

VARIATIONAL PRINCIPLES FOR BPS BLACK HOLE ENTROPY

Toronto, May 2005

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BPS \Rightarrow extremal black holes
& $N=2,4$ supersymmetry

'96 Strominger, Vafa :

- String/M theory microstates
 \rightarrow entropy (microscopic)
 - black hole solution \rightarrow area
 \rightarrow entropy (macroscopic)
- ↑ ↓
 MATCH
- area law

$$S_{\text{macro}} = \frac{1}{4} \frac{A}{G_N}$$

associated with the
first law of black hole mechanics

CHARACTERISTICS OF BORN-HOKE SOLUTIONS

$N=2$ supersymmetry

- solitonic interpolations between
 - at $r=\infty$: flat Minkowski space, $N=2$ susy
 - at horizon: $\text{AdS}_2 \times S^2$, $N=2$ susy
↑
unique
 - BPS: $N=1$ globally
- ⇒ **attractor mechanism** '96 Ferrara, Kallosh, Strominger
- horizon: moduli fixed in terms of charges
- $r=\infty$: moduli free integration constants
(relevant for M_{BH})

important details:

- $N=2$ supergravity + vector supermultiplets encoded in holomorphic homogeneous fct.
 $F(X)$: $F(\lambda X) = \lambda^2 F(X)$
- subject to electric/magnetic duality
(abelian charges)
- symplectic periods (special geometry)
 $(X^I, F_I(X))$ rotate under $\text{Sp}(2n+2, \mathbb{R})$
isomorphism classes

X^I : projective \rightarrow rescale to Y^I

Attractor equations consistent with e/m duality

$$X^I - \bar{X}^I = i p^I \quad \text{magnetic charges}$$

$$F_I - \bar{F}_I = i q_I \quad \text{electric charges}$$

$$\frac{\text{Area}}{G_N} = 4\pi |Z|^2$$

$$|Z|^2 = p^I F_I(Y) - q_I Y^I$$

$$\propto \sqrt{Q^4}$$

example $CY_3 \times S^1$

↪ 5-brane wrapped over 4-cycle

2-forms $CY_3 \rightarrow p^A$ wrapping # $C_{p,[\text{atb}]}$

momentum $S^1 \rightarrow q_0$ KK photon $g_{\mu\nu}$

$$F(Y) = -\frac{1}{6} \left(\frac{C_{ABC} Y^A Y^B Y^C}{Y^0} \right)$$

intersection #

area \rightarrow entropy:

$$S_{\text{macro}}(p, q) = \pi |Z|^2 = 2\pi \sqrt{\frac{1}{6} q_0 C_{ABC} p^A p^B p^C}$$

$\hat{q}_0 = q_0 + \text{corrections due}$
 $\text{to electric charges } q_A$

(MSW effect)

Variational principle (BCKLM '96)

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define:

$$\Sigma(Y, \bar{Y}, p, q) = -K(Y, \bar{Y}) - W(Y, p, q) - \bar{W}(\bar{Y}, p, q)$$

$K(Y, \bar{Y}) \sim \text{'K\"ahler potential' } i(\bar{Y}^I F_I - Y^I \bar{F}_I)$

$W(Y, p, q) \sim \text{'holomorphic BPS mass'}$

$$q_I Y^I - p^I F_I$$

(functions under symplectic transformations)

$$\begin{aligned}\delta \Sigma = & i(Y^I \bar{Y}^I - i p^I) \delta(F_I + \bar{F}_I) \\ & - i(F_I - \bar{F}_I - i q_I) \delta(Y^I + \bar{Y}^I)\end{aligned}$$

$\delta \Sigma = 0 \rightarrow$ attractor equations

$$\Sigma_{\text{attractor}} = \frac{i}{\pi} S_{\text{macro}}(p, q)$$

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$$+ 4 \operatorname{Im}(T F_T) \quad (T = 64)$$

Corrections subleading in charges 5

① 'second Chern class' etc

② non-holomorphic corrections

(i.e. non-Wilsonian)

ad 1: how to break homogeneity?

