Bivariant Chern Character and Connes' Index Theorem.

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1 Statement of the Main Theorem.

Let $\pi : P \to B$ be a submersion. Assume that fibers $\pi^{-1}(b)$, $b \in B$ are complete Riemannian manifolds. Let Γ be an etale groupoid (e.g. a discrete group) acting on P and B, so that π is Γ-equivariant. Assume that the action preserves the fiberwise metric. Let D be a fiberwise family of Dirac-type operators, acting on sections of a $\mathbb{Z}_2$-graded bundle $\mathcal{E}$ (to simplify notations assume that $\mathcal{E}^{\pm} = S^{\pm} \otimes V$, where S is the fiberwise spinor bundle). We assume that the family D is equivariant with respect to the action of Γ.

Let $\Omega_*(B)$, $\Omega_*(P)$ denote the complexes of currents (differential forms twisted with the orientation bundle) on B, P respectively. Using the action of Γ on P we can form complexes $C^*(\Gamma, \Omega_*(P))$, $CC^*(C_0^\infty(P) \rtimes \Gamma)$ and A. Connes' map

$$\Phi : C^*(\Gamma, \Omega_*(P)) \to CC^*(C_0^\infty(P) \rtimes \Gamma)$$

as well as a similar map for B. The family D defines an element $[D]$ of KK-theory $KK(C_0(P) \rtimes \Gamma, C_0(B) \rtimes \Gamma)$. Moreover, one can define V. Nistor's bivariant Chern character

$$\mathrm{Ch}[D] : CC_*(C_0^\infty(P) \rtimes \Gamma) \to CC_*(C_0^\infty(B) \rtimes \Gamma).$$

Theorem 1. *The following diagram is commutative up to homotopy:*

$$\begin{tikzcd}
C^*\left(\Gamma, \Omega_*\left(P\right)\right) \arrow[r, "\Phi"] & CC^*\left(C_0^\infty\left(P\right) \rtimes \Gamma\right) \\
C^*\left(\Gamma, \Omega_*\left(B\right)\right) \arrow[r, "\Phi"] \arrow[u, "\widehat{\mathrm{A}}_\Gamma \, \mathrm{Ch}_\Gamma V \wedge \pi^*(\cdot)"] & CC^*\left(C_0^\infty\left(B\right) \rtimes \Gamma\right) \arrow[u, "(\mathrm{Ch}[D])^t"']
\end{tikzcd}$$

Here $\widehat{\mathrm{A}}_\Gamma = \widehat{\mathrm{A}}_\Gamma\left(TP|B\right)$ is the equivariant $\widehat{\mathrm{A}}$-genus of the vertical tangent bundle, and $\mathrm{Ch}_\Gamma V$ is the equivariant Chern character of the coefficient bundle V.

2 Examples

- Γ – trivial, P – compact. In this case there is an index of the family D, an element of $K^0(B) = K_0(C_0^\infty(B))$. We can define

$$\mathrm{Ch}(\mathrm{index}\, D) \in CC_*(C_0^\infty(B)).$$

It is related to the bivariant Chern character $\mathrm{Ch}[D]$ as follows. In the cyclic complex $CC_*(C_0^\infty(P))$ there is an element 1. Then $\mathrm{Ch}(\mathrm{index}\, D) = \mathrm{Ch}[D](1)$. Our theorem in this case gives

$$\mathrm{Ch}(\mathrm{index}\, D) = \int_{P|B} \widehat{\mathrm{A}}(TP|B)\, \mathrm{Ch}\, V$$

