Local Likelihood Point Process Intensity Estimation

(Joint work with Jennifer Asimit)

Outline

- **Point process intensity**
- **Point process likelihood**
- **Local point process likelihood**
- **Application: Local Constant Intensity for Ontario 1990-2003.**
- **Future work**

Point process intensity $\lambda(t; \mathcal{H}_t)$

- **e.g. Homogeneous Poisson process $N(t)$ (with rate λ):**

$$E[N(t)] = \int_0^t \lambda du = \lambda t$$

- **e.g. Inhomogeneous Poisson $N(t)$ (with time-varying rate $\lambda(t)$):**

$$E[N(t)] = \int_0^t \lambda(u) du$$

- **Alternate interpretation:**

$$\lambda(u) = \lim_{h \to 0} \frac{1}{h} P(N(u+h) - N(u) > 0)$$

- **For non-Poisson processes, the intensity will depend on the past $\mathcal{H}_t$.**

Point Process Likelihood

- **Consider observed points**
  $0 < b_1 < b_2 < \cdots < b_{n-1} < b_n < T$

- **The point process log-likelihood is given by**

$$L(\lambda) = \sum_j \log\left(\lambda(b_j)\right) - \int_0^T \lambda(u)du.$$

Local Point Process Likelihood

- Loader (1996) and Hjort and Jones (1996) have defined a local likelihood which can be used for, among other things, estimating densities.

  Advantages of this approach:

  1. Local linear (and other adjustments) are possible.

  2. Covariates (usually factors) can be included

- We imitate Loader (1996) and define a version of local point process likelihood as

$$L(\lambda_N, x) = \sum_j \log\left(\lambda_N(N_j)\right) K(x, \mathcal{H}_{N_j}, h) - \int_0^T \lambda_N(u) K(x, \mathcal{H}_u, h) du.$$

- The argument $\mathcal{H}_u$ denotes the history of the ignition point process up to time u

Application to Ignition intensity (in progress)

- **A simple example:**
  $K(x, \mathcal{H}_u, h) = K_h(x - (u - L_{N(u)}))$

  A possible (but not serious!) interpretation: associated with each lightning stroke L_i is a delay V_i (holdover time). The ignitions $N_i = L_i + V_i$ are thinned if $N_i > L_{i+1}$.

  the local likelihood becomes

  $$\sum_j \log((\lambda_N(N_j))\, K_h\left(x - (N_j - L_{N(N_j)})\right) - \int_0^T \lambda_N(u) K_h\left(x - (u - L_{N(u)})\right) du.$$

  Maximizing this gives an estimate of the intensity of ignitions x time units after the previous lightning stroke.

The local constant estimator:

Use

$$\log(\lambda_N(x)) = a_0$$

Maximize local likelihood at x w.r.t. a_0:

$$\frac{\sum_j K_h\left(x - (N_j - L_{N(N_j)})\right)}{\int_0^T K_h\left(x - (u - L_{N(u)})\right) du}. \qquad (1)$$

We might interpret the above estimator of $\lambda_N(x)$ as the conditional density of V_i, given $N_i \leq L_{i+1}$.

- **If we wish to study the dependence of a given response on all previous flashes, we might use a K function of the form**

$$\sum_{i:L_i<u} K_h\left(x-(u-L_i)\right).$$

The resulting local constant estimator is given by

$$\frac{\sum_j \sum_{i:L_i<N_j} K_h\Big(x-(N_j-L_i)\Big)}{\int_0^T \sum_{i:L_i<u} K_h\left(x-(u-L_i)\right)du}.$$

- **Allowing for possible dependence on all of the lightning strokes gives the following:**

$$\frac{\sum_j \sum_i K_h\Big(x-(N_j-L_i)\Big)}{\int_0^T \sum_i K_h\left(x-(u-L_i)\right)du}.$$

[This is related to Brillinger's (1976) estimate of the cross-intensity function between lightning strokes and ignitions.]

- **This methodology allows for linear and quadratic adjustments to be made to this kind of estimator.**

$$\log(\lambda_N(u); x) = a_0 + a_1(u - L(x,u) - x)$$

or the local quadratic

$$\log(\lambda_N(u); x) = a_0 + a_1(u - L(x,u) - x) + a_2(u - L(x,u) - x)^2.$$

We have to maximize the local likelihood at x w.r.t. a_0, a_1 and a_2. The intensity estimate at x is given by $\widehat{a_0}$.