extra chiral multiplet $T \sim T^2$

('graviphoton')²

attractor value $T = -64$

independent of charges

holomorphic homogeneous fct $F(Y, T)$

$$F(\lambda Y, \lambda^2 T) = \lambda^2 F(Y, T)$$

T -dependence leads to R^2 terms
in the action

entropy formula

$$S_{\text{macro}}(p, g) = \pi \left[|Z|^2 - 256 \operatorname{Im} F_T \right] \Big|_{T=-64}$$

\propto area/ g Cardoso, dW, Mohanty

Violates area law!

based on Wald's definition of entropy
 \sim first law ensured

variational principle ✓

again $CY_3 \times S^1$

$$F(Y, T) = -\frac{1}{6} \frac{C_{ABC} Y^A Y^B Y^C}{Y^0} - \frac{c_{2A} Y^A}{24.64 \cdot Y^0} T$$

$$S_{\text{macro}}(p, q) = 2\pi \sqrt{\frac{1}{6} |q_0| (C_{ABC} p^A p^B p^C + c_{2A} p^A)}$$

2 subleading

microscopic: Maldacena, Strominger, Witten, Vafa '97
 macroscopic: Cardoso, daW, Mohaupt '98

Area Law

$$\frac{1}{4} \frac{A(p, q)}{G_N} = \frac{C_{ABC} p^A p^B p^C + \frac{1}{2} c_{2A} p^A}{C_{ABC} p^A p^B p^C + c_{2A} p^A} , S_{\text{macro}}(p, q)$$



$$\frac{c_R}{c_L} = \frac{\text{right}}{\text{left}} - \text{moving central charges}$$

R: susy sector

significance?

Another example heterotic black holes

- correspondence with $N=4$ perturbative string states
- nonholomorphic corrections
- dyonic degeneracy formula
'96 Dijkgraaf, Verlinde, Verlinde)

Heterotic black holes

lowest order $F(Y) = - \frac{Y'}{Y^0} Y^a \gamma_{ab} Y^b \quad a, b = 2, \dots, n$

symmetry: $\left[\frac{SU(1,1)}{U(1)} \right]_S \times \left[\frac{SO(2, n-1)}{SO(2) \times SO(n-1)} \right]_T$

extension to $N=4$ $\left[\frac{SO(6, 22)}{SO(6) \times SO(22)} \right]_T$

$$S_{\text{macro}}(P, q) = \frac{1}{2} \frac{A}{G_N} = \pi \sqrt{q^2 P^2 - (q \cdot P)^2}$$

$T \times S$ invariant

PM: p^2 and q^2 not positive definite

$N=4$ extension: 2 types of BPS states

- $1/4$ -BPS 'dyonic' $q^2 P^2 - (q \cdot P)^2 > 0$
- $1/2$ -BPS 'electric' $q^2 P^2 - (q \cdot P)^2 = 0$
(zero classical area)

Corrections affect S^1 duality

→ beyond Wilsonian action
no holomorphicity ('anomaly')

Invariance of the Wilsonian action under
 $Y \rightarrow Y'(Y)$ implies that the changes of Y
on the periods $(Y^I, F_I(Y))$ induce a
symplectic transformation, which acts
correspondingly on the charges (P^I, q_I)

Nonholomorphic corrections

$$F(Y, T) = -\frac{Y'}{Y^0} Y^a \eta_{ab} Y^b + T f(S)$$

$$iS = \frac{Y'}{Y^0} \quad \text{dilaton}$$

S-duality

$$Y^a \rightarrow dY^a - \frac{1}{2} c \eta^{ab} F_b$$

$$Y^0 \rightarrow dY^0 + c Y'$$

$$Y' \rightarrow a Y' + b Y^0 \quad ad - bc = 1$$

$$S \rightarrow \frac{aS - ib}{icS + d}$$

induces:

$$F_a \rightarrow a F_a - 2b \eta_{ab} Y^b$$

$$F_0 \rightarrow a F_0 - b F_1 + \Delta_0(f')$$

$$F_1 \rightarrow d F_1 - c F_0 + \Delta_1(f')$$

$\Delta = 0 \rightarrow$ symplectic transformation

$\Delta \neq 0 \rightarrow$ adjust $f(S)$ or $F(Y, T)$ such that $\Delta \rightarrow 0$

in this case $f'(S)$ modular function of degree 2
 \rightarrow must be nonholomorphic

$$F(Y, \bar{Y}, T, \bar{T}) = -\frac{Y'}{Y^0} Y^a \eta_{ab} Y^b + \frac{i}{64\pi} T \log \eta^{12}(S) \\ + \frac{i}{128\pi} (T + \bar{T}) \log (S + \bar{S})^6$$

'gg Cardoso, dW, Mohaupt

Using the same rules as before one derives
the entropy, which is S-duality invariant

$$S_{\text{macro}}(p, q, s) = -\bar{\eta} \left[\frac{q^* - p \cdot q (s - \bar{s}) + p^* |s|^2}{s + \bar{s}} \right] - 2 \log [(s + \bar{s})^6 |\eta(s)|^2]$$

attractor eqs: $\frac{\partial S(p, q, s)}{\partial s} = 0$ SxT invariant
variational principle!

normalization type II $K_3 \times T^2 \leftrightarrow$ heterotic duality
asymptotic degeneracy $1/2$ -BPS states

$1/4$ -BPS states:

'microscopic' dyonic degeneracy formula

$$d(p, q) = \int_{3\text{-cycle}} d\Omega \frac{e^{i\pi(Q, \Omega Q)}}{\Phi_{10}(\Omega)}$$

Ω : period matrix
 $g=2$ R surface
 Φ autom. form

formally S -duality invariant

Dijkgraaf, Verlinde, Verlinde

saddle-point approximation (large charges)

→ coincides fully with macroscopic result!

by Cardoso, dW, Käppeli, Mohaupt

$1/2$ -BPS states:

$$p^2 q^2 - (p \cdot q)^2 = 0 \rightarrow \text{"subleading becomes leading"}$$

$$S_{\text{macro}} = \frac{1}{2} \frac{A}{G_N}$$

Dabholkar, Kallosh, Maloney, Sen
Rangamani, Hubeny, ... etc

$$s + \bar{s} \approx \sqrt{|q^2|/2}$$

weak string coupling

$$S_{\text{macro}} \approx 4\pi \sqrt{|q^2|/2} - 6 \log |q^2|$$

↑ normalization

related to perturbative string states

$$d(q) = \int_{1\text{-cycle}} d\tau \frac{e^{i\tau \sigma q^2}}{\eta^{24}(\sigma)} \approx \exp \left[4\pi \sqrt{|q|^2/2} - \frac{2\pi}{\gamma} \log |q|^2 \right]$$

↑
normalization ↑ discrepancy

Non-holomorphic corrections can be incorporated in the variational principle via

$$F_I \rightarrow F_I + 2i \partial_I \Omega$$

$$F_{\bar{I}} \rightarrow F_{\bar{I}} + 2i \partial_{\bar{I}} \Omega$$

$\Omega(Y, \bar{Y}, \Psi, \bar{\Psi})$ real homogeneous, defined up to imaginary part of a holomorphic function

$$\begin{aligned} \Omega = \frac{1}{128\pi} & [Y^I \log \eta^{12}(S) + \bar{Y}^{\bar{I}} \log \eta^{12}(\bar{S}) \\ & + \frac{1}{2} [r + \bar{r}] \log (S + \bar{S})^6] \end{aligned}$$

