AdS Solutions and Some Deformations

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Introduction

Much progress has been made in classifying supersymmetric solutions of supergravity theories.

Want:

1. A precise characterisation of solutions to the equations of motion admitting Killing spinors

2. Explicit solutions where possible.

Key Tool: G-Structures

Gauntlett, Martelli, Pakis, Waldram

Gauntlett, Pakis

Gauntlett, Gutowski, Hull, Pakis, Reall

....

Can apply the programme in 3 broad ways:

1. Classify the most general supergravity solutions in D=10/11 supergravity

2. Lower-Dimensional Supergravities

Can be much more explicit in $D = 4, 5, 6, 7$
Black rings, Gödel, ...
Black hole uniqueness theorems

3. Special classes of Solutions

Compactifications to $\mathbb{M}_4$ with flux
Compactifications to AdS_n.

PLAN:

1. Classification of $D = 11$ AdS_5 solutions

Gauntlett, Martelli, Sparks, Waldram

2. Construction of Sasaki-Einstein metrics in $D = 5$ and $D = 7$

Gauntlett, Martelli, Sparks, Waldram

3. Deformations of AdS_4 solutions in $D = 11$.
Gauntlett, Lee, Mateos, Waldram

Classifying AdS_5 solutions in $D = 11$

Until recently, surprisingly few explicit AdS_5 solutions:

Type IIB:

$$\begin{aligned} ds^2 &= AdS_5 \times Y_5 \\ F_5 &= Vol(AdS_5) + Vol(Y_5) \end{aligned}$$

where Y_5 is Sasaki-Einstein
Arise from D3-branes at the apex of Calabi-Yau three-fold cones:

$$ds^2 = dr^2 + r^2 ds^2(Y_5)$$

Two explicit examples: S^5 and $T^{1,1}$ - both homogeneous, field theories known.

$D = 11$:

$N = 1$ and 2 examples of Maldacena and Nunez. Field theories obscure.

We have classified the most general AdS_5 solutions of $D = 11$:

$$\begin{aligned} \mathrm{d}s^2 &= \mathrm{e}^{2\lambda(x)}[\mathrm{d}s^2(AdS_5) + \mathrm{d}s^2(M_6)(x)] \\ G_4 &= G_4(x) \end{aligned}$$

i.e. G is a 4-form on M_6. Ansatz preserves symmetries of AdS_5.

Explicit Solutions

Assume that M_6 is complex. Then can explicitly construct all compact regular solutions by solving ODEs.

$\star$ Topology: $S^2 \to M_6 \to B_4$

$\star$ Metric for M_6: completely explicit given metric on B_4 which can be in one of two classes:

(a) B_4 is Kähler-Einstein with positive scalar curvature (Kähler and $R_{ij} = \lambda g_{ij}$ with $\lambda > 0$). These have been classified by Tian and Yau:
explicit: $S^2 \times S^2$, CP^2
implicit: del Pezzo P_k $k = 3, \ldots 8$ (CP^2 blown up at k points).

(b) B_4 is a product.
All explicit: $S^2 \times S^2$, $S^2 \times H^2$, $S^2 \times T^2$

A special case of the S^2 bundle over $S^2 \times H^2$ case gives the $N = 1$ Maldacena Nunez solution.

Nice

$\star$ What is dual conformal field theory? Something to do with M5-branes.

$\star$ Where is the Maldacena $N = 2$ solution?

Consider D=11 solution with $S^2 \times T^2$ base:

Dimensional reduction on one of the S^1s of T^2 and then T-dualise on the other $S^1 \to$ type IIB solution:

$$AdS_5 \times X_5$$
$$F_5 \sim Vol(AdS_5) + Vol(X_5)$$

$\Rightarrow X_5$ must be Sasaki-Einstein, at least locally. In fact gives an infinite number of new explicit Sasaki-Einstein metrics on $S^2 \times S^3$!

Are all S^1 bundles over $S^2 \times S^2$ (like $T^{1,1}$), called $Y^{p,q}$ with integers $p > q$

Sasaki-Einstein

A SE X_5 is equivalent to the cone

$$ds^2 = dr^2 + r^2 ds^2(X_5)$$

being CY_3.

There is a canonical Killing vector:

$$(\partial_\psi)^j = r(\partial_r)^i J_i{}^j$$

This corresponds to the "U(1)" R-symmetry of the D=4 SCFT.

Locally, metric can be written

$$ds^2(X_5) = (d\psi + \sigma) + ds^2(B_4)$$

where B_4 is Kähler-Einstein and $d\sigma = 2J_4$

Three possibilities:

1. Regular SE:

Have a $U(1)$ R-symmetry and it is free.
B_4 is globally defined and hence can classify using Tian and Yau:
Explicit:
$B_4 = CP^2 \rightarrow S^5$
$B_4 = S^2 \times S^2 \rightarrow T^{1,1}$
Implicit: $B_4 = P_k$ del Pezzo $k = 3, \ldots 8$.

2. Quasi regular SE:

$U(1)$ R-symmetry with finite isotropy groups.
B_4 is an orbifold.

