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Set-Indexed Martingales: Tools for Multidimensional Stochastic Modelling and Analysis

Dean Slonowsky

Department of Statistics
University of Manitoba
Winnipeg, Manitoba, Canada

In this presentation:

– Introduce the notion of *set-indexed martingales*,

– Describe (potential) theoretical tools for modelling complex random spatio-temporal processes:
(1) Central Limit Theorems
(2) Set-Indexed Ito integral
(3) Set-Indexed Stopping Lines

Set-Indexed Processes

A *set-indexed process* is any collection

$$X = \{X_A\,;A \in \mathcal{A}\}$$

of random variables. Roughly,

$X_A =$ observation over the region A

$\mathcal{A}$ is a collection of "good" subsets of T, a fixed space (eg. $T \subseteq \mathbb{R}^d$).

Simplest example:

$T = [0, \infty)^d$ with the *lower rectangles*,

$$\mathcal{A} = \{[0, z] \, ; z \in T\} \, .$$

Identifying points, z with sets, $[0, z]$, we can capture:

(1) "Time-indexed" processes:

$$X = \{X_t \, ; t \in [0, \infty)\}$$

(2) "planar" processes:

$$X = \{X_{s,t} \, ; s, t \in [0, \infty)\}$$

Other “good” examples on $T = [0, \infty)^d$**:**

(convex) lower layers:

Early set-indexed work in 1980's: Empirical Processes
(K. Alexander, R. Dudley, R. Pyke, *et al.*)

$T = \mathbb{R}^d$, any $\mathcal{A} \subseteq \mathcal{B}(\mathbb{R}^d)$

Take an i.i.d. sample from dist'n,

$$F : \mathbb{R}^d \to [0,1]$$

$Y_1, Y_2, \cdots, Y_n \sim F$ and define:

$$\begin{aligned} X(A) &= n^{-1} \Sigma_{i=1}^{n} \mathbb{1}_{[Y_i \in A]} \\ &= n^{-1} \cdot (\#\text{ of } i \text{ with } \vec{Y_i} \in \mathcal{A}) \end{aligned}$$

Set-Indexed Martingales

Need additional structure on $\mathcal{A}$:

- A, B non-empty $\Rightarrow A \cap B \neq \emptyset$,
- $\exists\, g_n : \mathcal{A} \to \mathcal{A}_n$ ($\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \cdots$) s.t. $g_n(A) \downarrow A$ every $A \in \mathcal{A}$

Define collection of "rectangles",

$$\mathcal{C} \;=\; \{A \setminus \cup_{i=1}^{k} A_i \,;\, A, A_i \in \mathcal{A}\}$$

Can extend X to $\mathcal{C}$ ("increments"):

$$X_C = X_A - \sum_i X_{A \cap A_i} + \cdots$$

(i.e., inclusion-exclusion).

Also need an information structure, "filtration", $\{\mathcal{F}_A \,;\, A \in \mathcal{A}\}$. Roughly:

$$\mathcal{F}_A \;=\; \text{“info. up to the set } A\text{”}$$
$$(=\, \sigma\{X_B \,;\, B \subseteq A\}\,)$$

Definition $X = \{X(A)\ A \in \mathcal{A}\}$ is

(a) *additive* if X has unique additive extension to $\mathcal{C}$,

(b) *adapted* if X_A is $\mathcal{F}_A$-measurable for each $A \in \mathcal{A}$

(c) a *strong martingale* if X is additive, adapted and, given $C \in \mathcal{C}$,

$$E[X_C | \mathcal{G}^*_C] \;=\; 0$$

i.e., $E[$incr. over $C|$history of $C] = 0$.

$$\mathcal{G}^*_C \;=\; \vee\{\mathcal{F}_A\,; A \in \mathcal{A}, A \cap C = \emptyset\}$$

Examples of Strong Martingales:

- Weighted empirical process (Burke 98 for rectangles; Ivanoff and S., 99)

  $$X_n(A) = n^{-1} \sum_{i=1}^{n} Z_i \mathbb{1}_{[Y_i \in A]}$$

  (Z_i's are i.i.d. zero-mean weights).
- $X - \Lambda$, when X is a spatial ($\mathbb{R}^d$) Poisson process with intesity Λ (Ivanoff and Merzbach, 94)

Set-indexed Brownian motion
(Ivanoff and Merzbach, 96)

$B = \{B_A : A \in \mathcal{A}\}$ is a <u>set-indexed Brownian motion</u> if

$$(B_{C_1}, B_{C_2}, \cdots, B_{C_n}) \sim \mathbf{N}(\mathbf{0}, \mathbf{\Sigma})$$

for any $C_1, \cdots, C_n \in \mathcal{C}$ where $\mathbf{\Sigma}_{i,j} = \Lambda(C_i \cap C_j)$, Λ a measure on T.

Notes (1) B has "independent increments".

(2) If B has "outer-continuous sample paths", B a strong martingale with respect to its minimal filtration.

Contact infection model
(Dabrowski, Fang, and Ivanoff, 96).

