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Frobenius Manifolds and Integrable Hierarchies

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1. Introduction. First examples
2. Frobenius manifolds: a crash course
3. Integrable hierarchies: the classification project
4. Universal integrable hierarchy of the topological type

Refs:

- [B.D., Y.Zhang](#), Normal forms of hierarchies of integrable PDEs, Frobenius manifolds and Gromov - Witten invariants, math/0108160
- [S.-Q.Liu, Y.Zhang](#), Deformations of semisimple bihamiltonian structures of hydrodynamic type, math/0405146
- [B.D., S.-Q.Liu, Y.Zhang](#), On Hamiltonian perturbations of hyperbolic systems of conservation laws, math/0410027

Frobenius manifolds:

- [B.D.](#), Integrable systems in topological field theory, Nucl. Phys. **B379** (1992), 627–689.
- [B.D.](#), Geometry of 2D topological field theories, in: Integrable Systems and Quantum Groups, Montecatini, Terme, 1993. Editors: M.Francaviglia, S. Greco. Springer Lecture Notes in Math. **1620** (1996), 120–348.
- [B.D.](#), Painlevé transcendent in two-dimensional topological field theory. In: “The Painlevé property: 100 years later”, 287–412, CRM Ser. Math. Phys., Springer, New York, 1999.

Introduction. Historical comments

Frobenius manifolds appear in

- genus zero **Gromov - Witten invariants** of a compact symplectic manifold X (intersection numbers on $\bar{\mathcal{M}}_{g,n}(X, \beta)$, $g = 0$)
- base of universal unfolding of an isolated hypersurface **singularity** ([K.Saito theory of periods of primitive forms](#))

Integrable PDEs: the theory of nonlinear waves

Starting points for

Frobenius manifolds \leftrightarrow integrable systems

- matrix models of 2D gravity \rightarrow KdV

(Brézin, Kazakov; Douglas, Shenker; Gross, Migdal; 1990)

- 1991 - Witten's conjecture:

intersection of ψ -classes on $\bar{\mathcal{M}}_{g,n}$ via KdV

Proved by Kontsevich (other proofs:

Okounkov, Pandharipande; Mirzakhani)

Witten's project (1991):

$$\bar{\mathcal{M}}_{g,n}(X, \beta) \rightarrow \text{some (unknown?) integrable hierarchy}$$

Some contributions:

- B.D. 1992 bihamiltonian structure
- Eguchi et al 1994-95 some examples; 1997-98 Virasoro conjecture (with Hori, Jinzenji, Xiong, and S.Katz)
- B.D., Y.Zhang 1998 - integrable hierarchy at $g \leq 1$ approximation (using Dijkgraaf - Witten $g = 1$ TRR (1990) and Getzler's $\text{GW}_{g=1}$ defining relation (1997); 2001 - general classification project at all genera
- Givental 2001 quantization of quadratic Hamiltonians
- Eliashberg, Givental, Hofer 2000, symplectic field theory

First example: KdV and $\bar{\mathcal{M}}_{g,n}$

KdV hierarchy

$$u_{t_0} = u_x$$

$$u_{t_1} = u u_x + \frac{\epsilon^2}{12} u_{xxx}$$

$$u_{t_2} = \frac{1}{2} u^2 u_x + \frac{\epsilon^2}{12} (2u_x u_{xx} + u u_{xxx}) + \frac{\epsilon^4}{240} u^V$$

...

Commutativity

$$(u_{t_i})_{t_j} = (u_{t_j})_{t_i}$$

$$\Rightarrow u = u(x + t_0, t_1, t_2, \dots; \epsilon)$$

Construction: Lax operator

$$L = \frac{\epsilon^2}{2} \frac{d^2}{dx^2} + u$$

$$\epsilon \frac{\partial L}{\partial t_k} = [A_k, L], \quad k \geq 0$$

$$A_k = \frac{2^{\frac{2k+1}{2}}}{(2k+1)!!} \left(L^{\frac{2k+1}{2}} \right)_+$$

E.g.,

$$A_0 = \epsilon \partial_x, \quad A_1 = \frac{\epsilon^3}{3} \partial_x^3 + \epsilon u \partial_x + \frac{\epsilon}{2} u_x$$

etc.