- Γ – trivial, P – not necessarily compact. In this case $\operatorname{index} D$ is not necessarily defined. Let $\mathcal{F}_1$, $\mathcal{F}_2$ be two bundles on P which are isomorphic outside a compact subset of P (so they define a class $[\mathcal{F}_1 - \mathcal{F}_2] \in K^0(P)$). Then one can twist the family D with each bundle to construct the operators $D_{\mathcal{F}_1}$ and $D_{\mathcal{F}_2}$. These operators coincide outside the compact subset of P, and one can define their relative index $\operatorname{index}(D_{\mathcal{F}_1}, D_{\mathcal{F}_2}) \in K^0(B)$. In this case our theorem gives

$$\operatorname{Ch}(\operatorname{index}(D_{\mathcal{F}_1}, D_{\mathcal{F}_2})) = \int_{P|B} \widehat{\mathrm{A}}(TP|B)\operatorname{Ch} V(\operatorname{Ch}\mathcal{F}_1 - \operatorname{Ch}\mathcal{F}_2)$$

- Action of Γ on P is free, proper, cocompact. In this case we obtain A. Connes' index theorem for etale groupoid. As particular cases it contains Connes-Moscovici higher Γ-index theorem, V. Nistor's index theorem for foliated flat bundles, index theorem of X. Jiang. In this case there is an

$$\operatorname{index} D \in K_0\left(\left(C_0^\infty(B) \rtimes \Gamma\right) \otimes \mathcal{R}\right).$$

P/Γ is a compact manifold, and the algebras $C(P/\Gamma)$ and $C_0(P) \rtimes \Gamma$ are Morita equivalent, hence have the same K-theory. To the class $[1] \in K_0(C(P/\Gamma))$ corresponds a class in $K_0(C_0(P) \rtimes \Gamma)$ constructed as follows. Let $\phi \in C_0^\infty(P)$ be such that $\sum_{g\in\Gamma} \left(\phi^2\right)^g = 1$.

Define $p = \sum\limits_{g \in \Gamma} \phi \phi^g U_g$. Then p is a projector, representing the desired class. Then $\mathrm{Ch}[D]\,(\mathrm{Ch}\, p) = \mathrm{Ch}\,(\mathrm{index}\, D) \in HC_*\,(C_0^\infty\,(B) \rtimes \Gamma)$. Application of our theorem gives Connes' theorem: for any $c \in H^*\,(B \times_\Gamma \mathrm{E}\Gamma, \mathcal{O})$, $\mathcal{O}$ – orientation bundle on B, we have the following:

$$\langle \Phi\,(c)\,, \mathrm{index}\, D \rangle = \int\limits_{P/\Gamma} \widehat{\mathrm{A}}\,(T\mathcal{F})\,\mathrm{Ch}\, V \pi^*\,(c)$$

Here $\mathcal{F}$ is the foliation on P/Γ induced by the fibers of submersion π.

- Relation to foliations. A particular case of the previous result is the index theorem for foliations. Let M be a manifold with a foliation $\mathcal{F}$. Let D be a leafwise family of Dirac-type operators. Let G be the foliation groupoid, $r,\ s : G \to M$ – range and source maps. Let B be a complete transversal to the foliation (possibly disconnected). Set $P = \{\gamma \in G\ |r(\gamma) \in B\}$. Finally set Γ equal to the restriction of G to B. Then we are exactly in the situation described above.

3 Proof of the theorem.

Idea of the proof. We use a superconnection $\mathbb{A}$ to construct "McKean-Singer" type map $\Phi_{\mathbb{A}}$:

$$\begin{tikzcd}
C^*(\Gamma, \Omega_*(P)) \arrow[r, "\Phi"] & CC^*(C_0^\infty(P) \rtimes \Gamma) \\
C^*(\Gamma, \Omega_*(B)) \arrow[r, "\Phi"] \arrow[u, "\widehat{\mathrm{A}}_\Gamma \,\mathrm{Ch}_\Gamma V \wedge \pi^*(\cdot)"] \arrow[ur, dashed, "\Phi_{\mathbb{A}}"] & CC^*(C_0^\infty(B) \rtimes \Gamma) \arrow[u, "(\mathrm{Ch}[D])^t"']
\end{tikzcd}$$

In particular we obtain a superconnection proof of A. Connes' index theorem. A different superconnection proof was obtained earlier in our joint work with J. Lott.

