Discreteness vs. Continuity: From Music recordings to Cosmology

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The problem:

- Given a discretely defined function,

$$f : n \to f(n)$$

  we'd often find it easier to work with a continuous function:

$$f : t \to f(t)$$

- Because, e.g., integrals are often easier than sums!

Example:

(appears in the proof of an equation by Ramanujan):

$$C = \lim_{q \to 1^-} \sum_{m=1}^{\infty} \frac{q^m - q^{3m}}{m(1+q^m+q^{2m})^2} = ?$$

Can we write it as an integral?

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Can we write it as an integral?

Yes!

$$C = \int_{-\infty}^{\infty} \frac{\sinh(u)}{u(1+2\cosh(u))^2} du$$

or, conversely:

- Given a continuous function,

$$f : t \to f(t)$$

  we'd often find it preferable to work with:

$$f : n \to f(n)$$

- Because, e.g., we may have to count!

Example: Information Theory

see: C. Shannon, *The Mathematical Theory of Communication (1948)**

(*) NYT: one of the most influential text of 20th century

- <u>Discrete sources of information:</u>

"...erbf7834bieqc734nbcihfp34f8n9nf..."

… easy to quantify, counting bits and bytes

Example: Information Theory

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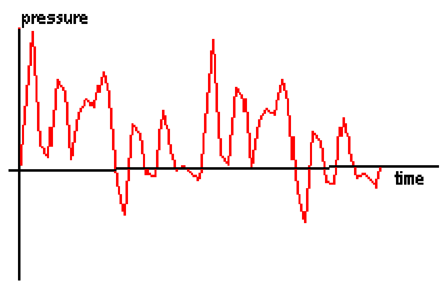
- <u>Continuous sources of information:</u>

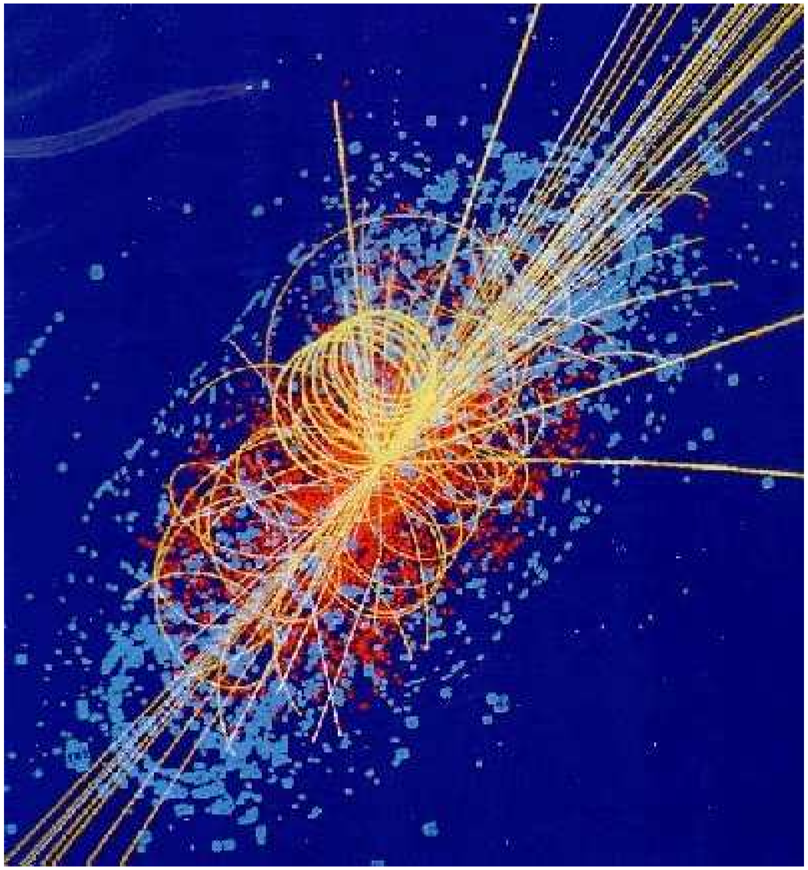
e.g. music:



- Is there an **uncountable amount of information ?**

Also, the problem can be:

We may not even know if a given case is discrete or continuous!



<u>Example:</u>

Space-time at
very short distances:
discrete or continuous?

Is the number of points in space countable or uncountable?