Solutions

$$u = u_0 + \epsilon^2 u_1 + \epsilon^4 u_2 + \dots$$

Tau-function of the solution

$$\epsilon^2 \partial_x^2 \log \tau = u$$

$$\log \tau = \sum_{g \geq 0} \epsilon^{2g-2} \mathcal{F}_g$$

More precisely,

$$\begin{aligned} \epsilon^2 \partial_{t_p} \partial_{t_q} \log \tau &= \frac{2^{p+q}}{(2p+1)!!(2q+1)!!} \\ &\times \partial_x^{-1} \text{res} \left[\left(L^{\frac{2p+1}{2}} \right)_+, L^{\frac{2q+1}{2}} \right] \end{aligned}$$

Here $\text{res} = \text{coefficient of } \partial_x^{-1}$

Galilean symmetry:

for KdV

$$u_t = u u_x + \frac{\epsilon^2}{12} u_{xxx}$$

the transformation

$$x \mapsto x + ct$$

$$t \mapsto t$$

$$u \mapsto u + c$$

(comoving frame). Action on τ (infinitesimal generator)

$$\delta\tau = L_{-1}\tau$$

$$L_{-1} = \sum_{k \geq 0} t_{k+1} \partial_{t_k} + \frac{1}{2\epsilon^2} t_0^2$$

Part of a representation of the Virasoro algebra

$$L_m = \sum_{ij} \epsilon^2 a_m^{ij} \frac{\partial^2}{\partial t_i \partial t_j} + b_m^{ij} t_i \frac{\partial}{\partial t_j} + \frac{1}{\epsilon^2} c_m^{ij} t_i t_j + \frac{1}{16} \delta_{m,0}$$

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{m^3 - m}{12} \delta_{m+n,0}$$

Vacuum tau-function

$$u = \epsilon^2 \partial_x^2 \log \tau_{\text{KdV}}^{\text{vac}}$$

$$L_m \tau_{\text{KdV}}^{\text{vac}} = 0, \quad m \geq -1$$

has the form

$$\begin{aligned} \tau_{\text{KdV}}^{\text{vac}} &= \frac{1}{(-t_1)^{1/24}} \exp \left\{ \frac{1}{\epsilon^2} \left[-\frac{t_0^3}{6 t_1} - \frac{t_0^4 t_2}{24 t_1^3} + O(t_0^5) \right] \right. \\ &\quad + \left[\frac{t_0 t_2}{24 t_1^2} - \frac{t_0^2 t_3}{48 t_1^3} + \frac{t_0^2 t_2^2}{24 t_1^4} + O(t_0^3) \right] \\ &\quad + \epsilon^2 \left[-\frac{t_4}{1152 t_1^3} + \frac{29 t_2 t_3}{5760 t_1^4} - \frac{7 t_2^3}{1440 t_1^5} + O(t_0) \right] \\ &\quad \left. + O(\epsilon^4) \right\}. \end{aligned}$$

Relationship between vacuum tau-function
and the topological tau-function

Theorem (Kontsevich - Witten)

$$\begin{aligned} \log \tau_{\text{KdV}}^{\text{top}}(t_0, t_1, t_2, \dots) &= \log \tau_{\text{KdV}}^{\text{vac}}(t_0, t_1-1, t_2, \dots) \\ &= \sum_{g \geq 0} \epsilon^{2g-2} \mathcal{F}_g(\mathbf{t}) \end{aligned}$$

where

$$\mathcal{F}_g = \sum \frac{1}{n!} t_{p_1} \dots t_{p_n} \int_{\bar{\mathcal{M}}_{g,n}} \psi_1^{p_1} \wedge \dots \wedge \psi_n^{p_n}$$