Construction of $\Phi_{\mathbb{A}}$ when Γ is trivial. E – an infinite dimensional bundle over B with fibers E_b – sections of $\mathcal{E}|_{\pi^{-1}(b)}$. Chose H – a horizontal distribution on P. At every point $p \in P$ H_p is a subspace of T_pP, complementary to the vertical subspace. Choice of H allows to define an Hermitian connection ∇^H on the bundle E. A superconnection on E is a (pseudo)differential operator $\mathbb{A}$ with differential form coefficients on P defined by

$$\mathbb{A} = D + \nabla^H + \mathbb{A}_2 + \dots$$

where $\mathbb{A}_i : \Omega^k(B, E) \to \Omega^{k+i}(B, E)$.

Define

$$\left(\phi_{\mathbb{A}}\left(c\right)\right)_l\left(a_0, a_1, \ldots, a_m\right) =$$
$$\left\langle c, \int\limits_{\Delta^m} \operatorname{Tr}_s a_0 e^{-t_0\mathbb{A}^2}[\mathbb{A}, a_1]\ldots[\mathbb{A}, a_m]e^{-t_m\mathbb{A}^2}dt_1\ldots dt_m\right\rangle$$

Here $c \in \Omega_*\left(B\right)$, $a_i \in C_0^\infty\left(P\right)$, $l = m + \deg c$ – even, and $\Delta^m = \{t_0, t_1, \ldots, t_m \mid t_i \geq 0, \sum\limits_{i=0}^{m} t_i = 1\}$. We view $\phi_{\mathbb{A}}$ as a cochain in the complex $\operatorname{Hom}\left(\Omega_*\left(B\right), CC^*\left(C_0^\infty\left(P\right)\right)\right)$.

Lemma 2. $\left(B + b + \partial\right)\phi_{\mathbb{A}}\left(c\right) = 0$

Problem: $\phi_{\mathbb{A}}\left(c\right)$ is an infinite cyclic cochain for every c. The definition of the periodic cyclic cohomological complex involves on the other hand only finite cochains. To correct this we need to truncate this cochain, following the method of Connes and Moscovici.

Rescaled superconnection $\mathbb{A}_s$ is defined by

$$\mathbb{A}_s = sD + \nabla^H + s^{-1}\mathbb{A}_1 + s^{-2}\mathbb{A}_2 + \ldots$$

Lemma 3. $\frac{d}{ds}\left(\phi_{\mathbb{A}_s}\right)_l = b\left(\tau_s\right)_{l-1} + \left(B + \partial\right)\left(\tau_s\right)_{l+1}$

for certain cochain $\tau_s \in \mathrm{Hom}\left(\Omega_*\left(B\right), CC^*\left(C_0^\infty\left(P\right)\right)\right)$.

Lemma 4. *There exists a number* N *such that for all*

$l > N$

$$\lim_{s\to 0}\left(\phi_{\mathbb{A}_s}\right)_l(c)\left(a_0, \ldots, a_m\right) = 0$$

and $\left(\tau_s\right)_l(c)\left(a_0, \ldots, a_m\right)$ *is integrable near* $s = 0$.

Proposition 5. *Choose any even $k \geq N$. Define the cochain $\Phi^0_{\mathbb{A}} \in \operatorname{Hom}\left(\Omega_*\left(B\right), CC^*\left(C^\infty_0\left(P\right)\right)\right)$:*

$$\left(\Phi^0_{\mathbb{A}}\right)_l = \begin{cases} \left(\phi_{\mathbb{A}}\right)_l & \text{for } l < k \\ \left(\phi_{\mathbb{A}}\right)_k - \int\limits_0^1 \left(B + \partial\right)\left(\tau_s\right)_{k+1} ds & \text{for } l = k \\ 0 & \text{for } l > k \end{cases}$$

Then this cochain is a cocycle. Cohomology class is independent of the choices made.

