Fast CDO Pricing in an Affine Markov Chain Model of Credit Risk

Tom Hurd Alexey Kuznetsov

Mathematics and Statistics
McMaster University

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Overview

Credit risk is the risk of financial losses due to

- unexpected changes in credit quality of a counterparty;
- failure of counterparty to service their debt;
- liquidation of the counterparty.

Fundamental Question

- How much interest should be charged to a default risky counterparty?
- Answer is expressed in terms of *credit spread*

$$S(t,T) = \frac{1}{T-t} \log\left(\frac{B_t^{(d)}(T)}{B_t(T)}\right)$$

where $B_t^{(d)}(T)$ is price of risky zero coupon bond and $B_t(T)$ is price of riskless zero coupon bond.

Understanding Credit Risk

Requires understanding and then modeling the following fundamental quantities:

- Risk-free interest rate r_t at time t;
- Time of a default "event" of the ith firm t_i^*;
- Recovery rate at time t conditioned on default: R_t^i;
- Premium investors demand as compensation for bearing credit risk.

Intensity Based (Reduced Form) Models

- Default time t^* modeled exogenously using stochastic intensity λ_t: a nonnegative process such that

$$P(t^* < t + dt | t^* > t) = \lambda_t dt$$

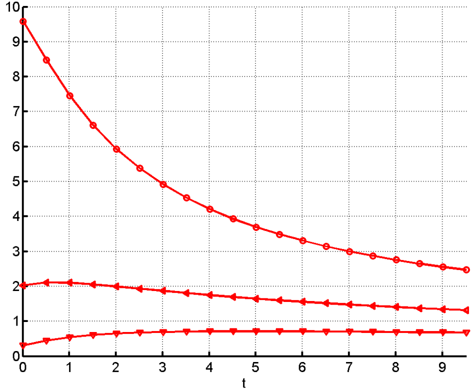
- It is convenient to introduce "default process"

$$Y_t = I\{t^* > t\}$$

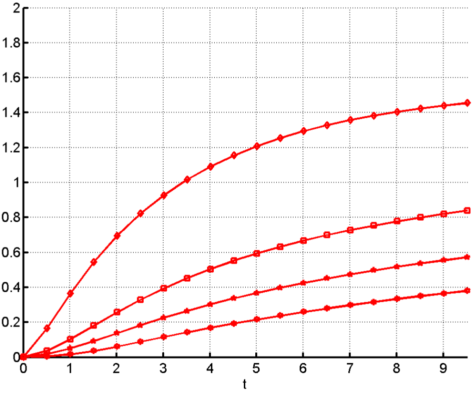
- Survival probability $P[t^* > t] = P[Y_t = 1] = E[e^{-\int_0^t \lambda_s ds}]$.
- Specify a market implied dynamics for λ_t.
- Bond price (zero recovery):

$$B_t^{(d)}(T) = E_t[e^{-\int_t^T r_s ds} I\{t^* > T\}] = E_t[e^{-\int_t^T [r_s + \lambda_s] ds}].$$

Typical Credit Spreads in Intensity Based Models



Typical Credit Spreads in Structural Models



Affine Markov Chain model

- Idea: generalize the process Y_t
- intensity based model: Y_t has two states: $\{0, 1\}$

$$P[Y_{t+dt} = 0 | Y_t = 1] = \lambda_t dt.$$

  and 0 is an absorbing state.
- let's introduce $K + 1$ states: $\{0, 1, 2, \ldots, K\}$ and stochastic intensity λ_t

$$P[Y_{t+dt} = j | Y_t = i] = L_{ij} \lambda_t dt.$$

  and 0 is an absorbing state.

Ingredients

- Spot interest rate r_t and recovery rate R_t;
- Default process Y_t, defined by
  - stochastic intensity $\lambda_t \geq 0$;
  - the matrix of transition intensities

$$\mathcal{L}_Y = (L_{ij})_{i,j=0\ldots K}, \qquad P[Y_{t+dt} = j | Y_t = i] = L_{ij}\lambda_t dt.$$

  0 is an absorbing state, thus $L_{0j} = 0$

- Default time t^*: first time Y_t hits absorbing state 0.