Topological KdV tau-function

$$\begin{aligned}
\log \tau_{\text{KdV}}^{\text{top}} = & \frac{1}{\epsilon^2} \left(\frac{t_0^3}{6} + \frac{t_0^3 t_1}{6} + \frac{t_0^3 t_1^2}{6} + \frac{t_0^3 t_1^3}{6} + \frac{t_0^3 t_1^4}{6} + \frac{t_0^4 t_2}{24} + \frac{t_0^4 t_1 t_2}{8} \right. \\
& + \frac{t_0^4 t_1^2 t_2}{4} + \frac{t_0^5 t_2^2}{40} + \frac{t_0^5 t_3}{120} + \frac{t_0^5 t_1 t_3}{30} + \frac{t_0^6 t_4}{720} + \dots \Big) \\
& + \left(\frac{t_1}{24} + \frac{t_1^2}{48} + \frac{t_1^3}{72} + \frac{t_1^4}{96} + \frac{t_0 t_2}{24} + \frac{t_0 t_1 t_2}{12} + \frac{t_0 t_1^2 t_2}{8} + \frac{t_0^2 t_2^2}{24} \right. \\
& \quad \left. + \frac{t_0^2 t_3}{48} + \frac{t_0^2 t_1 t_3}{16} + \frac{t_0^3 t_4}{144} + \dots \right) \\
& + \epsilon^2 \left(\frac{7 t_2^3}{1440} + \frac{7 t_1 t_2^3}{288} + \frac{29 t_2 t_3}{5760} + \frac{29 t_1 t_2 t_3}{1440} + \frac{29 t_1^2 t_2 t_3}{576} + \frac{5 t_0 t_2^2 t_3}{144} \right. \\
& + \frac{29 t_0 t_3^2}{5760} + \frac{29 t_0 t_1 t_3^2}{1152} + \frac{t_4}{1152} + \frac{t_1 t_4}{384} + \frac{t_1^2 t_4}{192} + \frac{t_1^3 t_4}{96} + \frac{11 t_0 t_2 t_4}{1440} \\
& \quad \left. + \frac{11 t_0 t_1 t_2 t_4}{288} + \frac{17 t_0^2 t_3 t_4}{1920} + \dots \right) + O(\epsilon^4).
\end{aligned}$$

(Extended) Toda hierarchy

Difference Lax operator

$$L = \Lambda + v + e^u \Lambda^{-1}, \quad \Lambda = e^{\epsilon \partial_x}$$

$$\begin{aligned}\epsilon \frac{\partial L}{\partial t_k} &= \frac{1}{(k+1)!} [(L^{k+1})_+, L] \\ \epsilon \frac{\partial L}{\partial s_k} &= \frac{2}{k!} \left[(L^k (\log L - c_k))_+, L \right] \\ c_k &= 1 + \frac{1}{2} + \dots + \frac{1}{k}\end{aligned}$$

Construction of $\log L$ (G.Carlet, B.D., Y.Zhang) uses
dressing operators

$$P = \sum_{k \geq 0} p_k \Lambda^{-k}, \quad Q = \sum_{k \geq 0} q_k \Lambda^k, \quad p_0 = 1$$

$$L = P \Lambda P^{-1} = Q \Lambda^{-1} Q^{-1}.$$

Put

$$\log L := \frac{1}{2} (P \epsilon \partial_x P^{-1} - Q \epsilon \partial_x Q^{-1}).$$

Coefficients of $\log L$ are differential ϵ -polynomials in u, v .

E.g.,

$$\epsilon \partial_{t_0} u = v(s_0) - v(s_0 + \epsilon)$$

$$\epsilon \partial_{t_0} v = e^{u(s_0 + \epsilon)} - e^{u(s_0)}.$$

($s_0 = x$ the spatial variable). Eliminating v and defining

$$u_n = u(n\epsilon), \quad t = \frac{t_0}{\epsilon}$$

\Rightarrow the standard **Toda lattice equation**

$$\ddot{u}_n = e^{u_{n-1} - u_n} - e^{u_n - u_{n+1}}.$$

Tau-function defined by

$$u = \log \frac{\tau(s_0 + \epsilon)\tau(s_0 - \epsilon)}{\tau^2(s_0)}$$

$$v = \epsilon \frac{\partial}{\partial t_0} \log \frac{\tau(s_0 + \epsilon)}{\tau(s_0)}.$$