Consider Quantum Theory + General relativity

decrease Δ x

=> increase Δ p (by uncertainty relation)

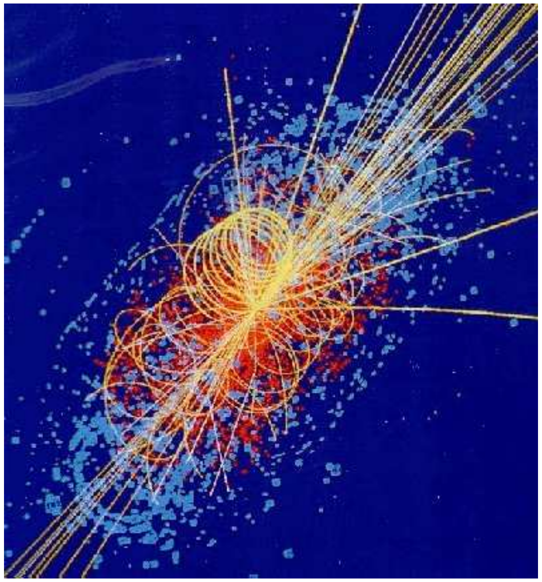
=> increase Δ R (momentum causes curvature)

=> increase Δ x (curvature affects distances)

=> **there exists a finite minimum Δx_{min}**

Estimate: Δx_{min} approx. 10^(-35)m

Current experiments: approx. 10^(-18)m



Is the number of points in space countable or uncountable, or else?

=> One expects a finite best position resolution in nature.

- **But, does this mean that:**

  (a) space might still be continuous?
  (b) space must be discrete?
  (c) space has a cardinality between $\aleph_0$ and **c** ?
  (d) else ?

- It's one of the deepest problems in mathematical physics !

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(c) Could the cardinality of physical space be in between $\aleph_0$ and **c** ?

Cantor (1877):

Continuum hypothesis (CH): “There is no set with an intermediate cardinality”

Hilbert (1900):

Lists (CH) as 1st in his list of problems for the 20th century.

Goedel (1938) & Cohen (1963):

Proof that CH is neither true nor false because (ZF) set theory is incomplete:

An axiom could be added to claim that CH is true or that it is false.

Implication for mathematical physics:

It is thought unlikely that axioms beyond regular set theory should be needed.

Any possibility of relating discrete and continuous in a simpler way?

Yes!

Key breakthrough in 1946:

- C. Shannon discovers significance of the "Sampling Theorem":
- succeeds in reducing continuous sources of information to discrete sources of information.

The basic sampling theorem (Shannon)

The basic sampling theorem (Nyquist)

The basic sampling theorem (Kotelnikov)

The basic sampling theorem (Whittaker)

The basic sampling theorem (Borel)

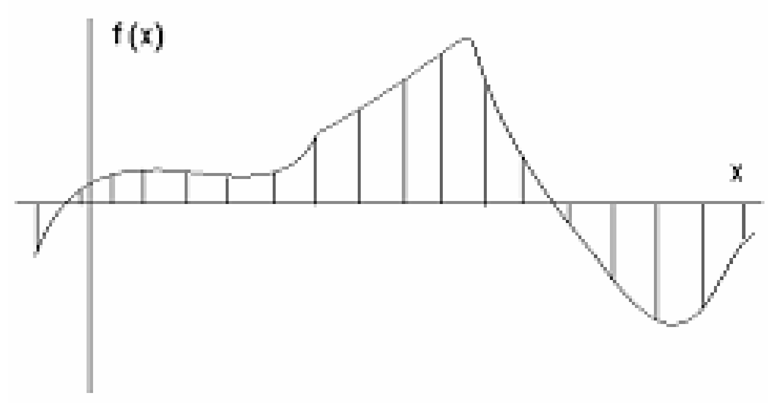
The basic sampling theorem (Cauchy 1840s)

The basic sampling theorem (SNKWBC et al)