A tree is growing at each point (n, m) in a forest. A disease spreads through the forest. Define:

$$X(n,m) = \begin{cases} 1 & \text{tree at } (n,m) \text{ is infected} \\ 0 & \text{tree at } (n,m) \text{ is healthy} . \end{cases}$$

Assumption: Health of tree at (n, m) depends (cond.) only on the health of trees at $(n-1, m)$ and $(n, m-1)$.

$$X(A) = \sum_{(n,m)\in A} X(n,m) \, , \quad A \in \mathcal{A}$$

where $\mathcal{A} = \{$ lower rectangles in $[0,\infty)^2\}$, or $\{$ lower layers in $[0,\infty)^2\}$.
When properly "centered", X a strong martingale with respect to its minimal filtration.

1. Central Limit Theorems

Ivanoff and S., 1999
Appeared in: Ivanoff and Merzbach 2000, *Set-Indexed Martingales*, Monographs on Statistics and Applied Probability **85**, Chapman & Hall / CRC.

Conditions on sequence (X_n) of set-indexed martingales ensuring

$$X_n \to B$$

"in distibution", where B is a set-indexed Brownian motion.

Moral: We can approximate X_n's with "simpler" process, B.

First ... need set-indexed analogue of *quadratic variation*.

Definition (S. 98).
A process $X^* = \{X^*(A) : A \in \mathcal{A}\}$ is a *-quadratic variation of a strong martingale $X = \{X(A) : A \in \mathcal{A}\}$ if

$$E[X(C)^2 - X^*(C) \,|\, \mathcal{G}_C^*] = 0 \text{ each } C \in \mathcal{C}$$

Notes (a) Since $X(C)^2 \neq X^2(C)$, $X^2 - X^*$ is not necessarily a strong martingale, even if X^* is adapted.

(b) Conditions for existence and uniquness of "predictable" *-quadratic variation in S. 98.

Example If $B = \{B(A) : A \in \mathcal{A}\}$ is an $\mathcal{A}$-indexed Brownian motion with variance measure Λ, then Λ is a *-quadratic variation of B.

Example (Ivanoff and S., 99).
A *-quadratic variation for weighted empirical process,
$U_n = \frac{1}{n^{1/2}} \Sigma_{k=1}^{n} Z_k \cdot \mathbb{1}[\vec{Y}_k \in A]$,

$$U_n^*(A) = \frac{1}{n} \sum_{k=1}^{n} \int_{R(\vec{Y}_k) \cap A} h_F(t)\, dF(t)$$

where $R(\vec{Y}_k) = \Pi_{i=1}^{d}[0, Y_k^i]$ and $h_F(t) = F(\Pi_{i=1}^{d}[t_i, 1])^{-1}$.

Theorem (S. 98).
If $X_1, X_2, \cdots$ are square-integrable strong martingales with corresponding *-quadratic variations $X_1^*, X_2^*, \cdots$ such that (i)

$$E[\,|X_n(T)|^{2+\delta}\,] \leq K < \infty$$

for some $\delta > 0$ and (ii) "jumps of X_n become asymptotically negligible as $n \to \infty$ and (iii) there is increasing continuous $\Lambda : \mathcal{A} \to [0, \infty)$ such that

$$X_n^*(A) \xrightarrow{P} \Lambda(A) \quad \text{as } n \to \infty$$

for each $A \in \mathcal{A}$, then $X_n \to_D B$ where B is set-indexed Brownian motion with variance measure Λ.

Asymptotic rarefaction of jumps

$$\max_{0 \le t \le 1} |\,\Delta\, X_n(f(t))\,| \xrightarrow{P} 0 \quad \text{as } n \to \infty$$

for every "flow" $f : [0,1] \to \mathcal{A}(u)$ where $\mathcal{A}(u) = \{$finite unions in $\mathcal{A}\}$.

Definition of "$X_n \to_D X$" for set-indexed process.

Either:

(a) <u>semi-functional convergence</u>
$f(X_n) \to_D f(X)$ on $D[0,1]$ for every "flow", $f : [0,1] \to \mathcal{A}(u)$, or

(b) <u>functional convergence</u>
measures on $D(\mathcal{A})$ induced by X_n's converge weakly to that induced on $D(\mathcal{A})$ by X.
Requires restrictions on "size" of $\mathcal{A}$.

2. The Set-Indexed Ito Integral

Saada & S., 2004,
in *Journal d'Analyse Mathematique*

Generalizes the classic notion of the Ito stochastic integral to the set-indexed setting.

Recall: classic situation:
Given a "time-indexed" martingale,

$$M = \{M_t\,; t \in [0, \infty)\}$$

and a "predictable" process,

$$X = \{X_t\,; t \in [0, \infty)\},$$

can define the *Ito integral* by

$$\int_0^t X_s\, dM_s = \lim_n \sum_{k=1}^{[t2^n]} X_{\frac{k-1}{2^n}} \left(M_{\frac{k}{2^n}} - M_{\frac{k-1}{2^n}} \right)$$

("random" Riemann-Stieltjes integral)

Set-indexed situation
Motivated by planar Ito integral (Cairoli and Walsh, 75).

Integrators: strong martingales.

Integrands: Predictable processes?