Vacuum tau-function

$$L_{-1} \tau_{\text{Toda}}^{\text{vac}} = 0$$

$$L_{-1} = \sum_{p \geq 0} t_{p+1} \frac{\partial}{\partial t_p} + s_{p+1} \frac{\partial}{\partial s_p} + \frac{s_0 t_0}{\epsilon^2}$$

Theorem (B.D., Y.Zhang, using Getzler; Okounkov, Pandharipande

$$\tau_{\text{Toda}}^{\text{top}} := \tau_{\text{Toda}}^{\text{vac}}(s_0, t_0, s_1 - 1, t_1, s_2, t_2, \dots; \epsilon)$$

is the total GW potential for \mathbf{P}^1

$$\phi_1 = 1 \in H^0(\mathbf{P}^1), \phi_2 \in H^2(\mathbf{P}^1), \int_{\mathbf{P}^1} \phi_2 = 1$$

$$\log \tau_{\text{Toda}}^{\text{top}}(s_0, t_0, s_1, t_1, \dots; \epsilon^2)$$

$$\begin{aligned} &= \sum_{g \geq 0} \epsilon^{2g-2} \mathcal{F}_g \\ \mathcal{F}_g &= \sum \frac{1}{n!} t_{\alpha_1, p_1} \dots t_{\alpha_n, p_n} \\ &\times \int_{[\bar{\mathcal{M}}_{g,n}(\mathbf{P}^1, \beta)]} \text{ev}_1^* \phi_{\alpha_1} \wedge \psi_1^{p_1} \wedge \dots \wedge \text{ev}_n^* \phi_{\alpha_n} \wedge \psi_n^{p_n} \end{aligned}$$

$$t_{1,p} = s_p, \quad t_{2,p} = t_p$$

Here

$$\mathcal{M}_{g,n}(\mathbf{P}^1, \beta) = \{ f : (C_g, x_1, \dots, x_n) \rightarrow \mathbf{P}^1$$

