Local index theorem for transversally elliptic operators

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    review of G-manifolds

    G-manifold and equivariant differential forms

    periodic cyclic (co)homology (of $\mathscr{A}$)

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  - Operator algebraic K-theory approach $(\mathscr{H}, F)$
- Spectral triples and Connes-Moscovici local index formula
  - τ_k for $(\mathscr{A}, \mathscr{H}, D)$
  - Wave front set
  - Oscillatory Integral
  - Resolution of Singularities
  - Renormalization

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- The only G-equivariant maps between G-points:

$$G/H \to G/K$$

are surjective ones for $(H) < (K)$.

This gives a partial order among types.

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(2) Attaching by G-equivariant map: $S^{n-1} \times (G/H) \longrightarrow X$ can be reduced to $S^{n-1} \longrightarrow X^H$.

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• Block and Getzler "globalized" the Cartan's version and showed that its cohomology is essentially the same as periodic cyclic homology of the smooth crossed product algebra $\mathscr{A}$, to be defined later.

(for a compact Lie group G acting on a compact manifold M)
An invariant (pseudo-)differential operator

$$P : \Gamma(E) \to \Gamma(F)$$

$$T_G^*M = \{(x, \xi) \in T^*M : \langle \xi, X_M \rangle = 0 \;\; \forall X \in \mathfrak{g}\}$$

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Transverse ellipticity (similar to a foliation): means principal symbol invertible on T_G^*M.

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- Wave operator with null directions not intersecting $T^*_G M$;
- Pseudo-Riemannian Dirac operators with null directions not intersecting $T^*_G M$.

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Atiyah also gave index theorem for torus action with finite isotropies (bigger orbit types). He used equivariant K-theory extensively.

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$$(a * b)(x,g) = \int_G a(x,h) b(h^{-1}x, h^{-1}g) d\mu(h).$$

Let the group action be $\rho : G \times M \longrightarrow M$.
Also use ρ for the equivariant map $\rho : G \times E \longrightarrow E$.
The algebra $\mathscr{A}$ acts on sections of a vector bundle E this way:

$$(\rho(a) \cdot s)(x) = \int_G a(x,g)(\rho(g)s)(g^{-1}x) dg.$$

The special case: $\mathscr{H}$, the (graded) L^2 sections $L^2(E) \oplus L^2(F)$,

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And we can express the index (for large enough n):

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Julg (1981) pointed out TEOs gives cycles in $KK(\mathscr{A}, \mathbb{C})$.

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We have seen the Connes-Moscovici local index formula involves traces of operators like

$$a^0(da^1)^{(k_1)} \ldots (da^n)^{(k_n)}$$

composed with powers of $|D|^{-1}$

For our purpose, we view those operators in the first line as elements in a much bigger algebra $\Psi(E) \rtimes G$ (See Appendix).

Wave front relation of $A \in \Psi(E) \rtimes G$,

$$WF'(K_A) \in \{(\xi, g_* \xi) : \xi \in T^*_{G,x} M\};$$

compared to $A \in \Psi(E)$,

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Using the composition and pushing rules of wave front sets. here is how a typical index formula looks like:

$$Trace(\rho(f)(1+P^*P+K_1)^{-s}) - Trace(\rho(f)(1+PP^*+K_2)^{-s})$$

with

(A) with $1+P^*P+K_1$,$1+P^*P+K_2$ elliptic;

(B) and K_1, K_2 are chosen so "small" as to be absorbed by $\rho(f)$ to smoothing operators.

