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Frobenius Manifolds and Integrable Hierarchies

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Lecture 3

Systems of Integrable PDEs

Bihamiltonian systems of n PDEs:

$$u^i_t = A^i_j(u)u^j_x + \epsilon\left(B^i_j(u)u^j_{xx} + \frac{1}{2}C^i_{jk}(u)u^j_x u^k_x\right) + O(\epsilon^2)$$

$$= \{u^i(x), H_1\}_1 = \{u^i(x), H_2\}_2$$

$i = 1, \dots, n$.

The rhs are differential ϵ-polynomials of degree 1,

$$\deg u^{(k)} = k, \quad k \geq 1, \quad \deg \epsilon = -1.$$

Hamiltonians are local functionals

$$H_k[u] = \int h_k(u; u_x, u_{xx}, \ldots; \epsilon)\, dx, \qquad k = 1, 2.$$

Compatible **local** Poisson brackets, i.e.

$$a_1\{\ ,\ \}_1 + a_2\{\ ,\ \}_2$$

is a Poisson bracket for any a_1, a_2

Locality

$$\{u^i(x), u^j(y)\}_{1,2} = \eta^{ij}_{1,2}(u(x))\delta'(x-y) + \Gamma^{ij}_{k\ 1,2}(u)u^k_x\delta(x-y) + O(\epsilon)$$

$$\det \eta^{ij}_{1,2}(u) \neq 0$$

Semisimplicity: characteristic roots of $(A^i_j(u))$ distinct for generic u

$u = (u^1, \ldots, u^n) \in M$ local coordinates (later: M=Frobenius manifold)

Classify wrt the group of Miura-type transformations

$$u^i \mapsto \tilde{u}^i = f_0^i(u) + \epsilon\, f_1^i(u; u_x) + \epsilon^2\, f_2^i(u; u_x, u_{xx}) + O(\epsilon^3)$$

$$\deg f_m^i(u; u_x, \ldots, u^{(m)}) = m, \quad \det\left(\frac{\partial f_0^i(u)}{\partial u^j}\right) \neq 0$$

Remark **One** Poisson bracket

$\{u^i(x), u^j(y)\} = \eta^{ij}(u(x))\delta'(x-y) + \Gamma^{ij}_k(u)u^k_x\delta(x-y) + O(\epsilon)$

$$\det \eta^{ij}(u) \neq 0$$

is equivalent to

$$\{v^\alpha(x), v^\beta(y)\} = \eta^{\alpha\beta}\delta'(x-y), \quad \eta^{\alpha\beta} = \text{const}$$

(B.D., S.Novikov, 1983: $\eta^{ij}(u)$ is a **flat metric**; E.Getzler, 2001: triviality of Poisson cohomology)

Bihamiltonian $\Rightarrow$ commuting Hamiltonians:
find reducing transformation for the **pencil**

$$\{u^i(x), u^j(y)\}_2 - \lambda \{u^i(x), u^j(y)\}_1$$

$$u^i \mapsto v^\alpha = f^\alpha_{-1}(v; v_x, \ldots; \epsilon) + \frac{f^\alpha_0(v; v_x, \ldots; \epsilon)}{\lambda} + \frac{f^\alpha_1(v; v_x, \ldots; \epsilon)}{\lambda^2} + \ldots$$

$$\{v^\alpha(x), v^\beta(y)\} = -\lambda\, \eta^{\alpha\beta}\, \delta'(x-y)$$

Theorem The Hamiltonians

$$F^\alpha_p = \int f^\alpha_p(v; v_x, \ldots; \epsilon), \quad \alpha = 1, \ldots, n, \quad p \geq -1$$

commute

$$\{F^\alpha_p, F^\beta_q\}_{1,2} = 0$$

Proof The functionals

$$F^\alpha(\lambda) := \int v^\alpha \, dx = F^\alpha_{-1} + \frac{F^\alpha_0}{\lambda} + \frac{F^\alpha_1}{\lambda^2} + \dots$$

are Casimirs of the Poisson pencil

$$\{v^\alpha(x), F^\beta(\lambda)\}_2 - \{v^\alpha(x), F^\beta(\lambda)\}_1$$

$$= -\lambda \, \eta^{\alpha\beta} \int \delta'(x-u) dy = 0$$

Hierarchy = commuting bihamiltonian flows on the space of vector functions $u(x) = (u^1(x), \dots, u^n(x))$ of the spatial variable x organized in n infinite chains

$$t = (t^{\alpha,p}), \quad \alpha = 1, \dots, n, \quad p = 0,\, 1,\, 2, \dots$$

$$\frac{\partial u}{\partial t^{\alpha,p}} = \{u(x), H_{\alpha,p}\}_1$$

$$H_{\alpha,p} = \int h_{\alpha,p}(u; u_x, \dots; \epsilon)\, dx$$

$$\left[\frac{\partial}{\partial t^{\alpha,p}}, \frac{\partial}{\partial t^{\beta,q}}\right] = 0$$