$\beta = \text{degree of the map } f \}$

$$\begin{aligned}
\mathcal{F}_0 = & \frac{s_0^2 t_0}{2!} + \frac{s_0^2 s_1 t_0}{2!} + \frac{s_0^3 t_1}{3!} + \frac{s_0^3 s_2 t_0}{3!} + \frac{s_0^4 t_2}{4!} + \frac{s_0^2 s_1^2 t_0}{2!} \\
& + 2 \frac{s_0^3 s_1 t_1}{3!} + \frac{s_0^4 s_3 t_0}{4!} + \frac{s_0^5 t_3}{5!} + 3 \frac{s_0^3 s_1 s_2 t_0}{3!} + 3 \frac{s_0^4 s_2 t_1}{4!} + 3 \frac{s_0^4 s_1 t_2}{4!} \\
& + e^{t_0} \left[1 - 2 s_1 + 2 \frac{s_1^2}{2!} - 2 s_0 s_2 + s_1 t_0 + s_0 t_1 - 2 \frac{s_0^2 s_3}{2!} \right. \\
& \quad - 2 \frac{s_1^2 t_0}{2!} + s_0 s_2 t_0 + \frac{s_0^2 t_2}{2!} - 2 \frac{s_0^2 s_1 s_3}{2!} - s_0 s_1 s_2 t_0 + \frac{s_0^2 s_3 t_0}{2!} \\
& \quad + 2 \frac{s_1^2 t_0^2}{(2!)^2} - \frac{s_0^2 s_2 t_1}{2!} + s_0 s_1 t_0 t_1 + \frac{s_0^2 t_1^2}{2!} + \frac{s_0^2 s_1 t_2}{2!} + \frac{s_0^3 t_3}{3!} \Big] \\
& + e^{2t_0} \left[-\frac{3}{4} s_3 + \frac{1}{4} t_2 + \frac{5}{4} \frac{s_2^2}{2!} + \frac{3}{4} s_1 s_3 + \frac{1}{4} s_3 t_0 - \frac{3}{4} s_2 t_1 + \frac{1}{2} \frac{t_1^2}{2!} \right. \\
& \quad - \frac{1}{4} s_1 t_2 + \frac{1}{4} s_0 t_3 + 2 s_0 s_2 s_3 - \frac{3}{2} \frac{s_2^2 t_0}{2!} - \frac{3}{2} s_1 s_3 t_0 - 2 s_0 s_3 t_1 \\
& \quad + \frac{1}{2} s_2 t_0 t_1 - s_0 s_2 t_2 + \frac{1}{2} s_1 t_0 t_2 + s_0 t_1 t_2 + 2 s_0 s_1 s_2 s_3 + 4 \frac{s_0^2 s_3^2}{(2!)^2} \\
& \quad + \frac{s_1 s_2^2 t_0}{2!} - 3 s_0 s_2 s_3 t_0 + \frac{s_2^2 t_0^2}{(2!)^2} + \frac{s_1 s_3 t_0^2}{2!} + \frac{s_0 s_2^2 t_1}{2!} \\
& \quad - 2 s_0 s_1 s_3 t_1 - s_1 s_2 t_0 t_1 + s_0 s_3 t_0 t_1 - 2 \frac{s_0 s_2 t_1^2}{2!} + \frac{s_1 t_0 t_1^2}{2!} \\
& \quad + 3 \frac{s_0 t_1^3}{3!} - s_0 s_1 s_2 t_2 - 3 \frac{s_0^2 s_3 t_2}{2!} + s_0 s_2 t_0 t_2 + s_0 s_1 t_1 t_2 \\
& \quad + 2 \frac{s_0^2 t_2^2}{(2!)^2} - \frac{s_0^2 s_2 t_3}{2!} + \frac{s_0 s_1 t_0 t_3}{2} + \frac{3 s_0^2 t_1 t_3}{2!} \Big] \\
& + e^{3t_0} \left[\frac{50 s_3^2}{27 2!} - \frac{7}{9} s_3 t_2 + \frac{1}{3} \frac{t_2^2}{2!} - \frac{2}{9} s_2 t_3 + \frac{1}{6} t_1 t_3 - 2 \frac{s_2^2 s_3}{2!} - \frac{14}{9} \frac{s_3^2 t_0}{2!} \right. \\
& \quad + 2 s_2 s_3 t_1 - 2 \frac{s_3 t_1^2}{2!} + \frac{s_2^2 t_2}{2!} + \frac{1}{3} s_3 t_0 t_2 - s_2 t_1 t_2 + \frac{t_1^2 t_2}{2!} - s_0 s_3 t_3 \\
& \quad + \frac{1}{6} s_2 t_0 t_3 + \frac{1}{2} s_0 t_2 t_3 - 4 \frac{s_0 s_2 s_3^2}{2!} + 5 \frac{s_2^2 s_3 t_0}{2!} + 4 \frac{s_1 s_3^2 t_0}{2!} + \frac{2 s_3^2 t_0^2}{3 (2!)^2} \Big]
\end{aligned}$$

$$\begin{aligned}
& +8 \frac{s_0 s_3^2 t_1}{2!} - 3 s_2 s_3 t_0 t_1 + \frac{s_3 t_0 t_1^2}{2!} + 3 s_0 s_2 s_3 t_2 - 2 \frac{s_2^2 t_0 t_2}{2!} \\
& - 2 s_1 s_3 t_0 t_2 - 5 s_0 s_3 t_1 t_2 + s_2 t_0 t_1 t_2 - 2 \frac{s_0 s_2 t_2^2}{2!} + \frac{s_1 t_0 t_2^2}{2!} \\
& + 3 \frac{s_0 t_1 t_2^2}{2!} + \frac{s_0 s_2^2 t_3}{2!} - s_0 s_1 s_3 t_3 - \frac{1}{2} s_1 s_2 t_0 t_3 + \frac{1}{2} s_0 s_3 t_0 t_3 \\
& - \frac{3}{2} s_0 s_2 t_1 t_3 + \frac{1}{2} s_1 t_0 t_1 t_3 + 2 \frac{s_0 t_1^2 t_3}{2!} + \frac{1}{2} s_0 s_1 t_2 t_3 + \frac{s_0^2 t_3^2}{(2!)^2} \Big] + \dots
\end{aligned}$$