Def'n. *Predictable rectangles,*

$$\mathcal{P}_0 = \{F \times [C]\,; C \in \mathcal{C}\,, F \in \mathcal{G}_C\}$$

where, given $C = A \setminus \cup_{j=1}^{k} A_i \in \mathcal{C}$,

$$[C] = [\emptyset, A] \setminus \bigcup_{i=1}^{n} [\emptyset, A_i]$$

with $[\emptyset, A] = \{B \in \mathcal{A}\,; B \subseteq A\}$ and

$$\mathcal{G}_C = \cap\{\mathcal{F}_A\,; A \in \mathcal{A}, A \cap C \neq \emptyset\}$$

(*weak history* at C).

That is: $F \times [C] \subseteq \Omega \times \mathcal{A}$ in $\mathcal{P}_0$ iff F is in weak history of rectangle C. (Close to that in: Ivanoff *et al.* 93)

To illustrate: If $\mathcal{A} = \{[0, z]\,; z \in [0, \infty)^d\}$, then identifying z's with $[0, z]$'s:

$$F \times [C] \;\equiv\; F \times (z', z]$$

$d = 1$ (time-ind.) Reduces to:

$d = 2$ (planar) Reduces to:

Def'n. (i) The $\mathcal{A}$-*predictable* σ-*algebra*

$$\mathcal{P} = \sigma(\mathcal{P}_0) \quad \text{on} \quad \Omega \times \mathcal{A}\,.$$

(ii) X is $\mathcal{A}$-predictable if it's $\mathcal{P}$-meas. as a map on $\Omega \times \mathcal{A}$.

Therefore, ... every "simple process",

$$X = \sum_{i=1}^{n} \alpha_i \mathbb{1}_{F \times [C_i]} \quad (F_i \in \mathcal{G}_{C_i}, \text{ each } i)$$

is $\mathcal{A}$-predictable.

Q: Other predictable processes?

Def'n (Analogue of "left-continuous").
$x : \mathcal{A} \to I\!R$ is *outer-continuous* at $A \in \mathcal{A}$ if, given (A_n) in $\mathcal{A}$,

$$A_n \subseteq A \text{ and } A_n \to_{d_H} A$$

implies $\lim_n x(A_n) = x(A)$.

Theorem (Saada and S., 2004)
Every inner-continuous, adapted process is $\mathcal{A}$-predictable.

Definition of set-indexed integral:
If M a strong martingale in L^2 and

$$X = \sum_{i=1}^{n} \alpha_i \mathbb{1}_{F_i \times [C_i]}$$

is simple and $\mathcal{A}$-predictable,

$$\int_{[\emptyset, A]} X_B \, dM_B \;=\; \sum_{i=1}^{n} \alpha_i \mathbb{1}_{F_i} M(C_i \cap A)$$

Bridge to more general integrands: *the set-indexed Ito isometry*. Employs:

Def'n. $Q = \{Q_A\,;\, A \in \mathcal{A}\}$ a *quadratic variation* of M if for every $C \in \mathcal{C}$,

(i) $Q_C \geq 0$ and

(ii) $E[(M_C)^2 | \mathcal{G}_C] = E[Q_C \,| \mathcal{G}_C]$

Note: $\mathcal{G}_C \subseteq \mathcal{G}_C^*$, hence *-quadratic variation is a quadratic variation.

Theorem (Saada and S., 2004)
Given any simple process X,

$$E[(\int_{[\emptyset,A]} X\, dM)^2] = E[\int_{[\emptyset,A]} X^2\, dQ],$$

where Q a quadratic variation of M.

Therefore, can extend integral to any $\mathcal{A}$-predictable X for which

$$||X||_M = E[\int_{[\emptyset,T]} X^2\, dQ] < \infty$$

Define:

$$L^2_M = \{X \in \mathcal{P} : ||X||_M < \infty\}$$

Note that

$$\int X\, dM = \{\int_{[\emptyset,A]} X_B\, dM_B\,; A \in \mathcal{A}\}$$

is a set-indexed process.

Properties of set-indexed integral:

(1) $\int X\, dM$ is a strong martingale,

(2) If Q a quadratic variation of M, then $\int X^2\, dQ$ a quadratic variation of $\int X\, dM$

(3) ξ a *stopping set* (in sense of Ivanoff and Merzbach, 95), then

$$\left(\int X\, dM\right)^{\xi} = \int \mathbb{1}_{[0,\xi]} X\, dM^{\xi}$$

(the key to localization.)

3. Set-indexed Stopping Lines

Saada and S., 2005.
To appear in *J. of Theoretical Probab.*

Provides means of “stopping” a set-indexed process in a flexible and general way.

Extends and compliments the notion of “stopping sets”, ξ, introduced in Ivanoff and Merzbach, 95.

Motivation: In the plane ...
(see Merzbach, 80; Merzbach and Zakai, 87).

Two notions of stopping.
(a) Stopping point: $S : \Omega \to I\!R^2_+$.

(b) Stopping line: $\lambda : \Omega \to L$ where L denotes set of all "decreasing lines" in $I\!R^2_+$.

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