3. Irregular SE:

Have a non-compact $\mathbb{R}$ R-symmetry.
B_4 is not a manifold.

The $Y^{p,q}$ metrics obtained from $D = 11$ provide the first explicit examples in the quasi-regular class, and the very first examples in the irregular class!

$\star$ Isometry group $\sim SU(2) \times U(1) \times U(1)$

$\star$ Topology: $S^2 \times S^3$ just as for $T^{1,1}$

Can generalise and construct new Sasaki-Einstein metrics in any odd dimension. Return to $D = 7$ case later.

Predictions for dual SCFTs

$\star$ Symmetries: $SU(2) \times U(1) \times U(1) \times U(1)_B$

$\star$ Central charges:

$$
\begin{aligned}
\frac{a(Y^{p,q})}{a(S^5)} &= Vol(S^5)/Vol(Y^{p,q}) \\
&= \frac{3p^2[3q^2 - 2p^2 + p(4p^2 - 3q^2)^{1/2}]}{q^2[2p + (4p^2 - 3q^2)^{1/2}]}
\end{aligned}
$$

$\star$ Baryons arise from D3-branes wrapped on supersymmetric 3-cycles [Martelli, Sparks; Herzog, Ejaz, Klebanov]. R-charges of baryons:

$$R \propto \frac{Vol(\Sigma_i)}{Vol(Y^{p,q})} = \ldots$$

Dual field theory

Quiver gauge theory plus superpotential is now known [Benvenuti, Franco, Hanany, Martelli, Sparks]

Using the procedure of a-maximisation [Intriligator, Wecht] can determine the central charge a and the R-charges of the baryons. Find exact agreement with that predicted from the geometry.

Now have an infinite number of AdS/CFT examples where both the geometry and the field theory are known. Further generalisations are being pursued.

Deformations of AdS_4 solutions in $D = 11$

Supersymmetric conformal field theories can have exactly marginal deformations. Basic reason: beta functions depend on gamma functions.

e.g. $N = 4$ SYM has three complex such deformations. One of them, the β-deformation, preserves $U(1) \times U(1)$ and also exists in other CFTs with a $U(1) \times U(1)$ symmetry such as $AdS_5 \times T^{1,1}$ and $AdS_5 \times Y^{p,q}$.

If we know the dual AdS solution, can we find the corresponding deformed solution?

Lunin and Maldacena: found a very clever way of generating β-deformations.

Idea:

$\star$ Consider $AdS_5 \times X_5$ where X_5 has $U(1) \times U(1)$ isometry.

$\star$ This is a solution of D=8 SUGRA which has $Sl(2,R) \times Sl(3,R)$ duality symmetry. The $Sl(2,R)$ acts on $\tau = B_{12} + i\sqrt{G(T^2)}$.

$\star$ The action

$$\tau \to \frac{\tau}{1 + \gamma\tau}$$

generates a new **regular** solution which uplifts to a new AdS_5 solution with additional fluxes.

$\star$ If $U(1) \times U(1)$ isometry commutes with susy (i.e. commutes with $U(1)_R$) then the deformed solution preserves susy.

Applied to $AdS_5 \times S^5$, $AdS_5 \times T^{1,1}$ and $AdS_5 \times Y^{p,q}$.

(Can also consider breaking susy and also deformations of non-conformal theories).

Can be generalised to AdS solutions of D=11 that have a $U(1)^3$ action on compact space. The important $Sl(2,R)$ action of the $D=8$ SUGRA is now acting on $\tau = C_{123}+i\sqrt{G(T^3)}$.

Lunin and Maldacena applied this to $AdS_4 \times S^7$. We have generalised:

Consider the supersymmetric solutions of D=11:

$$\begin{aligned} ds^2 &= AdS_4 \times H_7 \\ F_4 &\propto Vol_4 \end{aligned}$$

Dual to field theories on M2-branes sitting at the apex of special holonomy cones:

$$ds^2 = dr^2 + r^2(ds^2(H_7))$$

Spin(7) $\leftrightarrow$ H_7 is Weak G_2 $\leftrightarrow$ N=1 susy

CY_8 $\leftrightarrow$ H_7 is Sasaki-Einstein $\leftrightarrow$ N=2 susy

Hyper-Kähler $\leftrightarrow$ H_7 is Tri-Sasaki $\leftrightarrow$ N=3 susy

To find supersymmetric deformed solutions using Lunin and Maldacena need examples with a $U(1)^3$ isometry that preserve some supersymmetry.

$\star$ Homogeneous examples

Weak G_2:
N(k,l)=SU(3)/U(1)
Squashed 7-sphere

Sasaki-Einstein:
Q(1,1,1) - S^1 bundle over $S^2 \times S^2 \times S^2$
M(3,2) - S^1 bundle over $CP^2 \times S^2$

tri-Sasaki:
N(1,1)

$\star$ Inhomogeneous Examples

Our construction of D=5 SE $Y^{p,q}$ can be generalised to all odd dimensions. For D=7 we find infinite new families of cohomogeneity one Sasaki-Einstein manifolds that generalise M(3,2) and Q(1,1,1).