$$\begin{aligned}
\mathcal{F}_1 = & \frac{s_1}{12} + \frac{s_1^2}{24} - \frac{s_1^3}{18} + \frac{s_0 s_2}{12} + \frac{s_0 s_1 s_2}{6} + \frac{s_0^2 s_3}{24} - \frac{t_0}{24} - \frac{s_1 t_0}{24} - \frac{s_1^2 t_0}{24} \\
& - \frac{s_0 s_2 t_0}{24} - \frac{s_0 s_1 s_2 t_0}{8} - \frac{s_0^2 s_3 t_0}{48} \\
& - \frac{s_0 t_1}{24} - \frac{s_0 s_1 t_1}{12} - \frac{s_0^2 s_2 t_1}{16} - \frac{s_0^2 t_2}{48} - \frac{s_0^2 s_1 t_2}{16} - \frac{s_0^3 t_3}{144} \\
& - \frac{1}{96} e^{t_0} \left[-4 s_3 t_0 + 8 s_1 s_3 t_0 + 4 s_2^2 t_0^2 + 4 s_2 t_1 - 8 s_1 s_2 t_1 + 8 s_2 t_0 t_1 \right. \\
& + 24 s_1 s_2 t_0 t_1 + 8 s_0 s_3 t_0 t_1 + 8 s_1 s_2 t_0^2 t_1 + 16 s_1 t_1^2 - 24 s_1^2 t_1^2 \\
& + 24 s_0 s_2 t_1^2 + 4 s_1 t_0 t_1^2 + 12 s_1^2 t_0 t_1^2 + 12 s_0 s_2 t_0 t_1^2 + s_1^2 t_0^2 t_1^2 \\
& + 4 s_0 t_1^3 + 16 s_0 s_1 t_1^3 + 2 s_0 s_1 t_0 t_1^3 + s_0^2 t_1^4 - 4 t_2 + 4 s_1 t_2 + 8 s_1^2 t_2 \\
& + 8 s_0 s_2 t_2 + 8 s_0 s_2 t_0 t_2 + 8 s_0 t_1 t_2 + 24 s_0 s_1 t_1 t_2 + 8 s_0 s_1 t_0 t_1 t_2 \\
& \left. + 10 s_0^2 t_1^2 t_2 + 4 s_0^2 t_2^2 - 4 s_0 t_3 + 8 s_0 s_1 t_3 + 4 s_0^2 t_1 t_3 \right] \\
& + \frac{1}{48} e^{2t_0} \left[4 s_3 t_1^2 - 2 s_3 t_0 t_1^2 + 2 s_2 t_1^3 + 4 s_2 t_0 t_1^3 + 16 s_1 t_1^4 + 3 s_1 t_0 t_1^4 \right. \\
& + 3 s_0 t_1^5 - 14 s_3 t_2 - 4 s_3 t_0 t_2 + 4 s_2 t_1 t_2 + 4 s_2 t_0 t_1 t_2 - 2 t_1^2 t_2 + 18 s_1 t_1^2 \\
& + 3 s_1 t_0 t_1^2 t_2 + 7 s_0 t_1^3 t_2 + 4 t_2^2 - 8 s_1 t_2^2 + 4 s_0 t_1 t_2^2 - 6 s_2 t_3 - 4 s_2 t_0 t_3 \\
& \left. + 5 t_1 t_3 - 12 s_1 t_1 t_3 - 4 s_1 t_0 t_1 t_3 - 6 s_0 t_1^2 t_3 - 8 s_0 t_2 t_3 \right] \\
& + \frac{e^{3t_0}}{288} \left[29 t_1^6 + 66 t_1^4 t_2 + 33 t_1^2 t_2^2 - 40 t_2^3 + 6 t_1^3 t_3 - 48 t_1 t_2 t_3 + 18 t_3^2 \right] \\
& + \dots
\end{aligned}$$

$$\mathcal{F}_2 = \frac{7 t_2}{5760} + \frac{7 s_1 t_2}{1920} + \frac{7 s_0 t_3}{5760} + e^{t_0} \left(\frac{29 t_1^4}{11520} + \frac{t_1^2 t_2}{80} + \frac{13 t_2^2}{2880} + \frac{7 t_1 t_3}{1440} \right) .$$