Interpretation

- intensity based model:
  $Y_t = 1$: company is not in default by time t
  $Y_t = 0$: company has defaulted by time t
- AMC model:
  $Y_t = i$, $i \neq 0$: company is in the credit rating i at time t
  $Y_t = 0$: company has defaulted by time t
- if $K = 1$, then intensity based model $\equiv$ AMC with matrix $\mathcal{L}_Y$ given by

$$\mathcal{L}_Y = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}$$

Transition probabilities

- The rating transition probabilities are given by

$$P[Y_t = j | Y_0 = y] = \sum_{i=0}^{K} q_{yi} \tilde{q}_{ij} E_0 \left[e^{\alpha_i \int_0^t \lambda_s ds} \right].$$

- Intensity based model:

$$P[Y_t = 0 | Y_0 = 1] = 1 - E_0[e^{-\int_0^t \lambda_s ds}].$$

Defaultable Bonds

- Defaultable bond with zero recovery has price

$$B_t^{(d)}(T) = E_t\left[e^{-\int_t^T r_s ds} I\{t^* > T\}\right] =$$
$$= B_t(T) - \sum_{i=0}^{K} q_{yi}\tilde{q}_{i0} E_t\left[e^{-\int_t^T (r_s - \alpha_i \lambda_s) ds}\right]$$

- intensity based model:

$$B_t^{(d)}(T) = E_t[e^{-\int_t^T (r_s + \lambda_s) ds}].$$

Intensity and Stochastic time change

- Idea: define an increasing (continuous) process $\tau_t = \int_0^t \lambda_s ds$, thus $d\tau_t = \lambda_t dt$;
- AMC model: define a Markov chain $\tilde{Y}_t$ with generator $\mathcal{L}_Y$, independent of τ_t. Then

$$P[\tilde{Y}_{t+dt} = j | \tilde{Y}_t = i] = L_{ij} dt.$$

- Define $Y_t = \tilde{Y}_{\tau_t}$. Then

$$P[Y_{t+dt} = j | Y_t = i] = L_{ij} d\tau_t = L_{ij} \lambda_t dt.$$

- Intensity based models $\equiv$ doubly stochastic models: Y_t is a Markov chain subordinated by a stochastic time change τ_t.

Choice of underlying processes

- Key idea: in all the formulas we need to compute

$$E\left[e^{-\int_0^t (r_s - \alpha_i \lambda_s) ds}\right] = E\left[e^{-\int_0^t r_s ds + \alpha_i \tau_t}\right]$$

- Model r_t and λ_t as linear combination of affine processes Z_t^1 and Z_t^2

$$r_t = \mathbf{M}_r^1 Z_t^1 + \mathbf{M}_r^2 Z_t^2 = \langle \mathbf{M}_r \cdot Z_t \rangle, \qquad \lambda_t = \langle \mathbf{M}_\tau \cdot Z_t \rangle,$$

  for which

$$E\left[e^{-\int_0^t \alpha Z_s ds}\right] = e^{\Phi(t) + Z_0 \Psi(t)}$$

- Model time change process as $\tau_t = \int_0^t \lambda_s ds + m_\tau Z_t^3$, where Z_t^3 is a jump process.

Choice of $\mathbf{Z}_t = (Z_t^1, Z_t^2)$ and Z_t^3

Three processes with distinct characteristics:

- Z_t^1: CIR process (diffusion) with Markov generator

$$\mathcal{L}_{Z^1} f(x) = a(1-x)f'(x) + cxf''(x),$$

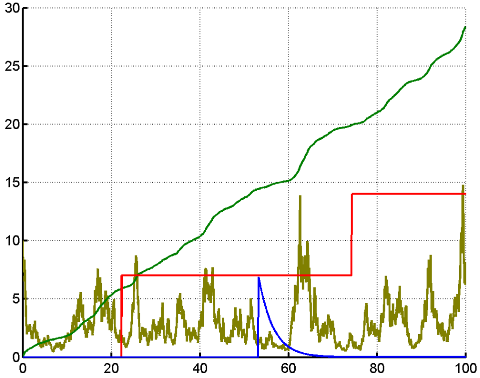
- Z_t^2: affine process with jumps defined by Markov generator

