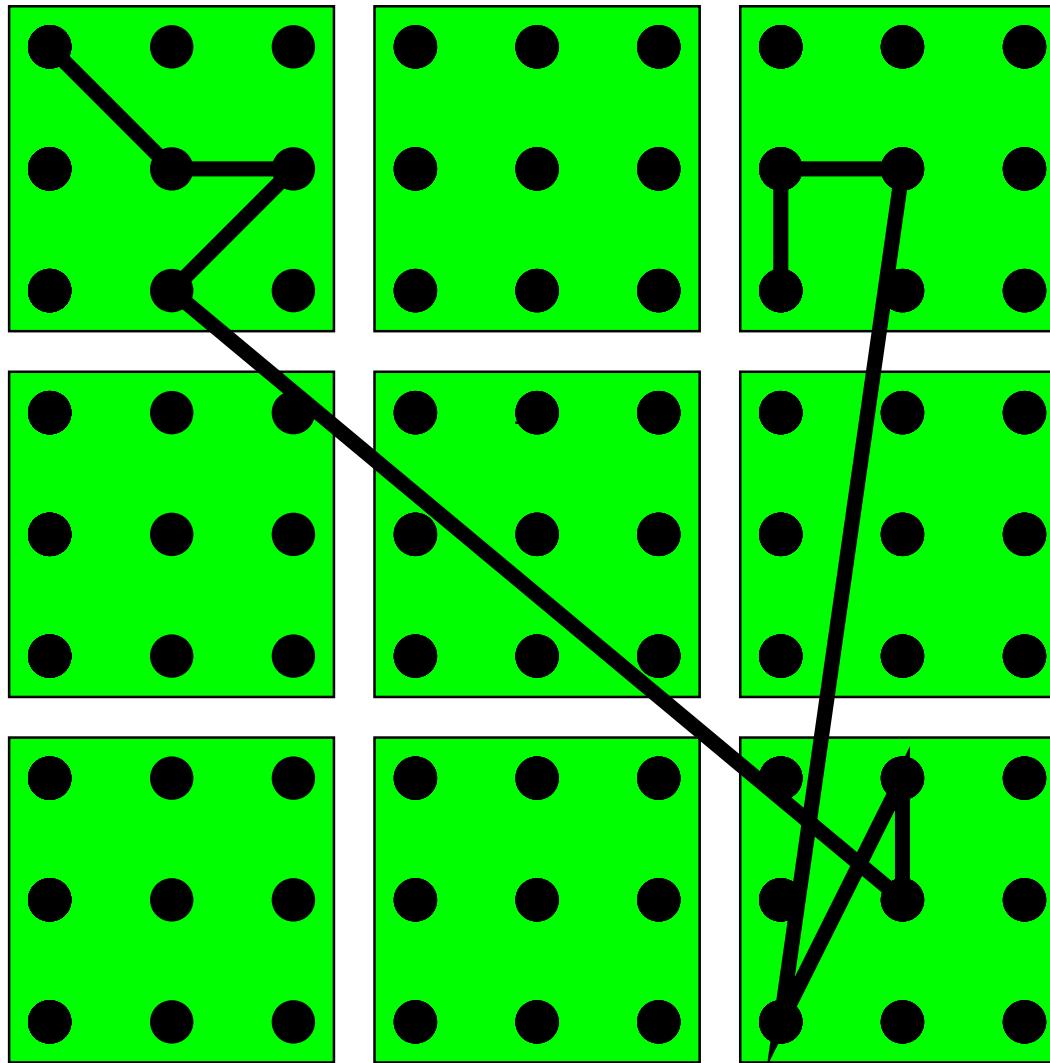
Hierarchical Lattice is a Group (+).