Small phase space:

$$t_p = s_p = 0, \quad p > 0$$

$$(\mathcal{F}_0)_{\text{small}} = \frac{1}{2} s_0^2 t_0 + e^{t_0}.$$

\Rightarrow Frobenius manifold

Less known examples

Example 1. Let us consider $\tau_{\text{KdV}}^{\text{vac}}$ as a functional on $z^{-1}\mathbb{C}[[1/z]]$

E.g.,

$$\tau_{\text{KdV}}^{\text{top}} = \tau_{\text{KdV}}^{\text{vac}}(t(z) - z^{-2})$$

$$t(z) := \frac{t_0}{z} + \frac{t_1}{z^2} + \dots \in \frac{1}{z}\mathbb{C}[[\frac{1}{z}]]$$

Denote

$$f(v) := v^{\frac{1}{2}} J_1(2\sqrt{v}) = \sum_{m=0}^{\infty} (-1)^m \frac{v^{m+1}}{m!(m+1)!}$$

Theorem (Kaufmann, Manin, Zagier; Zograf; Manin, Zograf)

$$\begin{aligned} & \log \tau_{\text{KdV}}^{\text{vac}} \left(\frac{1}{z} (x - f(1/z)); \epsilon^2 \right) \\ &= \sum_{g \geq 0} \left(\frac{\epsilon}{\pi^3} \right)^{2g-2} \sum_n Vol(\mathcal{M}_{g,n}) \left(\frac{x}{\pi^2} \right)^n \end{aligned}$$

where $Vol(\mathcal{M}_{g,n})$ is the Weil - Petersson volume of $\mathcal{M}_{g,n}$.

Example 2. Toda vacuum tau-function and enumeration of fat graphs/triangulations

Theorem (B.D., T.Grava) Substituting

$$\tau_{\text{Toda}}^{\text{vac}}(t_0, t_1, t_2, \dots; s_0, s_1, s_2, \dots; \epsilon)$$

$$\begin{aligned} t_0 &= 0, \quad \textcolor{blue}{t_1 = -1}, \quad t_k = (k+1)! \lambda_{k+1}, \quad k \geq 2 \\ s_0 &= x, \quad s_k = 0, \quad k \geq 1 \end{aligned}$$

one obtains

$$F := \log \tau_{\text{Toda}}^{\text{vac}}(0, -1, 3! \lambda_3, 4! \lambda_4, \dots; x, 0, \dots; \epsilon)$$

$$\begin{aligned} &= \frac{x^2}{2\epsilon^2} \left(\log x - \frac{3}{2} \right) - \frac{1}{12} \log x + \sum_{g \geq 2} \left(\frac{\epsilon}{x} \right)^{2g-2} \frac{B_{2g}}{2g(2g-2)} \\ &\quad + \sum_{g \geq 0} \epsilon^{2g-2} F_g(x; \lambda_3, \lambda_4, \dots) \end{aligned}$$

$$F_g(x; \lambda_3, \lambda_4, \dots)$$

$$= \sum_n \sum_{k_1, \dots, k_n} a_g(k_1, \dots, k_n) \lambda_{k_1} \dots \lambda_{k_n} x^h,$$

$$h = 2 - 2g - \left(n - \frac{|k|}{2} \right), \quad |k| = k_1 + \dots + k_n,$$

and

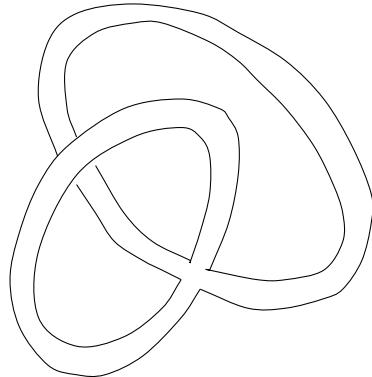
$$a_g(k_1, \dots, k_n) = \sum_{\Gamma} \frac{1}{\# \text{Sym } \Gamma}$$

where

Γ = a connected **fat graph** of genus g

with n vertices of the valencies k_1, \dots, k_n .