- Assume a function possesses a finite bandwidth, i.e.:

$$f(t) = \int_{-w_{\max}}^{w_{\max}} \tilde{f}(w) e^{-2\pi i w t} dw$$

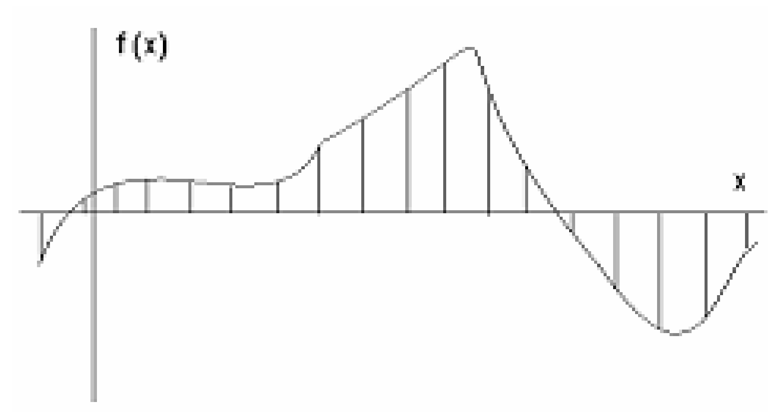
- Then, it can be fully reconstructed from discrete samples:



$$f(t) = \sum_{n=-\infty}^{\infty} f(t_n) \frac{\sin[2\pi w_{\max}(t - t_n)]}{2\pi w_{\max}(t - t_n)}$$

where $t_n = \dfrac{n}{2w_{\max}}$

Application: in information theory:



- It's the crucial link between discrete <-> continuous representations of information
- It's the reason why music can be stored on CDs!
- Applied also to imaging, scientific data taking etc...

Application: sums as integrals!

- Corollary: if the function is bandlimited then:

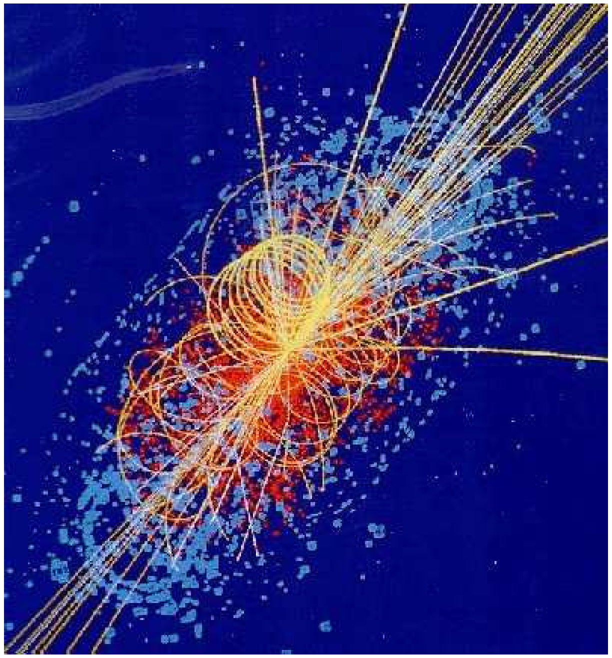
$$\int_{-\infty}^{\infty} f(t)^2\, dt = \frac{1}{2w_{\max}} \sum_{n=-\infty}^{\infty} f(t_n)^2$$

- (has been used occasionally in number theory)

Application: description of space-time at short distances?

Possibility:

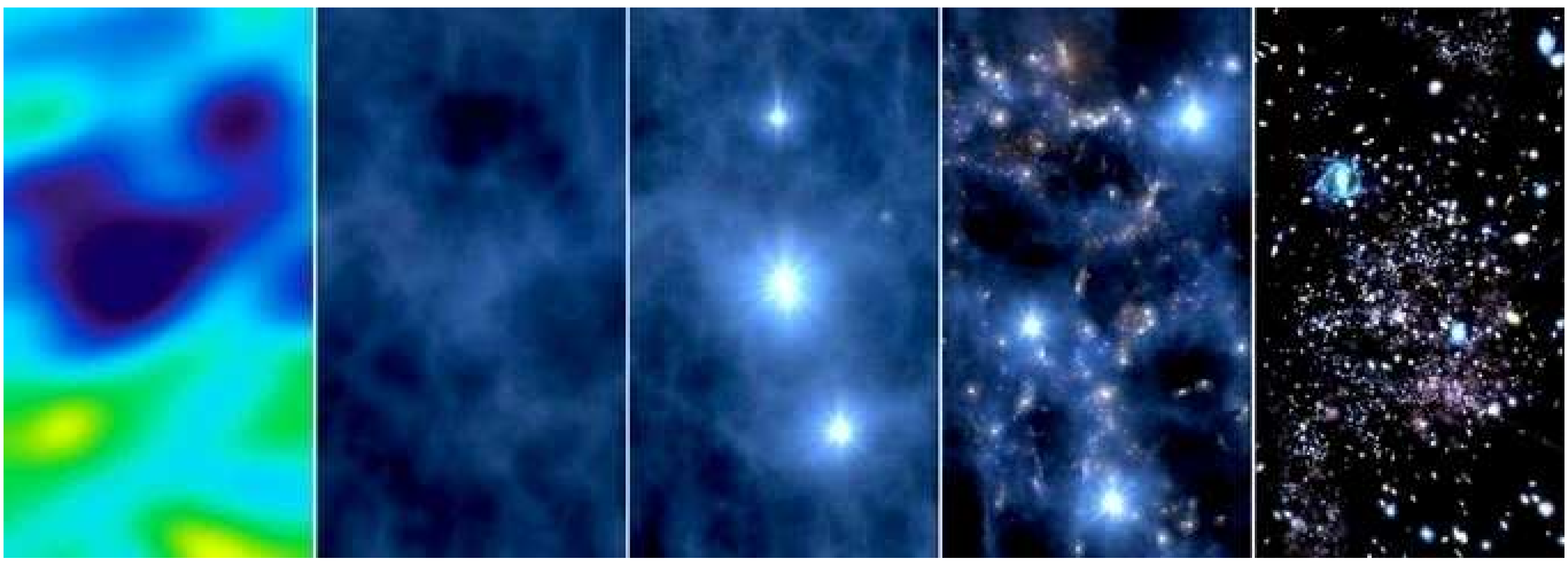
- The finite maximal spatial resolution is expressed through a generalized “bandwidth”.
- Electric, magnetic and other fields are reconstructible everywhere if known only at any discrete set of sufficiently densely spaced points.
- There is a finite bound to the density with which information can be physically represented.



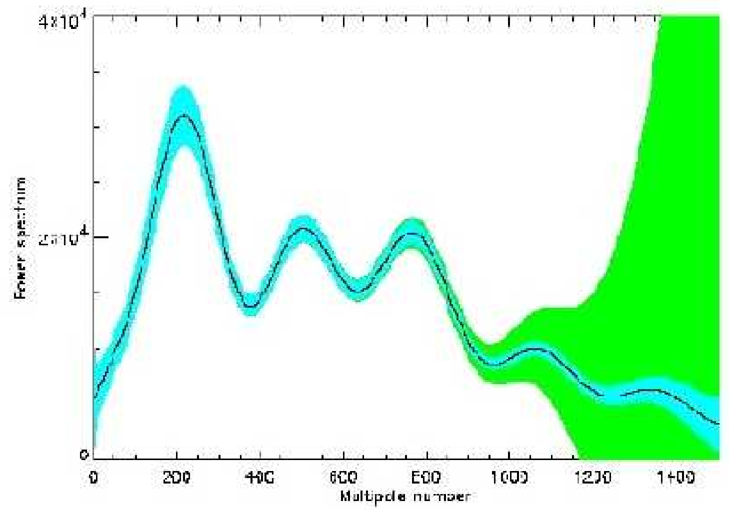
Can this be tested? Perhaps in cosmology!

Modern cosmology: Big bang acted as a microscope

- Tiny quantum fluctuations stretched very large
- then seeded the collapse of hydrogen into stars and galaxies.