Deformed solutions:

Tri-Sasaki:

$N(1,1)$

Has an exactly marginal deformation that breaks $N = 3 \rightarrow N = 1$

Sasaki-Einstein:

$M(3,2)$, $Q(1,1,1)$ and co-homogeneity one generalisations:

All have exactly marginal deformations that preserve $N = 2$ susy

Weak G_2:

N(k,l)=SU(3)/U(1) and squashed 7-sphere

Both have exactly marginal deformations that maintain $N = 1$ susy

Field Theory

For $AdS_5 \times X_5$ solutions of type IIB with $U(1) \times U(1)$ isometry, Lunin and Maldacena argued (using string field theory) that the dual version of the deformed geometries are obtained by adding some phases $e^{i\pi\gamma}$ into the Lagrangian.

In some cases this leads to a modified superpotential.

$AdS_5 \times T^{1,1}$ [Klebanov, Witten]

$SU(2)^2 \times U(1)_R$ global symmetry

$SU(N)^2$ quiver gauge theory

A_i in (2,1) and $(N, \bar{N})$
B_i in (1,2) and $(\bar{N}, N)$
$W = \epsilon^{ij}\epsilon^{kl}Tr(A_i B_k A_k B_l)$

Chiral primaries:
$Tr(A_{i_1}B_{j_1}...A_{i_k}B_{j_k})$ symmetrised over $SU(2)$ indices, i.e. in $(k+1, k+1)$, and $\Delta = 3k/2$

γ-deformation: Lunin and Maldacena
$W \to Tr(e^{i\pi\gamma}A_+B_+A_-B_- - e^{-i\pi\gamma}A_-B_+A_+B_-)$

For small γ:
$\Delta W \propto Tr(A_+B_+A_-B_- + A_-B_+A_+B_-)$

Unique $\Delta = 3$ chiral primary which breaks $SU(2)^2 \to U(1) \times U(1)$.

For the $AdS_4 \times H_7$ solutions of D=11 we know much less about the field theories living on the M2-branes which are strongly coupled gauge theories in the IR.

Nevertheless we have some understanding of the $Q(1,1,1)$, $M(3,2)$ [Fabbri, Fre, Gualtieri, Reina, Tomasiello, Zaffaroni, Zampa] and $N(1,1)$ [Billo, Fabbri, Fre, Merlatti, Zaffaroni] cases.
Chiral spectrum agreeing with Kaluza-Klein modes.
Supersymmetric 5-cycles agreeing with baryons (we did the $N(1,1)$ case).

In addition we can identify γ-deformation, for small γ by finding the unique superpotential that is:

Chiral with $\Delta = 2$
Preserves $U(1)^3$ global symmetry.

$AdS_4 \times Q(1,1,1)$

$SU(2)^3 \times U(1)_R$ global symmetry

$SU(N)^3$ quiver gauge theory

A_i in (2,1,1) and $(N, \bar{N}, 1)$
B_i in (1,2,1) and $(1, N, \bar{N})$
C_i in (1,1,2) and $(\bar{N}, 1, N)$

No superpotential!

Chiral primaries:
$Tr(ABC)^k$ symmetrised over all $SU(2)$ indices, i.e. in $(k+1, k+1, k+1)$, and $\Delta = k$

Note here we must *assume* that other $SU(2)$ reps decouple in the IR.

For small γ, what is superpotential deformation?
$Tr(ABC)^2$ has $\Delta = 2$ and is in $(3,3,3)$ rep, which has an element that preserves $U(1)^3$. Unique.

Analogous story for $M(3,2)$ and $N(1,1)$ case.

$AdS_4 \times M(3,2)$

$SU(3) \times SU(2) \times U(1)_R$ global symmetry

$SU(N)^2$ quiver gauge theory

U^i in (3,1) and
V^A in (1,2) and

No superpotential.

Chiral primaries:
$Tr(U^3V^2)^k$ symmetrised over all $SU(3)\times SU(2)$ indices, $\Delta = 2k$

Again we must *assume* that other $SU(3) \times SU(2)$ reps decouple in the IR.

For small γ, what is superpotential deformation?
$Tr(U^3V^2)$ has $\Delta = 2$ and is in $(10, 3)$ rep, which has an element that preserves $U(1)^3$.
Unique.

Conclusions

Classifying SUGRA solutions is a profitable endeavour. More to do on AdS side:

$\star$ Gravity duals for more general deformations for e.g. $AdS_5 \times S^5$?

$\star$ Field theories for $AdS_4 \times Y_{SE}$ solutions of D=11 with Y_{SE} generalising $Q(1,1,1)$ and $M(3,2)$. Check consistency with deformations.

$\star$ Field theories for $N = 1$ AdS_4 solutions of $D = 11$ e.g. $AdS_4 \times H_7$ when H_7 is weak G_2.

$\star$ Could classify AdS_n for other n in type IIB/D=11.

$\star$ New SE manifolds of [Cvetic, Lu, Page, Pope]

$\star$ Geometries describing renormalisation group flows between different field theories?

$\star$ Analogue of Calabi's theorem?