$$\mathcal{L}_{Z^2} f(x) = \lambda_2(f(x+h) - f(x)) - h\lambda_2 x f'(x).$$

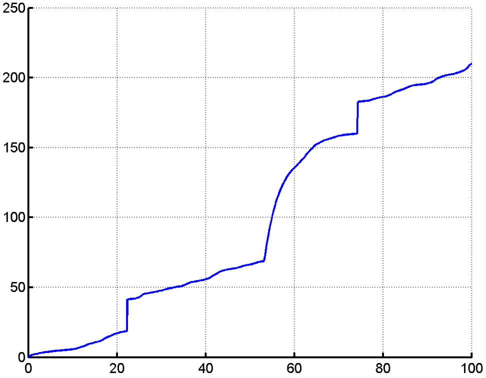
- Z_t^3 (jump part of the time change): Poisson process with jump size h_3 and intensity $\lambda_3 = 1/h_3$ and λ_3^{-1}:

$$Z_t^3 = h_3 \Pi(\lambda_3 t).$$

Components of time change process



Time change process τ_t



Building Blocks G_1, G_2, G_3

All essential computations come from *explicit* formulas for the following:

- $$G_1(t, \mathbf{z}; \mathbf{u}, \mathbf{v}) = E_{0,\mathbf{z}}\left[e^{-\int_0^t \langle \mathbf{u} \cdot \mathbf{Z}_s \rangle ds} e^{-\langle \mathbf{v} \cdot \mathbf{Z}_t \rangle}\right]$$

- $$\begin{aligned} G_2(t, \mathbf{z}; \mathbf{u}, \mathbf{v}, \mathbf{w}) &= E_{0,\mathbf{z}}\left[e^{-\int_0^t \langle \mathbf{u} \cdot \mathbf{Z}_s \rangle ds} \langle \mathbf{w} \cdot \mathbf{Z}_t \rangle e^{-\langle \mathbf{v} \cdot \mathbf{Z}_t \rangle}\right] \\ &= -\langle \mathbf{w}, \nabla_{\mathbf{v}} \mathbf{G_1} \rangle. \end{aligned}$$

- $$G_3(t; v) = E\left[e^{-v Z_t^3}\right] = \exp\left(\lambda_3 t (e^{-v/\lambda_3} - 1)\right).$$

Markov Generator $\mathcal{L}_Y$

- Markov states represent Standard and Poor's "rating class":

  $\{0, 1, \ldots, 7\} \leftrightarrow \{\text{'default'}, \text{CCC}, \text{B}, \text{BB}, \text{BBB}, \text{A}, \text{AA}, \text{AAA}\}.$

- Markov generator:

$$\mathcal{L}_Y = \begin{pmatrix} 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.2856 & -0.4318 & 0.0928 & 0.0250 & 0.0142 & 0.0142 & 0.0000 & 0.0000 \\ 0.0753 & 0.0479 & -0.1928 & 0.0568 & 0.0073 & 0.0034 & 0.0021 & 0.0000 \\ 0.0273 & 0.0144 & 0.1181 & -0.2530 & 0.0813 & 0.0089 & 0.0025 & 0.0005 \\ 0.0049 & 0.0020 & 0.0174 & 0.0701 & -0.1711 & 0.0713 & 0.0047 & 0.0007 \\ 0.0010 & 0.0000 & 0.0048 & 0.0107 & 0.0688 & -0.1172 & 0.0309 & 0.0010 \\ 0.0000 & 0.0000 & 0.0030 & 0.0030 & 0.0105 & 0.0787 & -0.1043 & 0.0091 \\ 0.0000 & 0.0000 & 0.0000 & 0.0031 & 0.0020 & 0.0083 & 0.1019 & -0.1153 \end{pmatrix}$$

- the columns of $Q = (q_{ij})$ are the eigenvectors of $\mathcal{L}_Y$ with eigenvalues α_i. $\tilde{Q} = (\tilde{q}_{ij}) = Q^{-1}$.

Other parameters

- interest rates: $\mathbf{M}_r = (0.05, 0)$
- stochastic time change: $\mathbf{M}_\tau = (0.6, 1.2)$ and $m_\tau = 0.2$
- CIR (Z^1) parameters: $a = c = 0.1$
- Z^2 parameters: $h_2 = 3$, $\lambda_2 = 0.3$
- Z^3 parameters: $h_3 = 3$, $\lambda_3 = 1/3$

Rating Transition Probabilities

Lemma

Rating transition probabilities for the process Y_t are

$$P_{0,\mathbf{z},y}(Y_t = j) = \sum_{i=0}^{K} q_{yi}\tilde{q}_{ij} G_1(t, \mathbf{z}; -\alpha_i \mathbf{M}_\tau, \mathbf{0}) G_3(t, -\alpha_i m_\tau).$$