E.g.: genus 1, one vertex, valency 4



$$\begin{aligned}
F = \epsilon^{-2} & \left[\frac{1}{2}x^2 \left(\log x - \frac{3}{2} \right) + 6x^3\lambda_3^2 + 2x^3\lambda_4 + 216x^4\lambda_3^2\lambda_4 + 18x^4\lambda_4^2 \right. \\
& + 288x^5\lambda_4^3 + 45x^4\lambda_3\lambda_5 + 2160x^5\lambda_3\lambda_4\lambda_5 + 90x^5\lambda_5^2 + 5400x^6\lambda_4\lambda_5^2 + 5x^4\lambda_6 \\
& + 1080x^5\lambda_3^2\lambda_6 + 144x^5\lambda_4\lambda_6 + 4320x^6\lambda_4^2\lambda_6 + 10800x^6\lambda_3\lambda_5\lambda_6 + 27000x^7\lambda_5^2\lambda_6 \\
& \quad \left. + 300x^6\lambda_6^2 + 21600x^7\lambda_4\lambda_6^2 + 36000x^8\lambda_6^3 \right] \\
& - \frac{1}{12} \log x + \frac{3}{2}x\lambda_3^2 + x\lambda_4 + 234x^2\lambda_3^2\lambda_4 + 30x^2\lambda_4^2 + 1056x^3\lambda_4^3 + 60x^2\lambda_3\lambda_5 \\
& + 6480x^3\lambda_3\lambda_4\lambda_5 + 300x^3\lambda_5^2 + 32400x^4\lambda_4\lambda_5^2 + 10x^2\lambda_6 + 3330x^3\lambda_3^2\lambda_6 \\
& + 600x^3\lambda_4\lambda_6 + 31680x^4\lambda_4^2\lambda_6 + 66600x^4\lambda_3\lambda_5\lambda_6 + 283500x^5\lambda_5^2\lambda_6 \\
& \quad + 2400x^4\lambda_6^2 + 270000x^5\lambda_4\lambda_6^2 + 696000x^6\lambda_6^3 \\
& + \epsilon^2 \left[-\frac{1}{240x^2} + 240x\lambda_4^3 + 1440x\lambda_3\lambda_4\lambda_5 + \frac{1}{2}165x\lambda_5^2 + 28350x^2\lambda_4\lambda_5^2 \right. \\
& + 675x\lambda_3^2\lambda_6 + 156x\lambda_4\lambda_6 + 28080x^2\lambda_4^2\lambda_6 + 56160x^2\lambda_3\lambda_5\lambda_6 + 580950x^3\lambda_5^2\lambda_6 \\
& \quad \left. + 2385x^2\lambda_6^2 + 580680x^3\lambda_4\lambda_6^2 + 2881800x^4\lambda_6^3 \right] + \dots
\end{aligned}$$

Proof uses Toda equations for the Hermitian matrix integral ('t Hooft; D.Bessis, C.Itzykson, J.-B.Zuber)

$$Z_N(\lambda; \epsilon) = \frac{1}{\text{Vol}(U_N)} \int_{N \times N} e^{-\frac{1}{\epsilon} \text{Tr } V(A)} dA$$

$$V(A) = \frac{1}{2} A^2 - \sum_{k \geq 3} \lambda_k A^k$$

where one has to replace

$$N \mapsto \frac{x}{\epsilon}$$

Remark This is the topological solution for the (extended) nonlinear Schrödinger hierarchy