# Hierarchical Lattice, $L = 2$

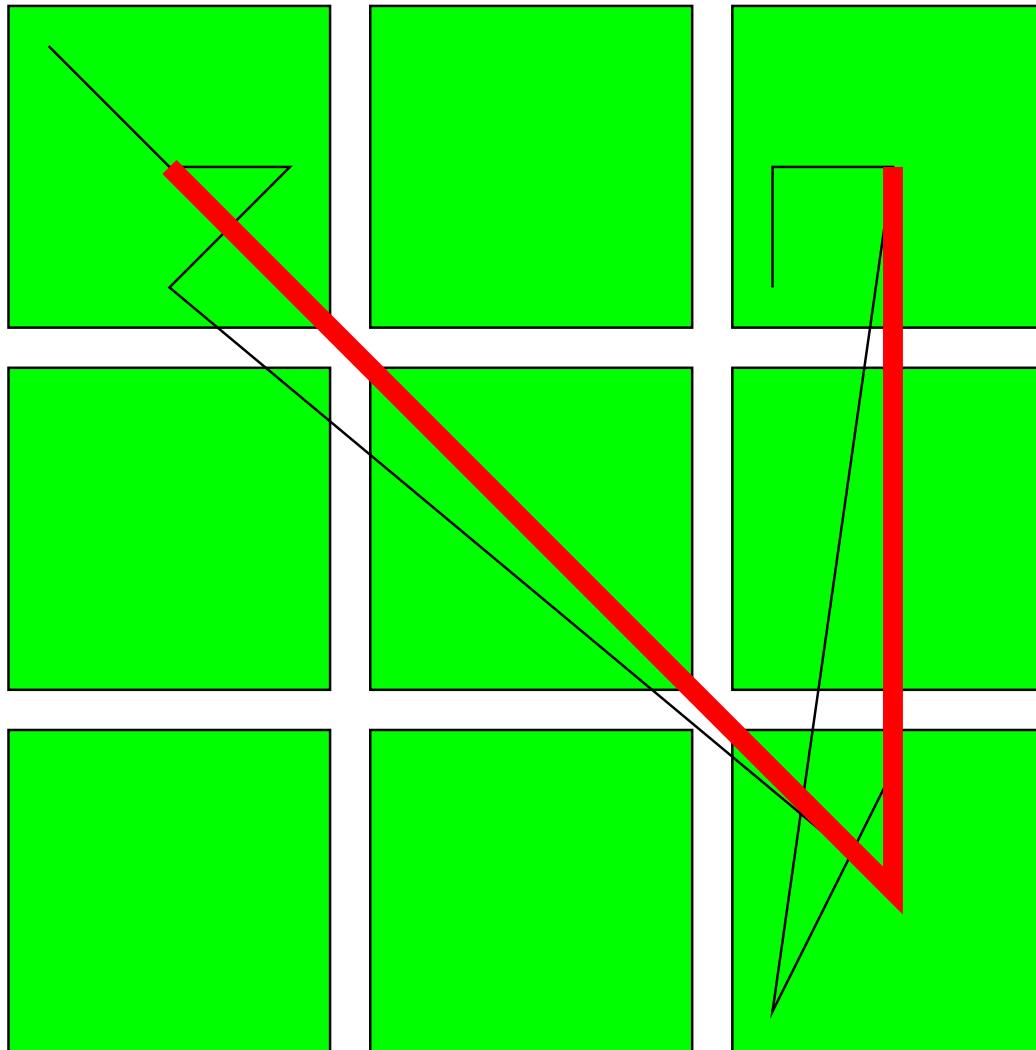


In this picture dimension of lattice  $D = 1$  because # points in ball grows as  $(\text{diameter})^D$ .

# Renormalisation Group



# Renormalisation Group



# Renormalisation Group Continued

- Choose the unique jump probability s.t. Green's function of non-interacting walk is  $|x - y|^{-2}$ .

# Renormalisation Group Continued

- Choose the unique jump probability s.t. Green's function of non-interacting walk is  $|x - y|^{-2}$ .
- Collapse the balls starting with smallest.

# Renormalisation Group Continued

- Choose the unique jump probability s.t. Green's function of non-interacting walk is  $|x - y|^{-2}$ .
- Collapse the balls starting with smallest.
- **Notation**  $x \mapsto L^{-1}x$ .