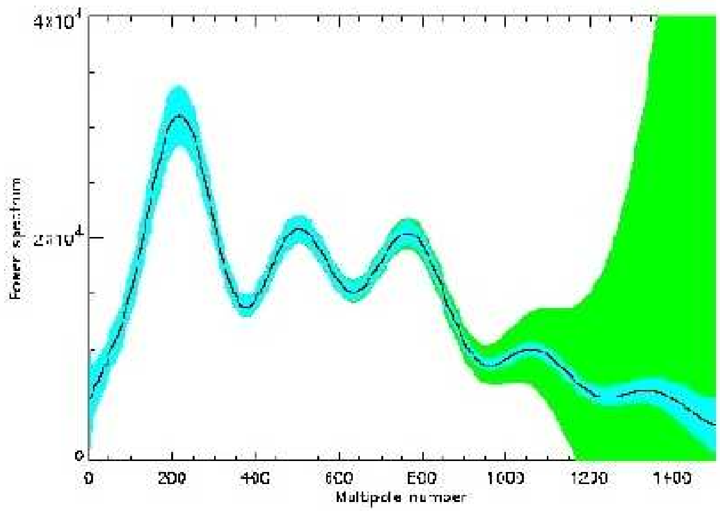


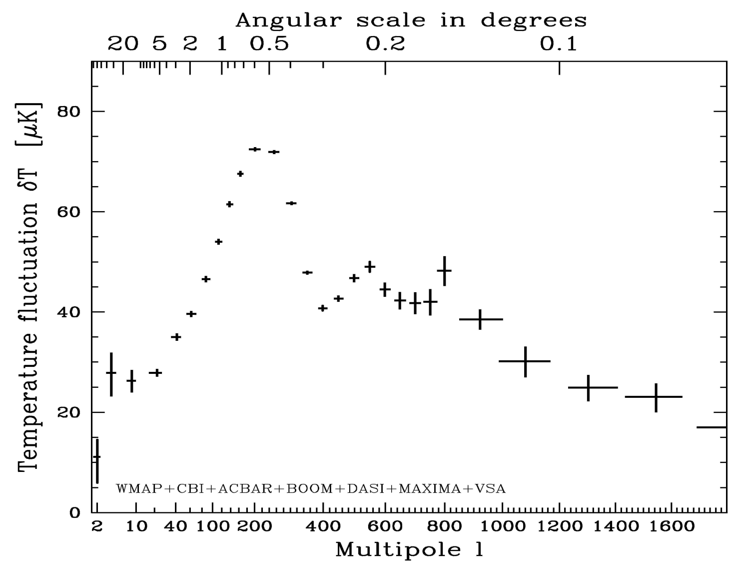
Theory of Inflation predicted (20 years ago):



Graph shows:
statistics of inhomogeneities

Satellite measurements in 2003:





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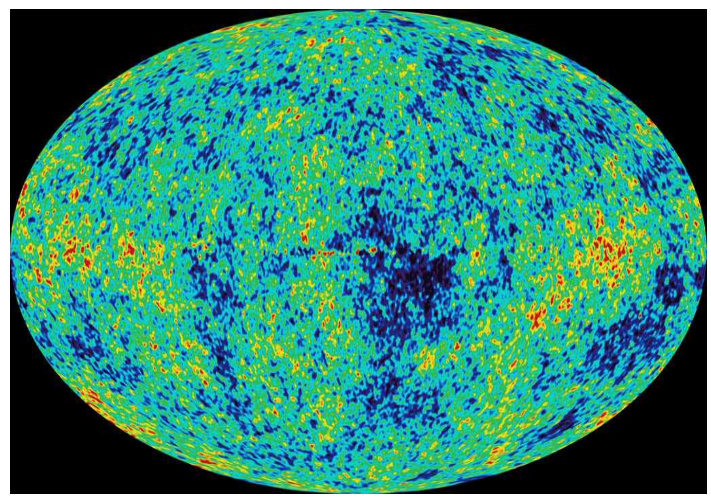
Bandwidth in the sky?

- More precision measurements of the early energy distribution in the sky (cosmic microwave background) are planned.
- The universe's fast inflation close to big bang should have stretched even 10^(-35)m cosmically large.
- Recent publications suggest it is possible we'll, see at better resolution, an imprint of space-time's
  - continuity ?
  - discreteness ?
  - finite bandwidth ?
  - or as the case may be... ?



Proof of the basic sampling theorem:

- Recall Fourier transform:

$$f(t) = \int_{-\infty}^{\infty} \tilde{f}(w) e^{-2\pi i w t} dw \quad \text{(A)}\ , \qquad \tilde{f}(w) = \int_{-\infty}^{\infty} f(t) e^{2\pi i w t} dt$$

- Recall Fourier series: assume $\tilde{f}(w) = 0$ whenever $|w| \geq w_{\max}$:

$$c_n = \frac{1}{2w_{\max}} \int_{-w_{\max}}^{w_{\max}} \tilde{f}(w) e^{-2\pi i w \frac{n}{2w_{\max}}} dw \quad , \qquad \tilde{f}(w) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i w \frac{n}{2w_{\max}}} \quad \text{for} \quad w \in (-w_{\max}, w_{\max})$$

- Notice: $c_n = \frac{1}{2w_{\max}} f(t_n)$ where we defined: $t_n = \frac{n}{2w_{\max}}$

- Thus: $\tilde{f}(w) = \sum_{n=-\infty}^{\infty} f(t_n) \frac{e^{2\pi i w t_n}}{2w_{\max}}$ Substitute this in Eq. A =>