Default Density Function

Proposition

Probability density function of default is

$$\frac{d}{dt}P_{0,\mathbf{z},y}(t^* < t) = \sum_{i=1}^{K} q_{yi}\tilde{q}_{i1}\Big[\alpha_i G_2(t,\mathbf{z};-\alpha_i\mathbf{M}_\tau,\mathbf{0},\mathbf{M}_\tau) +$$

$$+G_1(t,\mathbf{z};-\alpha_i\mathbf{M}_\tau,\mathbf{0})\lambda_3(e^{\alpha_i m_\tau/\lambda_3}-1)\Big]G_3(t,-\alpha_i m_\tau).$$

Default Free Bonds

Proposition

Price at time t of riskless zero–coupon bond with maturity T

$$B_t(T) = E_t\left[e^{-\int_t^T r_s ds}\right] = G_{\mathbf{1}}(T-t, \mathbf{Z}_t; \mathbf{M}_r, \mathbf{0}).$$

Defaultable Bonds

Proposition

1. *Defaultable bond with zero recovery has price*

$$B_t^{(d)}(T) = E_{t,\mathbf{z},y}\left[e^{-\int_t^T r_s ds} I\{t^* > T\}\right] =$$

$$B_t(T) - \sum_{i=0}^{K} q_{yi}\tilde{q}_{i1} G_1(T-t, \mathbf{z}; \mathbf{M}_r - \alpha_i \mathbf{M}_\tau, \mathbf{0}) G_3(T-t, -\alpha_i m_\tau).$$

2. *Defaultable bond with non-zero recovery are given by a more complicated formula of same type.*

Simulation of Credit Spread for a BB company

spreads

Simulation of Credit Spreads for all companies

spreads

Dependence of Credit Spreads on h_2

spreads

Model of M Firms

- Independent processes $\tilde{Y}_t^1, \ldots, \tilde{Y}_t^M$, which are Markov chains on $\{0, 1, 2, \ldots, K\}$, with 0 an absorbing state, and with identical generators $\mathcal{L}_{\tilde{Y}}$.
- Interest process r_t, recovery process R_t and stochastic time change process τ_t, as for one firm model.
- Credit migration processes

$$Y_t^1 = \tilde{Y}_{\tau_t}^1, \ldots, Y_t^M = \tilde{Y}_{\tau_t}^M.$$

- Default time t_i^* is first time corresponding process Y_t^i hits absorbing state 1.

Interpretation of Z^1, Z^2, Z^3

- $$\tau_t = \int_0^t \left(\mathbf{M}_\tau^1 Z_s^1 + \mathbf{M}_\tau^2 Z_s^2\right) ds + m_\tau Z_t^3.$$
- Z^1: "normal" economy
- Z^2: clustering of defaults
- Z^3: simultaneous defaults

Pairwise Joint Default Distributions

Lemma

The joint probability $P_{\mathbf{0},\mathbf{z},y_i,y_j}(t_i^* < s, t_j^* < t)$ *is given by*

$$P_{\mathbf{0},\mathbf{z},y_i,y_j}(t_i^* < s, t_j^* < t) = E_{\mathbf{0},\mathbf{z},y_i,y_j}\left[I\{Y_s^i = 1\}I\{Y_t^j = 1\}\right]$$

$$= \sum_{k,l=1}^{K} q_{y_i k}\tilde{q}_{k\mathbf{1}} q_{y_j l}\tilde{q}_{l\mathbf{1}} E_{\mathbf{0},\mathbf{z}}\left[e^{\alpha_k \tau_s + \alpha_l \tau_t}\right],$$

where the expectation $E_{\mathbf{0},\mathbf{z},y_i,y_j}\left[e^{\alpha_k \tau_s + \alpha_l \tau_t}\right]$ *can be computed explicitly.*

