Valuing the Option to Invest in an Incomplete Market

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Introduction I: The Problem

• Aim to extend real option valuation models to include incompleteness of capital markets. Specifically examine option to invest

• How does incompleteness and managerial risk-aversion to idiosyncratic risk impact on:
(i) Value of option to invest
(ii) Investment timing decision ?

• Incompleteness in our model arises via non-tradability of underlying real assets/project value

Introduction II: Background

• Typically real options theory relies upon market completeness via existence of a (perfect) spanning asset $\rightarrow$ risk-neutral valuation

• Alternatively, investors are assumed risk-averse to market risks but risk-neutral to idiosyncratic risks (McDonald and Siegel (1986) CAPM argument)

• Call these two approaches the *classic models*

Introduction III

- We introduce partial spanning asset to retain some idiosyncratic risk. Could be market asset, industry benchmark, individual stock
- Owner-manager facing irreversible investment decision over infinite horizon
- Owner-manager is entrepreneur in that only asset of firm is option to invest
- Manager is risk-averse to idiosyncratic risks
- Manager chooses investment time and trading strategy in partial spanning asset to maximize expected utility of wealth where wealth consists of option to invest and portfolio from trading (and thus maximizes value of firm)
- Value of option to manager found by certainty equivalence: compensation manager requires to give up right to option
- Both complete/risk-neutral and CAPM (McDonald and Siegel (1986)) models are special cases of our incomplete one

Main Results and Implications

- Risk-averse manager places less value on option to invest than under classic models
- Risk-aversion induces manager to invest earlier than classic real options models (reduces gap between NPV criteria and classic RO investment times)
- Qualitative difference in investment recommendation of incomplete model versus classic models - approximating an incomplete situation with a complete solution can result in an incorrect decision

Literature

- Vast literature on real options - Myers (1977), Brennan and Schwartz (1985), McDonald and Siegel (1986), Dixit and Pindyck (1994), ...

- Pinches (1998), Lander and Pinches (1998), Borison (2003) amongst others
- Rogers and Scheinkman (2003)
- Kadam, Lakner and Srinivasan (2003)
- Smith and Nau (1995)
- Henderson (2002), Henderson and Hobson (2002a, 2002b), Musiela and Zariphopoulou (2003)
- Miao and Wang (2004)
- Empirical ? Huddart and Lang (1996)

Modeling Assumptions I

• Manager can invest at cost $Ke^{r(\tau-t)}$ at time $\tau \geq t$, receives

$$(V_\tau - Ke^{r(\tau-t)})^+$$

where V, value of project cashflows follows

$$\frac{dV}{V} = \eta(\xi dt + dW) + rdt$$

where $\xi = \frac{\nu - r}{\eta}$ is project's Sharpe ratio, W Brownian motion.

• Manager invests in riskless bond with constant interest rate r and partial spanning asset P following

$$\frac{dP}{P} = \sigma(\lambda dt + dB) + rdt$$

where $\lambda = \frac{\mu - r}{\sigma}$ is Sharpe ratio. B and W are correlated with $-1 \leq \rho \leq 1$ and so for Z indept of B,

$$dW = \rho dB + \sqrt{1-\rho^2}dZ$$

Modeling Assumptions II

• Manager's portfolio X has dynamics

$$dX = \theta \frac{dP}{P} + r(X - \theta)dt$$

where θ is cash amount in P.

• Unless $\rho^2 = 1$, manager faces idiosyncratic risk via $\eta^2(1 - \rho^2)dZ$

• Manager is risk-averse towards idiosyncratic risks and has utility function $U(x) = -\frac{1}{\gamma}e^{-\gamma x}, \;\; \gamma > 0$ with CRRA

• Manager maximizes value of firm via utility maximization of value of option to invest. Value function given by optimal stopping problem

$$G(x, v) = \sup_{t \leq \tau} \sup_{\theta_u, t \leq u \leq \tau} \mathbb{E}_t \left[U_\tau \left(X_\tau + (V_\tau - Ke^{r(\tau - t)})^+ \right) | X_t = x, V_t = v \right]$$

where U_τ denotes that utility is for wealth at time τ

Time Consistency of Utility Functions I

• Consider the simpler problem of maximizing expected utility from wealth (no option) over finite horizon T'.