Proof of the basic sampling theorem:

$$f(t) = \int_{-w_{\max}}^{w_{\max}} \sum_{n=-\infty}^{\infty} f(t_n) \frac{1}{2w_{\max}} e^{-2\pi i w(t-t_n)} dw$$

Thus,

$$f(t) = \sum_{n=-\infty}^{\infty} f(t_n) \int_{-w_{\max}}^{w_{\max}} e^{-2\pi i w(t-t_n)} dw$$

and therefore, indeed:

$$\boxed{f(t) = \sum_{n=-\infty}^{\infty} f(t_n) \frac{\sin[2\pi w_{\max}(t-t_n)]}{2\pi w_{\max}(t-t_n)}}$$

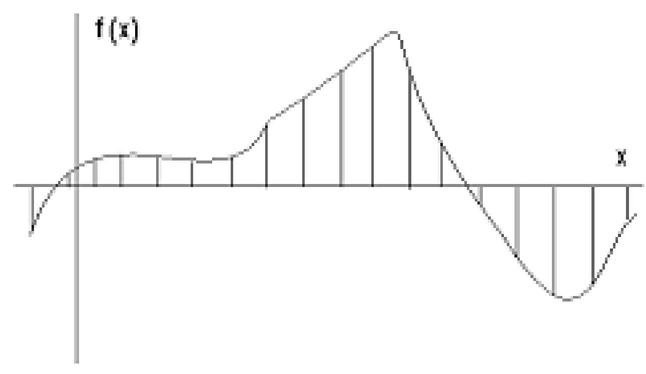
Generalized Sampling Theorems

- simple Fourier theory does not suffice!

- instead, use:
  - harmonic analysis
  - differential equations
  - functional analysis
  - differential geometry
  - spectral geometry
  - group theory
  - ...

Current research in sampling theory

Examples:

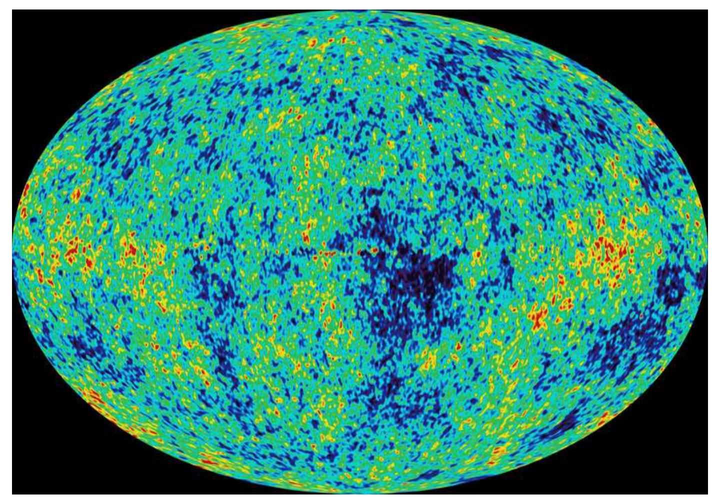


- Find generalized sampling theorems.
    - e.g. to turn sums into integrals
- Study stability of reconstruction with:
    - noise
    - lost samples
    - imprecise samples
- Use variable sample rates to adjust to varying bandwidth:
    - Fourier theory no longer suffices.
    - In this case, involves functional analysis: self-adjoint extensions of symmetric operators
    - application to data compression (patent)

Current research in sampling theory

Example:

- Sampling theory on curved manifolds:
  - model dinosaur skin etc...
  - study curved space-time
  - **calculate predictions for cosmology!**
- involves beautiful functional analysis & differential geometry:
  - the frequency bandwith becomes a cutoff of spectrum of the Laplacian of the manifold.



Current research in sampling theory

Example:

- In the field of Quantum computing & quantum communication:
  - Prospects of the field:
    - exponential speed-up of some computations
    - secured communication (eavesdropper “collapses the wave function”)
  - Role of sampling theory:
    - the focus so far has been on discrete information
    - establish the link between continuous and discrete information also within quantum information theory

Summary

- Sampling theory:
  - a bridge between discrete and continuous structures
  - involves various branches of mathematics.
- New results in this field potentially find a wide range of applications, e.g. in:
  - **quantum computing**
  - **digital audio/video**
  - **radar**
  - **...**
  - **and even in cosmology!**