The general case. We start by constructing an element $\Phi'_{\mathbb{A}} \in C^{*}\left(\Gamma, \mathrm{Hom}\left(\Omega_{*}\left(B\right), CC^{*}\left(C_{0}^{\infty}\left(P\right)\right)\right)\right)$. Given a superconnection $\mathbb{A}$,

$$\Phi_{\mathbb{A}}^{0} \in C^{0}\left(\Gamma, \mathrm{Hom}\left(\Omega_{*}\left(B\right), CC^{*}\left(C_{0}^{\infty}\left(P\right)\right)\right)\right)$$

but its boundary under the group differential is not 0. We construct higher components

$$\Phi_{\mathbb{A}}^{k} \in C^{k}\left(\Gamma, \mathrm{Hom}\left(\Omega_{*}\left(B\right), CC^{*}\left(C_{0}^{\infty}\left(P\right)\right)\right)\right)$$

so that a resulting total cochain is a cocycle. This cochain depends not only on the superconnection $\mathbb{A}$, but on its simplicial extension. Let $\mathbb{A}$ be a superconnection on the submersion $\pi : P \to B$. Then $\mathbb{A}^{g}$ is another superconnection. Connect them by a family of superconnections

$\mathbb{A}(g)(t_0, t_1)$, $0 \leq t_i \leq 1$, $t_0 + t_1 = 1$, so that

$$\mathbb{A}(g)(0,1) = \mathbb{A}^g \text{ and } \mathbb{A}(g)(1,0) = \mathbb{A}.$$

Now inductively construct a simplicial superconnection, i.e. a family of superconnections $\mathbb{A}(g_1, g_2, \ldots g_k)(t_0, \ldots t_k)$ on $P \to B$ such that

$$\mathbb{A}(g_1, g_2, \ldots g_k)(t_0, \ldots t_k)|_{t_i=0} =$$
$$\begin{cases} (\mathbb{A}(g_2, \ldots g_k)(t_1, \ldots t_k))^{g_1} & \text{if } i = 0 \\ \mathbb{A}(g_1, \ldots, g_i g_{i+1}, \ldots g_k)(t_0, \ldots, t_k) & \text{if } 0 < i < k \\ \mathbb{A}(g_1, \ldots g_{k-1})(t_0, \ldots t_{k-1}) & \text{if } i = k \end{cases}$$

Then for each $g_1, \ldots, g_k$ consider the submersion $\pi \times id :$ $P \times \Delta^k \to B \times \Delta^k$.

On this submersion construct a superconnection

$$\mathbb{A}_{g_1,\dots,g_k} = \mathbb{A}\left(g_1, g_2, \dots g_k\right)\left(t_0, \dots t_k\right) + d_{dR}.$$

Introduce also projections $p_P : P \times \Delta^k \to P$ and $p_B : B \times \Delta^k \to B$, and extension homomorphism $e_P = p_P^* : C_0^\infty(P) \to C^\infty\left(P \times \Delta^k\right)$ Then define

$$\Phi_{\mathbb{A}}^k\left(g_1, \dots, g_k\right) = e_P^* \circ \Phi^0_{\mathbb{A}_{g_1,\dots,g_k}} \circ p_B^*$$

Proposition 6. *The cochain $\Phi'_{\mathbb{A}} = \{\Phi^k_{\mathbb{A}}\}$ is a cocycle in the complex $C^*\left(\Gamma, \mathrm{Hom}\left(\Omega_*(B), CC^*\left(C_0^\infty(P)\right)\right)\right)$.*

We now use cup-product to define $\Phi_{\mathbb{A}}$ as follows:

$$\begin{tikzcd}
C^*\left(\Gamma, \Omega_*(B)\right) \arrow[r, "\cdot\cup\Phi'_{\mathbb{A}}"] \arrow[dr, dashed, "\Phi_{\mathbb{A}}"] & C^*\left(\Gamma, CC^*\left(C_0^\infty(P)\right)\right) \arrow[d, "\Phi"] \\
& CC^*\left(C_0^\infty(P) \rtimes \Gamma\right)
\end{tikzcd}$$