# Renormalisation Group Continued

- Choose the unique jump probability s.t. Green's function of non-interacting walk is  $|x - y|^{-2}$ .
- Collapse the balls starting with smallest.
- **Notation**  $x \mapsto L^{-1}x$ .
- Real Space RG (Rodríguez-Romo, 1995)

# Renormalisation Group Continued

- Choose the unique jump probability s.t. Green's function of non-interacting walk is  $|x - y|^{-2}$ .
- Collapse the balls starting with smallest.
- **Notation**  $x \mapsto L^{-1}x$ .
- Real Space RG (Rodríguez-Romo, 1995)
- $\tau$  Isomorphism

# Free Green's Function

$$G(\beta, b) := \int_0^\infty dT e^{-\beta T} \mathbb{E}_a^{[0,T]} (\delta_{b,X_T})$$

# Free Green's Function

$$G(\beta, b) := \int_0^\infty dT e^{-\beta T} \mathbb{E}_a^{[0,T]} (\delta_{b,X_T})$$

$$= \int_0^\infty dT \mathbb{E}_a^{[0,T]} \left( \delta_{b,X_T} \prod_{x \in Lattice} e^{-\beta \tau_x} \right)$$

# Free Green's Function

$$G(\beta, b) := \int_0^\infty dT e^{-\beta T} \mathbb{E}_a^{[0,T]} (\delta_{b,X_T})$$

$$= \int_0^\infty dT \mathbb{E}_a^{[0,T]} \left( \delta_{b,X_T} \prod_{x \in Lattice} e^{-\beta \tau_x} \right)$$

RG applied  $k$  times

$$G(\beta, b) = L^{-2k} G(L^{2k}\beta, L^{-k}b)$$

until  $L^{-k}b = 0$ .

# With Interaction

## With Self-Interaction

$$G(g, b) := \int_0^\infty dT e^{-\beta T} \mathbb{E}_a^{[0, T]} \left( \delta_{b, X_T} \right) \prod_{x \in Lattice} g(\tau_x)$$

# With Interaction

With Self-Interaction

$$G(g, b) := \int_0^\infty dT e^{-\beta T} \mathbb{E}_a^{[0, T]} \left( \delta_{b, X_T} \right) \prod_{x \in Lattice} g(\tau_x)$$

**Lemma 0.0** *RG applied  $k$  times*

$$G(\textcolor{red}{g}, b) = L^{-2k} G(\textcolor{red}{g_k}, L^{-k} b)$$

# With Interaction

With Self-Interaction

$$G(g, b) := \int_0^\infty dT e^{-\beta T} \mathbb{E}_a^{[0, T]} \left( \delta_{b, X_T} \right) \prod_{x \in Lattice} g(\tau_x)$$

**Lemma 0.0** *RG applied  $k$  times*

$$G(\textcolor{red}{g}, b) = L^{-2k} G(\textcolor{red}{g_k}, L^{-k} b)$$

defines an RG evolution of interaction

$$g_0 \mapsto g_1 \mapsto \dots$$

for  $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  smooth and bounded.

# RG evolution of interaction

Interaction  $g$  **defines**  $\beta, \lambda$  by

$$g(t) = e^{-\beta t - \lambda t^2} + O(t^3)$$

# RG evolution of interaction

More precisely, interaction  $g$  **defines**  $\beta, \lambda$  and  $r$  by

$$g(t) = e^{-\beta t - \lambda t^2} + r(t)$$

# RG evolution of interaction

More precisely, interaction  $g$  **defines**  $\beta, \lambda$  and  $r$  by

$$g(t) = e^{-\beta t - \lambda t^2} + r(t)$$

Then can prove that RG induces recursion

$$\lambda_{j+1} \approx \lambda_j - \frac{8B\lambda_j^2}{(1 + \beta_j)^2}$$

# RG evolution of interaction

More precisely, interaction  $g$  **defines**  $\beta, \lambda$  and  $r$  by

$$g(t) = e^{-\beta t - \lambda t^2} + r(t)$$

Then can prove that RG induces recursion

$$\lambda_{j+1} \approx \lambda_j - \frac{8B\lambda_j^2}{(1 + \beta_j)^2}$$

$$\beta_{j+1} \approx L^2 \left[ \beta_j + \frac{2B}{1 + \beta_j} \lambda_j \right]$$