- **An even simpler application: take $K(x, \mathcal{H}_u, h) = K_h(x-u)$, where u is the time of the most recent ignition before x.**

- **Local constant:**

$$\log(\lambda(x)) = a_0$$

- **Local likelihood:**

$$L = \sum a_0 K_h(x - N_j) - e^{a_0} \int_0^T K_h(x-u)du$$

- $$\widehat{e^{a_0}} = \frac{\sum K_h(x-N_j)}{hc(x)}$$

  where $c(x)$ is easily calculated.

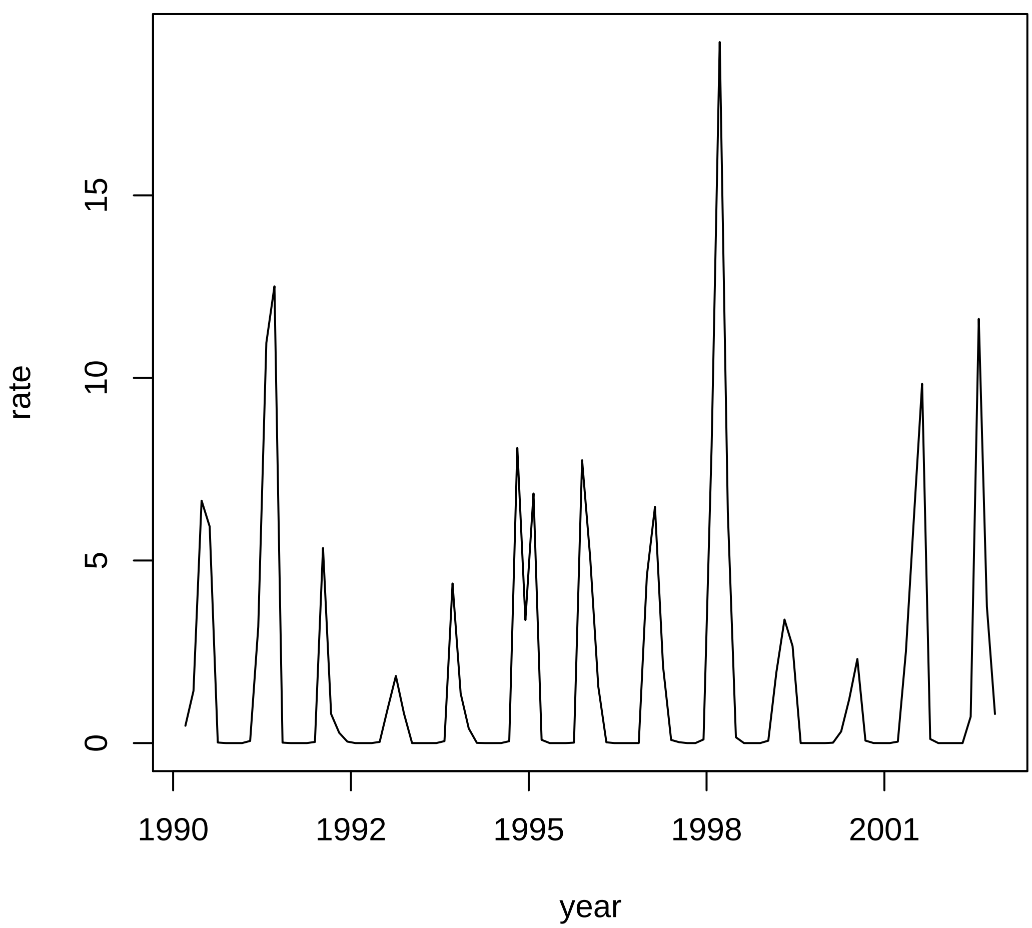
  This is closely related to a first order intensity function.

- **Including year as a covariate**

$$\log(\lambda(x)) = a_0 + a_1 \textbf{year}(x)$$

$\Rightarrow$ Equivalent to computing intensity estimates for each year separately

Ignitions in Ontario (1990-2002):



Future Work

- Implementation of local linear and quadratic first order intensity estimator, including year as a covariate.
- Higher order intensity/cross-intensity estimator, including year, district (where lightning occurred), and other covariates