Bismut superconnection and short-time limit. Let $P \to B$ be a submersion. Chose a horizontal distribution H. Bismut superconnection is defined by the equation

$$\mathbb{A}_{\text{Bismut}}(H) = D + \nabla^H - \frac{1}{4}c\left(T^H\right)$$

where T^H is the curvature of the distribution H. The space of horizontal distributions has a natural affine structure, with the underlying linear space $\operatorname{Hom}(TB, TP|B)$. Using this we define the simplicial Bismut superconnection

$$\mathbb{A}(g_1, g_2, \dots g_k)(t_0, \dots t_k)$$

$$= \mathbb{A}_{\text{Bismut}}\left(t_0 H + t_1 H^{g_1} + t_2 H^{g_1 g_2} + \dots + t_k H^{g_1 \cdots g_k}\right)$$

With $\mathbb{A}$ such defined we have the following

Proposition 7.

$$\lim_{s\to 0} \Phi^k_{\mathbb{A}_s}\left(g_1,\dots,g_k\right)\left(c\right)\left(a_0,a_1,\dots,a_m\right) = \int_P \omega\wedge\left(\pi^* c\right) a_0 da_1\dots da_m$$

where ω is a form on P given by

$$\omega = \int_{P\times\Delta^k|P} \widehat{\mathrm{A}}\left(p_P^* TP\right)\mathrm{Ch}\left(p_P^* V\right).$$

where p_P is the projection $P\times\Delta^k\to P$.

Theorem 8. *The following diagram commutes (up to homotopy)*

$$\begin{tikzcd}
C^*\left(\Gamma, \Omega_*\left(P\right)\right) \arrow[r, "\Phi"] & CC^*\left(C_0^\infty\left(P\right) \rtimes \Gamma\right) \\
C^*\left(\Gamma, \Omega_*\left(B\right)\right) \arrow[u, "\widehat{\mathrm{A}}_\Gamma \operatorname{Ch}_\Gamma V \wedge \pi^*(\cdot)"] \arrow[ur, "\Phi_{\mathbb{A}}"] &
\end{tikzcd}$$

Definition of the bivariant Chern character. Consider the algebra $\Psi = \Psi\left(\mathcal{E} \oplus \mathcal{E}\right)$ of the fiberwise pseudodifferential operators on the submersion $P \to B$ of order 0 acting on the sections of the bundle $\mathcal{F}$ whose principal (order zero) symbol has the form $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$, $a \in C_0^\infty\left(P\right)$ and whose Schwartz kernel is compactly supported. Let γ be the grading operator on $\mathcal{E}$ ($\gamma = \pm 1$ on $\mathcal{E}^{\pm}$). The grading on $\mathcal{E} \oplus \mathcal{E}$ is given by the operator $\begin{bmatrix} \gamma & 0 \\ 0 & -\gamma \end{bmatrix}$. We use this grading whenever we consider the cyclic complexes of this and related algebras. Groupoid G acts on Ψ, preserving the order filtration. We can then form the cross-product algebra $\Psi \rtimes \Gamma$ This algebra has a natural filtration by the order of pseudodifferential operators. This filtration induces corresponding filtration on the cyclic homology complex of the algebra Ψ. We denote by $F_{-k}CC_*(\Psi \rtimes \Gamma)$ the subcomplex of chains of total degree $\leq -k$. For k large enough $(k > \dim P - \dim B)$ there is a supertrace map:

$$\mathrm{Tr}_s : F_{-k}CC_*(\Psi \rtimes \Gamma) \rightarrow CC_*(C_0^\infty(B) \rtimes \Gamma)$$

Let Q be a family of proper pseudodifferential operators forming a parametrix of D. One can always chose Q to be Γ-equivariant. Then $S_0 = 1 - QD$ and $S_1 = 1 - DQ$ are proper smoothing G-equivariant fiberwise operators.