# RG evolution of interaction

More precisely, interaction  $g$  defines  $\beta, \lambda$  and  $r$  by

$$g(t) = e^{-\beta t - \lambda t^2} + r(t)$$

Then can prove that RG induces recursion

$$\lambda_{j+1} \approx \lambda_j - \frac{8B\lambda_j^2}{(1 + \beta_j)^2}$$

$$\beta_{j+1} \approx L^2 \left[ \beta_j + \frac{2B}{1 + \beta_j} \lambda_j \right]$$

The recursion for  $\lambda_k$  makes it decrease as  $\frac{1}{k}$

# RG evolution of interaction

More precisely, interaction  $g$  defines  $\beta, \lambda$  and  $r$  by

$$g(t) = e^{-\beta t - \lambda t^2} + r(t)$$

Then can prove that RG induces recursion

$$\lambda_{j+1} \approx \lambda_j - \frac{8B\lambda_j^2}{(1 + \beta_j)^2}$$

$$\beta_{j+1} \approx L^2 \left[ \beta_j + \frac{2B}{1 + \beta_j} \lambda_j \right]$$

The recursion for  $\lambda_k$  makes it decrease as  $\frac{1}{k}$  until  $\beta_k \geq 1$ . Then  $\lambda_k$  stabilises.

# A Main Result

Method to calculate Green's function  $G(g, b)$  with **Interaction**  
 $g =: g_0$

# A Main Result

Method to calculate Green's function  $G(g, b)$  with **Interaction**  
 $g =: g_0$

- Use  $G(g, b) = L^{-2N}G(g_N, L^{-N}b)$  with  $N = N(b)$  s.t.  
 $L^N b = 0.$

# A Main Result

Method to calculate Green's function  $G(g, b)$  with **Interaction**  
 $g =: g_0$

- Use  $G(g, b) = L^{-2N}G(g_N, L^{-N}b)$  with  $N = N(b)$  s.t.  
 $L^N b = 0$ .
- For  $b = 0$  interacting Green's  $\approx$  free Green's  
 $G(g_N, 0) \approx G(\beta_N, 0)$

# A Main Result

Method to calculate Green's function  $G(g, b)$  with **Interaction**  
 $g =: g_0$

- Use  $G(g, b) = L^{-2N}G(g_N, L^{-N}b)$  with  $N = N(b)$  s.t.  
 $L^N b = 0$ .
- For  $b = 0$  interacting Green's  $\approx$  free Green's  
 $G(g_N, 0) \approx G(\beta_N, 0)$
- Undo free RG:  $G(\beta_N, 0) = L^{-2N}G(L^{-2N}\beta_N, b)$

# A Main Result

Method to calculate Green's function  $G(g, b)$  with **Interaction**  
 $g =: g_0$

- Use  $G(g, b) = L^{-2N}G(g_N, L^{-N}b)$  with  $N = N(b)$  s.t.  
 $L^N b = 0$ .
- For  $b = 0$  interacting Green's  $\approx$  free Green's  
 $G(g_N, 0) \approx G(\beta_N, 0)$
- Undo free RG:  $G(\beta_N, 0) = L^{-2N}G(L^{-2N}\beta_N, b)$

**Theorem 0.1** Let  $g(\tau) = e^{-\beta\tau - \lambda\tau^2}$ . For all small  $\lambda$ ,

# A Main Result

Method to calculate Green's function  $G(g, b)$  with **Interaction**  
 $g =: g_0$

- Use  $G(g, b) = L^{-2N}G(g_N, L^{-N}b)$  with  $N = N(b)$  s.t.  
 $L^N b = 0$ .
- For  $b = 0$  interacting Green's  $\approx$  free Green's  
 $G(g_N, 0) \approx G(\beta_N, 0)$
- Undo free RG:  $G(\beta_N, 0) = L^{-2N}G(L^{-2N}\beta_N, b)$

**Theorem 0.1** Let  $g(\tau) = e^{-\beta\tau - \lambda\tau^2}$ . For all small  $\lambda$ ,

$$|G(g, b) - G_0(\beta_{\text{eff}}, b)| \leq \mathcal{O}(\lambda_{N(b)})|G_0(\beta_{\text{eff}}, b)|,$$