$$\frac{\partial}{\partial t^{1,0}} = \frac{\partial}{\partial x}$$

satisfying certain **additional properties**

• triangular bihamiltonian recursion

$$\{\ .\ , H_{\alpha,p-1}\}_2 = \sum_{q\le p} R^{\beta,q}_{\alpha,p}\{\ .\ , H_{\beta,q}\}_1$$

starting from the **Casimirs** of $\{\ ,\ \}_1$

$$H_{\alpha,-1} = \int u_\alpha\, dx, \qquad \{\cdot\,, H_{\alpha,-1}\}_1 \equiv 0$$

(recursion for F^α_p: $\quad \{\ \cdot\ , F^\alpha_{p-1}\}_2 = \{\ \cdot\ , F^\alpha_p\}_1$)

• existence of tau-function.
Hamiltonian densities

$$h_{\alpha,p}(u; u_x, \ldots; \epsilon) = \epsilon^2 \frac{\partial^2 \log \tau}{\partial x \partial t^{\alpha,p+1}}$$

$\Rightarrow$ strange symmetry

$$\frac{\partial h_{\alpha,p-1}}{\partial t^{\beta,q}} = \frac{\partial h_{\beta,q-1}}{\partial t^{\alpha,p}} = \partial_x \Omega_{\alpha,p;\beta,q}(u; u_x, \ldots; \epsilon)$$

Tau-structure: choice of a symmetric matrix
$\Omega_{\alpha,p;\beta,q}(u; u_x, \ldots; \epsilon) \quad = \quad \Omega_{\beta,q;\alpha,p}(u; u_x, \ldots; \epsilon)$
satisfying the above properties

In particular

$$u_\alpha = \epsilon^2 \frac{\partial^2 \log \tau}{\partial x \partial t^{\alpha,0}}, \qquad \alpha = 1, \ldots, n.$$

(recall: $x = t^{1,0}$, $h_{\alpha,-1} = u_\alpha$)

Tau-function $\tau(\mathbf{t}; \epsilon)$ *depends* on the choice of a solution

$$u(x, \mathbf{t}; \epsilon) = u_0(\mathbf{t}) + \epsilon\, u_1(x, \mathbf{t}) + \epsilon^2 u_2(x, \mathbf{t}) + \ldots$$

$$\epsilon^2 \frac{\partial^2 \log \tau}{\partial t^{\alpha,p} \partial t^{\beta,q}} = \Omega_{\alpha,p;\beta,q}(u; u_x, \ldots; \epsilon)$$

Results (B.D., Y.Zhang)

Theorem 1 At $\epsilon = 0$
hierarchies $\leftrightarrow$ semisimple Frobenius manifolds M^n (or their degenerations)

Must also allow the operation of changing the spatial direction

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial t^{1,0}} \mapsto \frac{\partial}{\partial \tilde{x}} = \sum b^i \frac{\partial}{\partial t^{i,0}}$$

$\Rightarrow \frac{n(n-1)}{2}$ parameters* of dispersionless ($=$ no ϵ) integrable hierarchies with n dependent variables.

*moduli $=$ holonomy of the deformed flat connection $\tilde{\nabla}$ on $M^n \times \mathbb{C}^*$

Construction of the **Principal Hierarchy** associated with M (B.D. 1992)

$$\frac{\partial v}{\partial t^{\alpha,p}} = \nabla\theta_{\alpha,p}(v)\cdot v_x = \{v(x), H_{\alpha,p}\}_1$$

$$H_{\alpha,p} = \int \theta_{\alpha,p+1}(v)\,dx$$

Here the functions

$$f_\alpha(v,z) = \sum_{p\geq 0}\theta_{\alpha,p}(v)z^{p+\mu_\alpha}, \qquad \alpha = 1,\dots,n$$

are such that

$$\tilde{\nabla}\, df_\alpha = 0$$

$\mu_1,\mu_2,\dots,\mu_n =$ eigenvalues of $\frac{d-2}{2} - \nabla E$ (here assuming semisimplicity and nonresonancy of ∇E, $\mu_\alpha - \mu_\beta \notin \mathbb{Z}_{>0}$).