Define then U_D by the formula

$$V_D = \begin{bmatrix} D & S_1 \\ \\ S_0 & -(1+S_0)\,Q \end{bmatrix}$$

Its inverse V_D^{-1} is given by the following explicit formula:

$$V_D^{-1} = \begin{bmatrix} (1+S_0)\,Q & S_0 \\ \\ S_1 & -D \end{bmatrix}$$

Define homomorphisms ϕ_0, $\phi_1 : C_0(P) \rtimes \Gamma \to \Psi \rtimes \Gamma$ by

$$\phi_0(aU_g) = \begin{bmatrix} a & 0 \\ \\ 0 & 0 \end{bmatrix} U_g$$

$$\phi_1(aU_g) = \left(V_D \begin{bmatrix} a & 0 \\ \\ 0 & 0 \end{bmatrix} V_D^{-1} \right) U_g$$

Then

$$\phi_0(aU_g) - \phi_1(aU_g) \in \Psi^{-1}(\mathcal{E} \oplus \mathcal{E}) \rtimes \Gamma.$$

Then define the map

$$c_1(D) : CC_*\left(C_0^\infty(P) \rtimes \Gamma\right) \to F_{-1}CC_*\left(\Psi \rtimes \Gamma\right)$$

by the formula $c_1(D) = \frac{1}{2}\left((\phi_1)_* - (\phi_0)_*\right)$. Then V. Nistor's construction shows how one can construct for every k a map

$$c_k(D) : CC_*\left(C_0^\infty(P) \rtimes \Gamma\right) \to F_{-k}CC_*\left(\Psi \rtimes \Gamma\right)$$

such that $c_1(D)$ and $c_k(D)$ are homotopic as maps

$$CC_*\left(C_0^\infty(P) \rtimes \Gamma\right) \to F_{-1}CC_*\left(\Psi \rtimes \Gamma\right).$$

The bivariant Chern character is now defined as

$$\mathrm{Ch}[D] = \mathrm{Tr}_s \circ c_k(D) : CC_*\left(C_0^\infty(P) \rtimes \Gamma\right) \to CC_*\left(C_0^\infty(B) \rtimes \Gamma\right)$$

Commutativity of the lower triangle. We can write the formula for $\Phi_{\mathbb{A}}$ using throughout connection ∇^H instead of the simplicial superconnection $\mathbb{A}$. This formula defines a map

$$\Phi_\nabla : C^*\left(\Gamma, \Omega_*\left(B\right)\right) \to \left(F_{-k}CC_*\left(\Psi \rtimes \Gamma\right)\right)'$$

for k large enough.

Lemma 9. *The diagram*

$$\begin{tikzcd}
 & \left(F_{-k}CC_*\left(\Psi \rtimes \Gamma\right)\right)' \\
C^*\left(\Gamma, \Omega_*\left(B\right)\right) \arrow[ur, "\Phi_\nabla"] \arrow[r, "\Phi"] & CC^*\left(C_0^\infty\left(B\right) \rtimes \Gamma\right) \arrow[u, "\left(\mathrm{Tr}_s\right)^t"']
\end{tikzcd}$$

is commutative up to homotopy.

Lemma 10. *The diagram*

$$\begin{tikzcd} & CC^*\left(C_0^\infty\left(P\right)\rtimes\Gamma\right) \\ C^*\left(\Gamma,\Omega_*\left(B\right)\right) \arrow[ur, "\Phi_{\mathbb{A}}"] \arrow[r, "\Phi_{\nabla}"] & \left(F_{-k}CC_*\left(\Psi\rtimes\Gamma\right)\right)' \arrow[u, "\left(c_k(D)\right)^t"'] \end{tikzcd}$$

is commutative up to homotopy.

Theorem 11. *The diagram*

$$\begin{tikzcd} & CC^*\left(C_0^\infty\left(P\right)\rtimes\Gamma\right) \\ C^*\left(\Gamma,\Omega_*\left(B\right)\right) \arrow[ur, "\Phi_{\mathbb{A}}"] \arrow[r, "\Phi"] & CC^*\left(C_0^\infty\left(B\right)\rtimes\Gamma\right) \arrow[u, "\left(\mathrm{Ch}[D]\right)^t"'] \end{tikzcd}$$

is commutative up to homotopy.