Joint default density

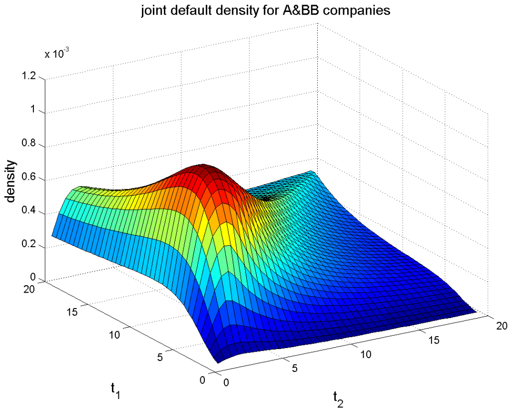


Figure: Joint BB + BB default density, no jumps

Joint default density

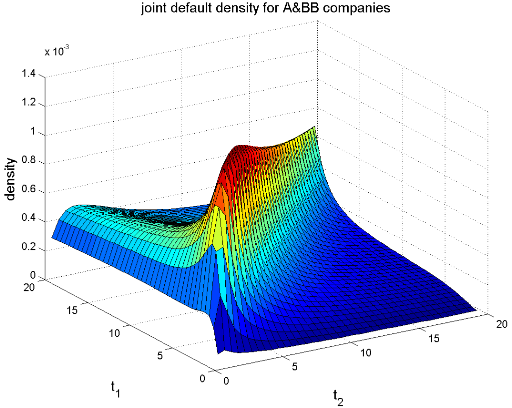


Figure: Joint BB + BB default density, jumps in Z^2

CDOs: What are they?

A large portfolio of similar bonds written on different firms is sliced into "tranches" ordered by "seniority". Each CDO tranche is a separate investment vehicle with its characteristic risk-reward.

Synthetic CDO Tranche

- Credit swap between two parties, insured and insurer.
- Two components, "premium leg" and "insurance leg" are basic credit derivatives on total default loss of portfolio.
- Fractional loss at time t

$$L_t = \sum_{i=1}^{M} (1 - R_0) \frac{N_i}{N} I\{t_i^* < t\}$$

- Here:
  - M: number of firms;
  - N_i: face value of bond ("notional") of firm i;
  - $N = \sum_{i=1}^{M} N_i$: total notional.

CDO Formulas

CDO tranche for fractional losses in a range $[\underline{x}, \bar{x}] \subset [0, 1]$:

$$U(x) = \frac{1}{\bar{x} - \underline{x}} \left[(\bar{x} - x)^{+} - (\underline{x} - x)^{+} \right]$$

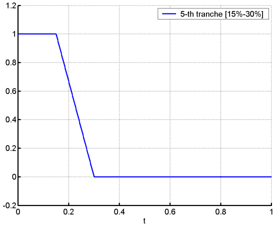


Figure: $U(l)$ for the fifth tranche [15%-30%]

CDO Formulas

- Premium leg is paid by insured at stochastic rate $U_t = U(L_t)$. Price:

$$V = E_{0,\mathbf{z}}\left[\int_0^T e^{-\int_0^t r_s ds} U(L_t) dt\right].$$

- Insurer pays tranche of losses by default of firms. Price:

$$W = -E_{0,\mathbf{z}}\left[\int_0^T e^{-\int_0^t r_s ds} dU_t\right]$$

Theorem on CDOs

Theorem (CDO Pricing)

- *Price of the premium and insurance legs:*

$$V = \int_0^\infty H^U(x) F^P(x, \mathbf{z}) dx, \qquad W = \int_0^\infty H^S(x) F^I(x, \mathbf{z}) dx.$$

- *Here* $H^U(x), H^S(x)$ *depend on parameters of loss* $\tilde{L}_t$ *and payoff functions* $U, S = 1 - U$ *(i.e. on tranche)*
- $F^P(x, \mathbf{z}), F^I(x, \mathbf{z})$ *depend only on interest rate and time change processes parameters.*

Remarks

- Functions $F^P(x,\mathbf{z}), F^I(x,\mathbf{z})$ are computed once (and stored); all CDO tranches are obtained by integrating tranche–dependent functions $H^U(x), H^S(x)$ against tranche–independent $F^P(x,\mathbf{z}), F^I(x,\mathbf{z})$.
- Formulas separate effects of stochastic time change (hidden in $F^P(x,\mathbf{z}), F^I(x,\mathbf{z})$) from all information about Markov chains $\tilde{\mathbf{Y}}$, conditional loss process $\tilde{L}_t$ and payoff functions U, S (hidden in $H^U(x), H^S(x)$).
- For equal notionals, we can compute H^U, H^S exactly. For unequal or stochastic notionals, we have a number of high speed approximation schemes.