• At T', we assume

$$U_{T'}(x) = -\frac{A_{T'}}{\gamma_{T'}} e^{-\gamma_{T'} x}$$

where $A_{T'}$ is some constant and the constant absolute risk aversion $\gamma_{T'}$ reflects risk aversion at date T'. Value function is

$$F^a_{T'}(t,x) = \sup_{\theta} \mathbb{E}_t U_{T'}(X_{T'})$$

• Merton (1969) shows

$$\begin{aligned} F^a_{T'}(t,x) &= -\frac{A_{T'}}{\gamma_{T'}} e^{-\gamma_{T'} e^{r(T'-t)} x} e^{-\frac{1}{2}\lambda^2 (T'-t)} \\ \theta_t &= \frac{\lambda e^{-r(T'-t)}}{\gamma_{T'} \sigma} \end{aligned}$$

Time Consistency of Utility Functions II

• Now think about an earlier intermediate date $t \leq T \leq T'$

• How to value wealth at T ? Consider choosing *any* strategy over $[t, T]$ and the optimal strategy on $(T, T']$. This optimal strategy is the Merton (1969) strategy and

$$\sup_{\theta_u, t \leq u \leq T'} \mathbb{E} U_{T'}(X_{T'}) = \sup_{\theta_u, t \leq u \leq T} \mathbb{E}\left[-\frac{A_{T'}}{\gamma_{T'}} e^{-\gamma_{T'} e^{r(T'-T)} X_T} e^{-\frac{1}{2}\lambda^2 (T'-T)} \right]$$

• The right hand side is now an optimization problem over the sub-horizon $[t, T]$. To value consistently with T' cashflows

$$U_T(x) = -\frac{A_T}{\gamma_T} e^{-\gamma_T x}$$

where A_T is constant and γ_T reflects risk aversion for time T. We require

$$\gamma_{T'} e^{rT'} = \gamma_T e^{rT} = \gamma e^{rt} \tag{1}$$

and

$$\frac{A_{T'}}{\gamma_{T'}} e^{-\frac{1}{2}\lambda^2 T'} = \frac{A_T}{\gamma_T} e^{-\frac{1}{2}\lambda^2 T} = \frac{A}{\gamma} e^{-\frac{1}{2}\lambda^2 t} \tag{2}$$

where in both (1) and (2), A is a constant and γ is the CARA parameter for today, t.

- Time consistent utility for T must be

$$U_T(x) = -\frac{A}{\gamma} e^{-\gamma e^{-r(T-t)} x} e^{\frac{1}{2}\lambda^2 (T-t)}.$$

Note T' has disappeared...

Proposition 1 *The time consistent exponential utility function is given by*

$$U_\tau(x) = -\frac{A}{\gamma} e^{-\gamma e^{-r(\tau-t)} x} e^{\frac{1}{2}\lambda^2(\tau-t)}$$

The Bellman Equation

Proposition 2 *The value function for the manager's investment problem solves the following non-linear Bellman equation. In the continuation region,* $G(x,v) > -\frac{A}{\gamma}e^{-\gamma(x+(v-K)^+)}$ *and* G *solves*

$$0 = \frac{1}{2}\lambda^2 G + \xi\eta v G_v + \frac{1}{2}\eta^2 v^2 G_{vv} - \frac{1}{2}\frac{(\lambda G_x + \rho\eta v G_{xv})^2}{G_{xx}} \tag{3}$$

with boundary, value matching and smooth pasting conditions

$$\begin{aligned} G(x,0) &= -\frac{A}{\gamma}e^{-\gamma x} \\ G(x,\tilde{V}^{(\rho,\gamma)}) &= -\frac{A}{\gamma}e^{-\gamma(x+(\tilde{V}^{(\rho,\gamma)}-K)^+)} \\ G_v(x,\tilde{V}^{(\rho,\gamma)}) &= AI_{\{\tilde{V}^{(\rho,\gamma)}>K\}}e^{-\gamma(x+(\tilde{V}^{(\rho,\gamma)}-K)^+)}. \end{aligned}$$

In the stopping region,

$$G(x,v) = -\frac{A}{\gamma} e^{-\gamma(x+(v-K)^+)}.$$

The optimal investment time τ^* is given by

$$\tau^* = \inf\left\{u \geq t : V_u \geq \tilde{V}^{(\rho,\gamma)} e^{r(u-t)}\right\}$$

so investment takes place when the discounted project value reaches some constant level $\tilde{V}^{(\rho,\gamma)}$