Return to variational principle

$$\begin{aligned} \delta \Sigma = i (Y^I \bar{Y}^J - i p^I) \delta (F_I + \bar{F}_{\bar{I}}) \\ - i (F_I - \bar{F}_{\bar{I}} - iq_I) \delta (Y^I + \bar{Y}^{\bar{I}}) \end{aligned}$$

consistent reductions:

$$\text{e.g. impose } Y^I \bar{Y}^I - i p^I = 0 \rightarrow Y^I = \frac{\phi^I}{2\pi} + \frac{i p^I}{2}$$

$$\pi \Sigma(\phi, p, q) = \mathcal{F}(\phi, p) - q_I \phi^I$$

Legendre transform

$$q_I = \frac{\partial \mathcal{F}(p, \phi)}{\partial \phi^I}$$

$$\mathcal{F} = 4\pi \text{Im } F(Y, r)$$

$$e^{\mathcal{F}} = Z_{\text{BH}}(\phi, p) = \sum_{\{q\}} d(p, q) e^{q_i \phi^i}$$

mixed partition function microscopic degeneracies

$$\text{PM} \quad e^{\mathcal{F}} = |e^{-2\pi i F}|^2$$

topological string partition fct.

inverse: Laplace transform

$$d(q, p) = \int d\phi^i e^{\mathcal{F}(p, \phi) - q_i \phi^i}$$

$\pi \Sigma(p, \phi)$

- well defined (integration contours, etc) ?
- satisfies appropriate symmetries ?
- holomorphic anomaly ?

'05 DDMR: Dabholkar, Denet, Moore, Pioline ($\frac{1}{2}$ -BPS)

\Rightarrow the OSV conjecture in its original form requires modifications

To resolve some of the difficulties we propose to return to an unreduced formulation

$$d \sim \int dY d\bar{Y} e^{\pi \Sigma(Y, \bar{Y}, p, q)}$$

\rightarrow OSV in saddle-point approximation

ignore (for the moment) choice of integration contours and possible non-holomorphic corrections

\Rightarrow full saddle-point approximation

$$\Rightarrow \left\{ \begin{array}{l} \text{stabilization eqs} \\ S_{\text{macro}}(p, q) \end{array} \right.$$

However, not consistent with electric/magnetic duality

$$\delta\Sigma \propto \left\{ \begin{array}{l} \delta(F_I + \bar{F}_I) \\ \delta(Y^I + \bar{Y}^I) \end{array} \right.$$

suggests to choose real integration variables $\left\{ \begin{array}{l} F_I + \bar{F}_I \\ Y^I + \bar{Y}^I \end{array} \right.$

i.e.

$$\prod_{I,J} d(Y + \bar{Y})^I d(F + \bar{F})_J = \underbrace{\left| \det(2 \operatorname{Im} F_{KL}) \right|}_{M} \prod_{I,J} dY^I d\bar{Y}^J$$

symplectically invariant measure !

upon integrating out $\operatorname{Im}(Y^I)$

$$\xrightarrow[\text{saddle-point}]{\quad} d(p, q) \sim \int d\phi^I \sqrt{\det(2 \operatorname{Im} F_{IJ})} e^{\pi\Sigma(\phi, p, q)}$$

for 'slowly-varying' determinant

$$\phi\text{-integral} \longrightarrow \exp[S_{\text{macro}}(p, q)]$$

Consistent with the situation for $1/4$ -BPS states where a saddle-point approximation of the DVV formula (in the limit of large charges) showed precisely the features:

- a measure which cancelled against the semiclassical integral
- whose derivatives vanish

Our proposal goes beyond the modifications proposed by DDMP, who

- introduce a p -dependent prefactor
- modify the topological partition function

return to $\frac{1}{2}$ -BPS states

$$\det(2 \operatorname{Im} F_{ij}) = 0 \quad \text{when nonperturbative terms } \sim O(e^{-2\pi s}) \text{ are suppressed}$$

this remains when introducing the non-holomorphic terms

at the same time the integral diverges !

this makes the $\frac{1}{2}$ -BPS case qualitatively different from the $\frac{1}{4}$ -BPS case.

DDMP: 'regulate' the divergence by restricting the integration contour to $\operatorname{Im} s \in (0, 2\pi)$.

This is only justified when S-duality is manifest, which seems to require non-holomorphic corrections

to be continued