(First) Hamiltonian structure

$$\{v^\alpha(x), v^\beta(y)\}_1 = \eta^{\alpha\beta}\delta'(x-y)$$

i.e.

$$\frac{\partial v^\beta}{\partial t^{\alpha,p}} = \partial_x \eta^{\beta\gamma}\frac{\delta H_{\alpha,p}}{\delta v^\gamma(x)} = \partial_x \eta^{\beta\gamma}\frac{\partial \theta_{\alpha,p+1}}{\partial v^\gamma}$$

Proof uses the *horizontality equations* $\tilde{\nabla} df_\alpha = 0$, i.e.

$$\partial_\lambda \partial_\mu \theta_{\alpha,p+1} = c^\nu_{\lambda\mu}(v)\partial_\nu \theta_{\alpha,p}$$

Second Hamiltonian structure

$$\{u^\alpha(x), u^\beta(y)\}_2 = g^{\alpha\beta}(v(x))\delta'(x-y) + \Gamma^{\alpha\beta}_\gamma(v) v^\gamma_x \delta(x-y)$$

with

$$g^{\alpha\beta}(v) := E^\gamma(v) c^{\alpha\beta}_\gamma(v)$$

another flat (outside *discriminant* $\det g = 0$) metric on M

Example Dispersionless KdV

$$u_{t_p} = \frac{u^p}{p!} u_x$$

Here $n = 1$

$$\Omega_{pq} = \frac{u^{p+q+1}}{p!\, q!\, (p+q+1)}$$

$$\frac{\partial \Omega_{pq}}{\partial t^r} = \frac{u^p}{p!} \frac{u^q}{q!} \frac{u^r}{r!} u_x$$

$$\{u(x), u(y)\}_1 = \delta'(x-y)$$
$$\{u(x), u(y)\}_2 = u(x)\delta'(x-y) + \frac{1}{2} u_x \delta(x-y)$$

$$H_p = \int \frac{u^{p+2}}{(p+2)!}\, dx$$

$$\{\ .\ , H_{p-1}\}_2 = \left(p + \frac{1}{2}\right) \{\ .\ , H_p\}_1$$

Solutions $u = u(\mathbf{t})$ given in implicit form

$$\sum_{p \geq 0} (t_p - c_p) \frac{u^p}{p!} = 0$$

arbitrary constants c_p

Tau-function

$$\log \tau = \frac{1}{2} \sum \Omega_{pq}(u(\mathbf{t}))(t_p - c_p)(t_q - c_q)$$

Theorem 1'. The dispersionless limit of any tau-structure $\leftrightarrow$ trivialization of the deformed flat connection on $M \times \mathbb{C}^*$

$$\tilde{\nabla}_a b = \nabla_a b + z\, a \cdot b, \qquad a,\, b \in T_x M$$

$$\tilde{\nabla}_{\frac{d}{dz}} b = \partial_z b + E \cdot b + \frac{\mu}{z} b$$

$$\mu = \frac{d-2}{2} - \nabla E$$

(= *calibrated* Frobenius manifold in the Givental's terminology)

Conversely: from tau-structures to Frobenius manifolds.

$$\Omega_{\alpha,0;\beta,0}|_{\epsilon=0} = \frac{\partial^2 F(u)}{\partial u^\alpha \partial u^\beta}$$

The function $F(u)$ satisfies $WDVV$ $\Rightarrow$ a structure of semisimple Frobenius manifold on the space of fields $(u^1, \ldots, u^n)$

(must also allow the operation of change of the spatial direction

$$\frac{\partial}{\partial t^{1,0}} = \frac{\partial}{\partial x} \mapsto \sum b^\alpha \frac{\partial}{\partial t^{\alpha,0}} = \frac{\partial}{\partial x}$$

for constant b^α)

E.g., for the (extended) Toda the change

$$\frac{\partial}{\partial s_0} \leftrightarrow \frac{\partial}{\partial t_0}$$

corresponds to

Toda $\leftrightarrow$ NLS

Next step: classification of tau-structures with the **given** dispersionless limit
(i.e., associated with a given calibrated semisimple Frobenius manifold)

Deformation theory of bihamiltonian structures:

two differentials ∂_1 and ∂_2 on multivectors (Schouten - Nijenhuis brackets with $\{\ ,\ \}_1$ and $\{\ ,\ \}_2$ resp.)

$$\partial_1^2 = \partial_2^2 = \partial_1\partial_2 + \partial_2\partial_1 = 0$$

In our case both ∂_1 and ∂_2 are acyclic $\Rightarrow$ the space of infinitesimal deformations $H^2(\partial_1, \partial_2)$ equals

$$H^2(\partial_1, \partial_2) = \mathrm{Ker}\,\partial_1\partial_2 \,/\, \mathrm{Im}\,\partial_1 + \mathrm{Im}\,\partial_2$$

(Here $\partial_1\partial_2$ acts on vector fields).