The Solution

Proposition 3 *Let* $\beta_1^{(\rho,\gamma)} = 1 - \frac{2(\xi-\lambda\rho)}{\eta}$. *If* $\beta_1^{(\rho,\gamma)} > 0$ *(correspondingly* $\xi < \lambda\rho + \frac{\eta}{2}$*), the firm will invest at time* τ^* *given in Proposition 1. The optimal investment trigger,* $\tilde{V}^{(\rho,\gamma)}$*, is the solution to*

$$\tilde{V}^{(\rho,\gamma)} - K = \frac{1}{\gamma(1-\rho^2)} \ln\left[1 + \frac{\gamma \tilde{V}^{(\rho,\gamma)}(1-\rho^2)}{\beta_1^{(\rho,\gamma)}}\right] \tag{4}$$

If $\beta_1^{(\rho,\gamma)} \leq 0$ *(or equivalently* $\xi \geq \lambda\rho + \frac{\eta}{2}$*) then smooth pasting fails and there is no solution. In this case, the firm postpones investment indefinitely.*
The value function $G(x,v)$ *is given by* $G(x,v) =$

$$\begin{cases} -\frac{1}{\gamma}e^{-\gamma x}\left[1-\left(1-e^{-\gamma(\tilde{V}^{(\rho,\gamma)}-K)(1-\rho^2)}\right)\left(\frac{v}{\tilde{V}^{(\rho,\gamma)}}\right)^{\beta_1^{(\rho,\gamma)}}\right]^{\frac{1}{1-\rho^2}} & v < \tilde{V}^{(\rho,\gamma)} \\ -\frac{1}{\gamma}e^{-\gamma x}e^{-\gamma(v-K)} & v \geq \tilde{V}^{(\rho,\gamma)} \end{cases}$$

Proof of Proposition 3

• Transform to remove non-linearity and propose power-type solution

• Proposing $G(x,v) = -\frac{A}{\gamma}e^{-\gamma x}J(v)$, setting $J(v) = \Gamma(v)^g$ gives

$$0 = \left[v\Gamma_v \eta\,(\xi - \lambda\rho) + \frac{1}{2}\eta^2 v^2 \Gamma_{vv} + \frac{1}{2}\frac{\Gamma_v^2}{\Gamma}\eta^2 v^2 (g(1-\rho^2) - 1) \right]$$

Choosing $g = \frac{1}{1-\rho^2}$,

$$0 = \left[v\Gamma_v \eta\,(\xi - \lambda\rho) + \frac{1}{2}\eta^2 v^2 \Gamma_{vv} \right]$$

with

$$\begin{aligned} \Gamma(0) &= 1 \\ \Gamma(\tilde{V}^{(\rho,\gamma)}) &= e^{-\gamma(\tilde{V}^{(\rho,\gamma)} - K)^+(1-\rho^2)} \\ \frac{\Gamma_v(\tilde{V}^{(\rho,\gamma)})}{\Gamma(\tilde{V}^{(\rho,\gamma)})} &= -\gamma I_{\{\tilde{V}^{(\rho,\gamma)} > K\}}(1-\rho^2) \end{aligned}$$

• Propose a solution of the form $\Gamma(v) = L^{(\rho,\gamma)} v^{\psi}$,

$$0 = \psi(\psi - \beta_1^{(\rho,\gamma)})$$

where $\beta_1^{(\rho,\gamma)} = 1 - \frac{2(\xi - \lambda\rho)}{\eta}$. Solutions are

$$\psi = 0, \psi = \beta_1^{(\rho,\gamma)} = 1 - \frac{2(\xi - \lambda\rho)}{\eta}$$

• Now $\Gamma(v) = L^{(\rho,\gamma)} v^{\beta_1^{(\rho,\gamma)}} + B$ and boundary cdn gives $B = 1$.
If $\beta_1^{(\rho,\gamma)} \leq 0$ ($\xi \geq \lambda\rho + \frac{\eta}{2}$) then $L^{(\rho,\gamma)} = 0$, smooth pasting fails and there is no solution $\rightarrow$ firm postpones investment.
• If $\beta_1^{(\rho,\gamma)} > 0$ ($\xi < \lambda\rho + \frac{\eta}{2}$), value matching gives an expression for $L^{(\rho,\gamma)}$, and smooth pasting gives $\tilde{V}^{(\rho,\gamma)}$ solves (4) in the proposition.