Functions H^S and F^I

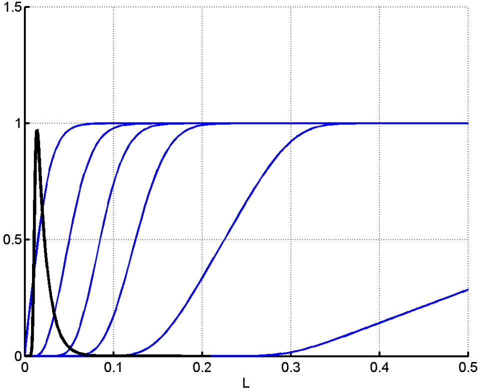


Figure: Computing the price of insurance leg

Normal Approximation Scheme

- When $T \times \lambda$ is not too small the normal approximation is reasonable:

$$\tilde{L}_t \overset{d}{\approx} L(t,\xi) = m(t) + \xi\sigma(t),$$

  where ξ is Gaussian $N(0,1)$.
- Mean $m(t)$ and variance $\sigma^2(t)$

$$\begin{aligned} m(t) = \sum_{k=0}^{K} \alpha_k p_{k1}(t), \quad \alpha_k &= \sum_{i=1}^{M} (1-R_0) I\{\tilde{Y}_0^i = k\} \frac{N_i}{N} \\ \sigma^2(t) = \sum_{k=1}^{K} \beta_k p_{k1}(t)(1-p_{k1}(t)), \quad \beta_k &= \sum_{i=1}^{M} (1-R_0)^2 I\{\tilde{Y}_0^i = k\} \frac{N_i^2}{N^2} \end{aligned}$$

Normal Approximation Scheme

- It follows that

$$H^U(\tau) = 1 - H^S(\tau) = \frac{\sigma(\tau)}{\bar{x} - \underline{x}} \left[\tilde{\Phi}\left(\frac{\bar{x} - m(\tau)}{\sigma(\tau)} \right) - \tilde{\Phi}\left(\frac{\underline{x} - m(\tau)}{\sigma(\tau)} \right) \right]$$

- where

$$\tilde{\Phi}(x) = \int_{-\infty}^{x} \Phi(y)dy = x\Phi(x) + \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}},$$

- and $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy$ (CDF of $N(0,1)$).

Short Maturity

Beta vs Poisson vs Normal Approximations, Short Maturity

Long Maturity

Beta vs Poisson vs Normal Approximations, Long Maturity

Exact and approximate schemes: relative errors

For equal notionals, we compare exact and approximate computations:

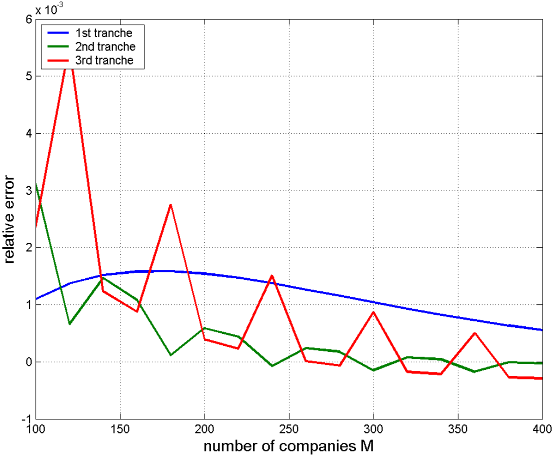


Figure: Relative error

Exact and approximate scheme: computation times

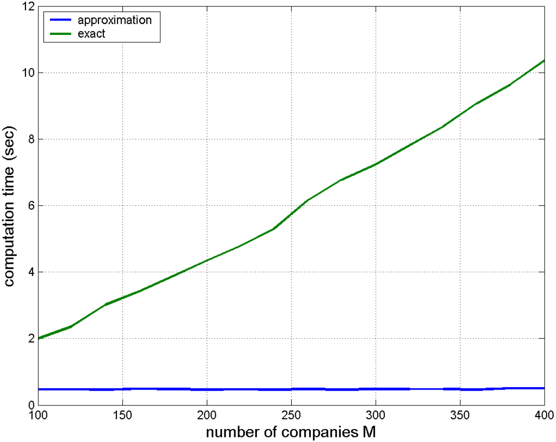


Figure: Computation time

Default Parameters

- $\mathcal{L}_Y$ as before;
- interest rates: $\mathbf{M}_r = (0.05, 0)$; stochastic time change: $\mathbf{M}_\tau = (0.6, 1.2)$ and $m_\tau = 0.2$;
- CIR (Z^1) parameters: $a = c = 0.1$; Z^2 parameters: $h_2 = 3$, $\lambda_2 = 0.3$; Z^3 parameters: $h_3 = m_\tau 3$, $\lambda_3 = 1/3$;
- Number of firms in each rating class: [0 0 0 0 50 50 50 50];
- Equal notionals (we haven't got around to running unequal notionals!).