Theorem 2 The deformation space of a *given* Principal Hierarchy with n dependent variables is at most n-dimensional

Construction of the parameters $c_1, \ldots, c_n$ for a bihamiltonian structure

$$\{w^\alpha(x), w^\beta(y)\}_1 = \eta^{\alpha\beta}\delta'(x-y) + \epsilon^2 A_1^{\alpha\beta}(w(x))\delta'''(x-y) + \ldots$$

$$\{w^\alpha(x), w^\beta(y)\}_2 = g^{\alpha\beta}(w(x))\delta'(x-y) + \Gamma^{\alpha\beta}_\gamma(w)w^\gamma_x\delta(x-y)$$

$$+\epsilon^2 A_2^{\alpha\beta}(w(x))\delta'''(x-y) + \ldots$$

• Canonical coordinates $u_1, \ldots, u_n$ on M: roots of

$$\det(g^{\alpha\beta}(v) - \lambda\,\eta^{\alpha\beta}) = 0$$

Main property:

$$\partial/\partial u_i \cdot \partial/\partial u_j = \delta_{ij}\partial/\partial u_i$$

The two metrics are diagonal in the canonical coordinates

$$\eta = \sum_{i=1}^n \eta_{ii}(u)du_i^2, \quad g = \sum_{i=1}^n \eta_{ii}(u)\frac{du_i^2}{u_i}.$$

- Define

$$B_k^{ij}(u) := \frac{\partial u^i}{\partial w^\alpha}\frac{\partial u^j}{\partial w^\beta} A_k^{\alpha\beta}(w), \quad k = 1, 2$$

- Then

$$c_i := \frac{1}{3}\eta_{ii}^2(u)(B_2^{ii}(u) - u_i B_1^{ii}(u)), \ i = 1, \ldots, n.$$

This is the complete set of invariants of the bihamiltonian structure with the given leading term at $\epsilon = 0$.

Example For $n = 1$ all tau-structures are equivalent to KdV

$$u_t = u\, u_x + 2c\, \epsilon^2 u_{xxx}$$

Proof uses results of S.-Q.Liu, Y.Zhang on deformations of bihamiltonian structures and also **Quasitriviality Theorem**

Quasitriviality Theorem: any tau-structure can be obtained from the dispersionless limit by a transformation

$$v_\alpha \mapsto w_\alpha = v_\alpha + \sum_{k\geq 1} \epsilon^k F_\alpha^{[k]}(v; v_x, v_{xx}, \ldots)$$

with some functions $F_\alpha^{[k]}$ **rational** in the derivatives (B.D., S.-Q.Liu, Y.Zhang, math/0410027)

Example 1

Riemann wave $\mapsto$ KdV

$$v_t + v\, v_x = 0 \quad \mapsto \quad w_t + w\, w_x + \frac{\epsilon^2}{12} w_{xxx} = 0$$

The substitution

$$w = v + \frac{\epsilon^2}{24}\partial_x^2 (\log v_x)$$
$$+\epsilon^4 \partial_x^2 \left(\frac{v^{IV}}{1152\, v_x^2} - \frac{7\, v_{xx} v_{xxx}}{1920\, v_x^3} + \frac{v_{xx}^3}{360\, v_x^4} \right) + O(\epsilon^6).$$

Baikov, Gazizov, Ibragimov, 1989

Example 2

Riemann wave $\mapsto$ Camassa-Holm

$$v_t = \frac{3}{2} v\, v_x \;\mapsto\; w_t = (1-\epsilon^2\partial_x^2)^{-1}\left(\frac{3}{2} w\, w_x - \epsilon^2\left[w_x w_{xx} + \frac{1}{2} w\, w_{xxx}\right]\right)$$

$$\begin{aligned} w = v &+ \epsilon^2\,\partial_x\left(\frac{v\, v_{xx}}{3\, v_x} - \frac{v_x}{6}\right) \\ &+\epsilon^4\partial_x\left(\frac{7\, v_{xx}^2}{45\, v_x} + \frac{45\, v\, v_{xx}^3}{16\, v_x^3} - \frac{45\, v^2\, v_{xx}^4}{32\, v_x^5} - \frac{v_{xxx}}{8} - \frac{59\, v\, v_{xx}\, v_{xxx}}{90\, v_x^2}\right. \\ &\quad + \frac{37\, v^2\, v_{xx}^2\, v_{xxx}}{30\, v_x^4} - \frac{7\, v^2\, v_{xxx}^2}{30\, v_x^3} + \frac{5\, v\, v^{IV}}{18\, v_x} \\ &\left. - \frac{31\, v^2\, v_{xx}\, v^{IV}}{90\, v_x^3} + \frac{v^2\, v^V}{18\, v_x^2}\right) + O(\epsilon^6) \end{aligned}$$

Lorenzoni, 2002

From Frobenius Manifolds to Integrable Systems of the Topological Type

Hierarchy of the topological type:

$$c_1 = c_2 = \ldots = c_n = \frac{1}{24}$$

Main goal: for any n construct **universal** integrable hierarchy of the topological type