For $A \in \Psi \rtimes G$ and D the part of the spectral triple:

Following Grubb and Seeley (1995) we have the asymptotic behavior of resolvent along a ray

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Trace formula in $\Psi \rtimes G$ amounts to the asymptotic expansion (as $\mu \to \infty$) of oscillatory integrals locally of the form:

$$\mu^{-m} \int_{\mathbb{R}^{2n}} \int_G e^{i\mu(x-gx,\xi)} q(x,g,\xi)\, dg d\xi\, dx$$

(Malgrange) Let ϕ be a real valued nonzero analytic function on $\mathbb{R}^N$. For $u \in C_c^\infty(\mathbb{R}^N)$ (real or complex valued) a test function, let $I(\tau)$ be the oscillatory integral

$$I_{\phi,u}(\tau) = \int_{\mathbb{R}^N} e^{i\tau\phi(x)} u(x) dvol(x).$$

Then for $\tau \to \infty$

$$I(\tau) = \sum_{\alpha,p,q} c_{\alpha,p,q}(u) \tau^{\alpha-p} (ln\,\tau)^q,$$

where $\alpha \leq 0$ runs through a finite set of rational numbers, $p, q \in \mathbb{N}$ and $0 \leq q < n$. Moreover $c_{\alpha,p,q}$ are all distributions with support inside

$$S_\phi = \{x \in \mathbb{R}^N : d\phi(x) = 0\},$$

and with finite orders not exceeding N.

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(2) Recall the functionals defined in (CM95) (needed for local index formula)

$$\tau_k^{|D|}(A) = Res_{z=0} z^k Trace(A|D|^{-z}).$$

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(3) $\tau_k^{|D|}$ are decided by the symbol near a conic neighborhood of T_G^*M in $T^*M \backslash \{0\}$.

(4) Therefore, renormalization, as in CM95, is not needed.

Appendix 1:

The algebra $\Psi(E) \rtimes G$: an element is a continuous function : $G \to \Psi(E)$, with product

$$(P * Q)(g) = \int_G P(h) \cdot \left[((h^{-1})_* Q)(h^{-1}g)\right] d\mu(h),$$

There is an action on the sections $\Psi^k(E,E) \rtimes G$ on $\Gamma(E)$, defined as:

for $P = P(g)$ in $\Psi^\infty(E,E) \rtimes G$,

$$(Ps)(x) = \int_G P(g)\rho(g)(s(g^{-1}x))d\mu(g),$$

$\Psi^\infty(E,E) \rtimes G$ is a $\Psi^\infty(E,E)$ bimodule (but be careful of the module map)

Operators in $\Psi^k(E,E) \rtimes G$ has order k compatible to $\Psi^{k'}(E,E)$ (the $\mathbb{Z}$-grading of Ψ)

Appendix 2:

Let $(\mathscr{A},\mathscr{H},D)$ be an even spectral triple defined by a first order transversally elliptic pseudo-differential operator D and with all the above conditions.

The Connes character $ch(\mathscr{A},\mathscr{H},D)$ in periodic cyclic cohomology is represented by the following even cocycle in the periodic cyclic cohomology:

$$\phi_{2m}(a_0,\ldots,a_{2m}) = \sum_{k\in\mathbb{N}^{2m},q\geq 0} c_{2m,k,q}\cdot \tau_q\left(\gamma a^0(da_1)^{(k_1)}\cdots(da_{2m})^{(k_{2m})}|D|^{-2|k|-2m}\right) \quad (1)$$

for $m > 0$ and

$$\phi_0(a^0) = \tau_{-1}(\gamma a^0). \quad (2)$$

Appendix 3:

In the above formula $k=(k_1,\ldots,k_{2m})\in\mathbb{N}^{2m}$ are multi-indices and $c_{2m,k,q}$ are universal constants given by

$$c_{2m,k,q}=\frac{(-1)^{|k|}}{k!\tilde{k}!}\sigma_q(|k|+m), \tag{3}$$

where $k!=k_1!\ldots k_{2m}!$, $\tilde{k}!=(k_1+1)(k_1+k_2+2)\ldots(k_1+\ldots+k_{2m}+2m)$, and for any $N\in\mathbb{N}$, $\sigma_q(N)$ is the q-th elementary polynomial of the set $\{1,2,\ldots,N-1\}$.