Dependence on $Z^1(0)$

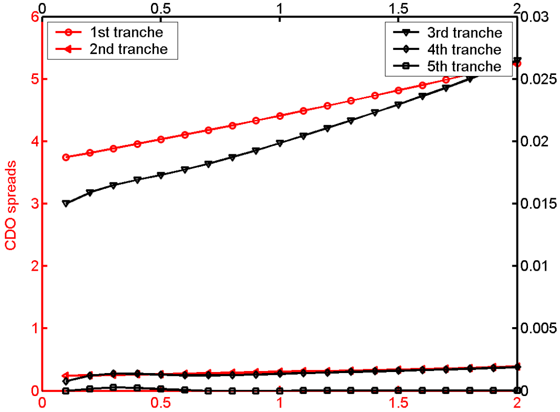


Figure: Dependence of CDO spreads on $Z^1(0)$

Dependence on $Z^2(0)$

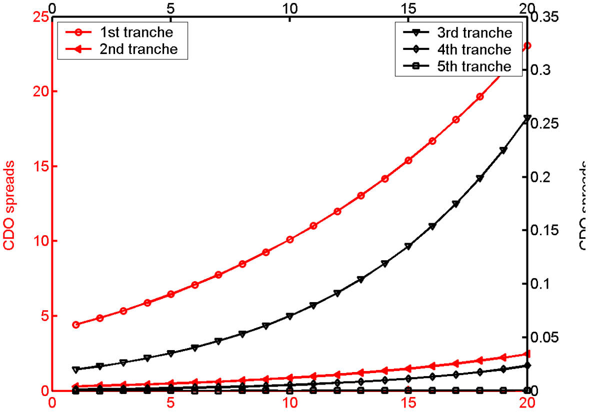


Figure: Dependence of CDO spreads on $Z^2(0)$

Dependence on jump size h_2

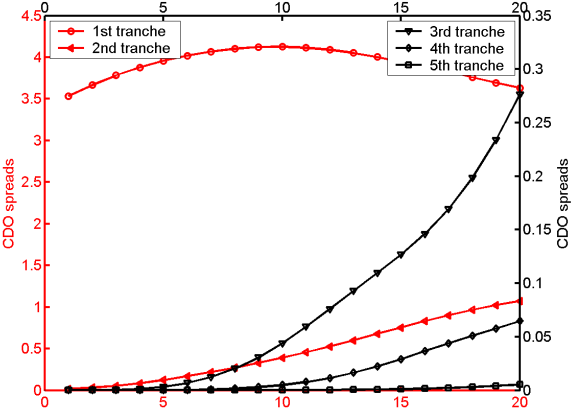


Figure: Dependence of CDO spreads on jump size h_2

Dependence on jump size h_3

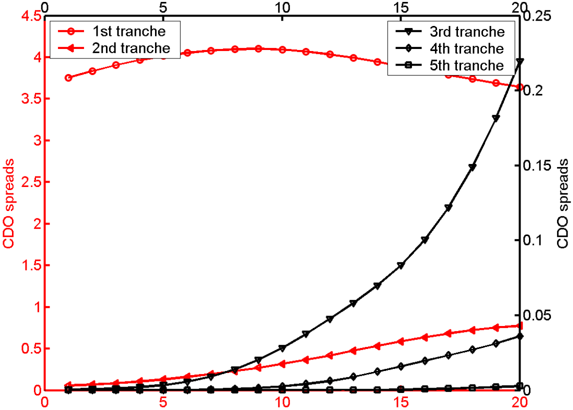


Figure: Dependence of CDO spreads on jump size h_3

Dependence on CIR volatility c

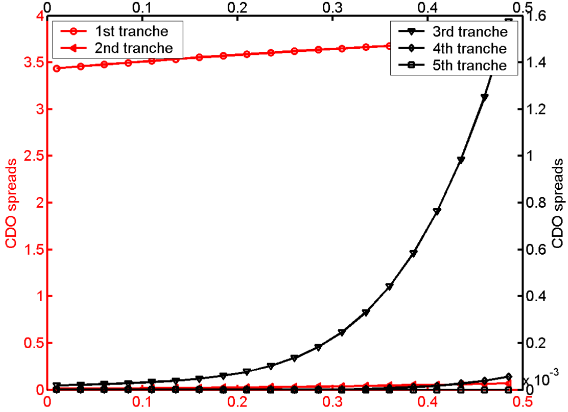


Figure: Dependence of CDO spreads on CIR volatility c

Dependence on maturity T

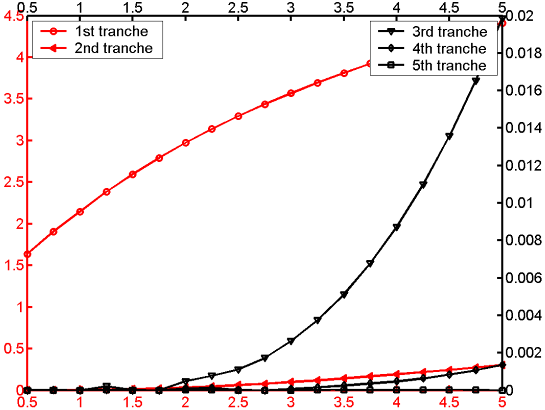


Figure: Dependence of CDO spreads on maturity T

Dependence on interest rates

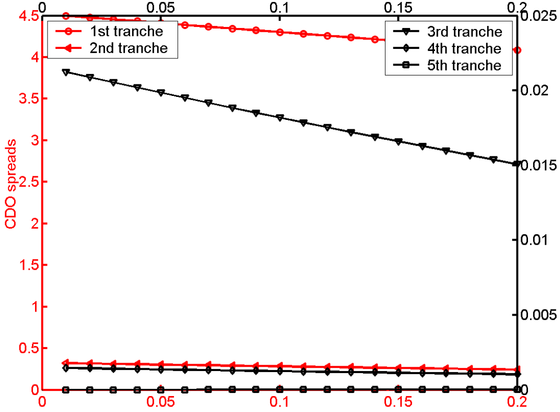


Figure: Dependence of CDO spreads interest rate

Sensitivities

- Security prices are sensitive to dynamic risk factors $\tilde{Y}, Z^1, Z^2, Z^3$ and model parameters $(\mathcal{L}_Y, a, c, \lambda_2, h_2, h_3, \mathbf{M}_r, \mathbf{M}_\tau, m_\tau)$.
- Delta hedging Z^1, Z^2 means hedging general market risk; requisite derivatives of both premium and insurance legs are explicitly computable:

$$(\Delta_{V,1}, \Delta_{V,2}) = \partial_{\mathbf{z}} V \quad = \quad \int_0^\infty H(x) \partial_{\mathbf{z}} F^P(x, \mathbf{z}) dx, \qquad (1)$$

- Hedging risk factors Y amounts to protecting against risk of any individual migrations or defaults: such “firm specific risks” are difficult to hedge and are of secondary importance.

Calibration Issues

Not yet addressed!

- Suitable input data set is complicated, huge and difficult to obtain: corporate bonds are not exchange traded, trade relatively infrequently, come in many flavours, etc.
- Many model parameters to fit;
- $\vec{Z}_t$ is a vector-valued *unobserved* process driving credit spreads;
- $\mathcal{L}_Y$: should we use risk-neutral (they need to be calibrated) vs. historical probabilities (easy to use, but not reliable)?
- Extensions to non-minimal models will be needed.

Conclusion

- AMC framework gives *complete* dynamical models of multifirm credit migration and default.
- AMC is a generalization of reduced form or doubly stochastic models but includes "structural" characteristics.
- Computations are very efficient:
  - Speed for one-two firm models is comparable with intensity based models,
  - For CDO computations the speed is independent of the number of companies M,
  - Errors across tranches decrease as M increases,
  - Typical error is less than one basis point.

Conclusion

- Flexible correlation structure;
- Excellent engine for scenario generation/stress testing;
- Analytical computation of the greeks;
- Model easily includes:
  - stochastic interest rates;
  - stochastic recovery (possibly correlated with credit spreads, interest rates);
  - multi-factor models;
  - nonexchangeable firms