# A Main Result

Method to calculate Green's function  $G(g, b)$  with **Interaction**  
 $g =: g_0$

- Use  $G(g, b) = L^{-2N}G(g_N, L^{-N}b)$  with  $N = N(b)$  s.t.  
 $L^N b = 0$ .
- For  $b = 0$  interacting Green's  $\approx$  free Green's  
 $G(g_N, 0) \approx G(\beta_N, 0)$
- Undo free RG:  $G(\beta_N, 0) = L^{-2N}G(L^{-2N}\beta_N, b)$

**Theorem 0.1** Let  $g(\tau) = e^{-\beta\tau - \lambda\tau^2}$ . For all small  $\lambda$ ,

$$|G(g, b) - G_0(\beta_{\text{eff}}, b)| \leq \mathcal{O}(\lambda_{N(b)})|G_0(\beta_{\text{eff}}, b)|,$$

where  $\beta_{\text{eff}} = \lim_{n \rightarrow \infty} (\text{RG})_{\text{free}}^{-n} (\text{RG})^n \beta$ .

# End-to-Distance

- For each small  $\lambda$ ,  $\exists! \beta = \beta^c$  s.t.  $\beta_k \rightarrow 0$ . Theorem implies  $G(g, b)$  decays as massless free Green's function.

# End-to-Distance

- For each small  $\lambda$ ,  $\exists! \beta = \beta^c$  s.t.  $\beta_k \rightarrow 0$ . Theorem implies  $G(g, b)$  decays as massless free Green's function.
- Origin of Log correction in end-to-end is behaviour of Green's function as  $\beta \rightarrow \beta^c$ : by Theorem

$$\sum_b G(g, b) |b|^{2a} \sim \beta_{\text{eff}}^{-1-a}$$

# End-to-Distance

- For each small  $\lambda$ ,  $\exists! \beta = \beta^c$  s.t.  $\beta_k \rightarrow 0$ . Theorem implies  $G(g, b)$  decays as massless free Green's function.
- Origin of Log correction in end-to-end is behaviour of Green's function as  $\beta \rightarrow \beta^c$ : by Theorem

$$\sum_b G(g, b) |b|^{2a} \sim \beta_{\text{eff}}^{-1-a}$$

- From the  $\beta, \lambda$  recursion

$$\beta_{\text{eff}} \approx (\beta - \beta^c) \log^{-1/4}(\beta - \beta^c)^{-1}.$$

# End-to-Distance

**Theorem 0.2** Fix  $0 < \alpha < 2$ . If  $\lambda$  is sufficiently small with  $|\arg \lambda| < \frac{\pi}{3}$ , then

# End-to-Distance

**Theorem 0.2** Fix  $0 < \alpha < 2$ . If  $\lambda$  is sufficiently small with  $|\arg \lambda| < \frac{\pi}{3}$ , then

$$E_{0,\lambda}^T(|X_T|^\alpha)^{\frac{1}{\alpha}} = \left(1 + \frac{O(\lambda)}{\ell(T^{-1})}\right) E_0 \left( \left| X_{\left(T\ell(T^{-1})^{\frac{1}{4}}\right)} \right|^\alpha \right)^{\frac{1}{\alpha}},$$

# End-to-Distance

**Theorem 0.2** Fix  $0 < \alpha < 2$ . If  $\lambda$  is sufficiently small with  $|\arg \lambda| < \frac{\pi}{3}$ , then

$$E_{0,\lambda}^T(|X_T|^\alpha)^{\frac{1}{\alpha}} = \left(1 + \frac{O(\lambda)}{\ell(T^{-1})}\right) E_0 \left( \left| X_{\left(T\ell(T^{-1})^{\frac{1}{4}}\right)} \right|^\alpha \right)^{\frac{1}{\alpha}},$$

where with  $T > 1$ , the logarithmic factor is

$$\ell(T^{-1}) = 1 + O(\lambda) + B\lambda(4 \log T + \log |1 + \lambda \log T|).$$