Value of Option to Invest

The value achievable by investing in P and the riskless asset and receiving amount $p^{(\rho,\gamma)}$ for the option is compared with the value achievable by having the option

Proposition 4 *The manager's certainty equivalence valuation of the option to invest is given by*

$$p^{(\rho,\gamma)}(v) = -\frac{1}{\gamma(1-\rho^2)} \ln\left(1 - (1 - e^{-\gamma(\tilde{V}^{(\rho,\gamma)}-K)(1-\rho^2)})\left(\frac{v}{\tilde{V}^{(\rho,\gamma)}}\right)^{\beta_1^{(\rho,\gamma)}}\right).$$

where $\tilde{V}^{(\rho,\gamma)}$ *solves (4) and* $\beta_1^{(\rho,\gamma)}$ *is given in Proposition 3*

Optimal Stopping Representation - Risk Neutral and McDonald and Siegel Models

• In a classic risk-neutral model where V is perfectly spanned by P, the value of the option to invest can be expressed as

$$p^{(1)}(v) = \sup_{t \leq \tau < \infty} \mathbb{E}_t^{\mathbb{Q}}[e^{-r(\tau - t)}(V_\tau - Ke^{r(\tau-t)})^+ | V_t = v] \tag{5}$$

where $\mathbb{Q}$ is the risk-neutral pricing measure.

A similar representation holds under the model of McDonald and Siegel (1986) albeit involving the equilibrium expected rate of return on the investment, μ_e,

$$p^{(\rho)}(v) = \sup_{t \leq \tau < \infty} \mathbb{E}_t[e^{-\mu_e(\tau - t)}(V_\tau - Ke^{r(\tau-t)})^+ | V_t = v]. \tag{6}$$

Optimal Stopping Representation II

Proposition 5 *The manager's certainty equivalence valuation of the option to invest can be represented as*

$$p^{(\rho,\gamma)}(v) = \sup_{t \le \tau < \infty} -\frac{1}{\gamma(1-\rho^2)} \ln \mathbb{E}^{\mathbb{Q}^0}(e^{-\gamma(1-\rho^2)e^{-r(\tau-t)}(V_\tau - Ke^{r(\tau-t)})^+} | V_t = v)$$

where $\mathbb{E}^{\mathbb{Q}^0}$ *denotes expectation with respect to pricing measure* $\mathbb{Q}^0$*. Under* $\mathbb{Q}^0$*,*

$$\frac{dP}{P} = rdt + \sigma dB^0$$

where $B^0 = B + \lambda t$ *is a* $\mathbb{Q}^0$*-Brownian motion and the independent Brownian motion* Z *is unchanged under* $\mathbb{Q}^0$*.*
Project value V *follows under* $\mathbb{Q}^0$

$$\frac{dV}{V} = (\nu - \lambda\eta\rho)dt + \eta(\rho dB^0 + \sqrt{1-\rho^2}dZ) \qquad (7)$$

Optimal Stopping Representation III

- Representation of option value similar but -

(i) pricing measure is $\mathbb{Q}^0$ not $\mathbb{Q}$

(ii) value is non-linear function of payoff

- Other utility specifications would change valuation formula but not our later qualitative conclusions
- $\mathbb{Q}^0$ is the pricing measure compensating for market but not idiosyncratic risks (Z unchanged) - [minimal martingale measure]
- In fact the McDonald-Siegel valuation can be written as an expectation under $\mathbb{Q}^0$ of the option payoff

Recovering the Risk-Neutral and McDonald and Siegel valuations

Proposition 6 Risk-Neutral Valuation
Under the assumption $|\rho| = 1$, the risk-neutral Bellman equation is

$$\frac{1}{2}\eta^2 v^2 p_{vv}^{(1)}(v) + \eta(\xi - \lambda)v p_v^{(1)}(v) = 0 \tag{8}$$

with boundary, value matching and smooth pasting conditions

$$\begin{aligned} p^{(1)}(0) &= 0 && (9) \\ p^{(1)}(\tilde{V}^{(1)}) &= \tilde{V}^{(1)} - K && (10) \\ p_v^{(1)}(\tilde{V}^{(1)}) &= 1. && (11) \end{aligned}$$

The optimal investment time τ^* is given by

$$\tau^* = \inf\left\{u \geq t : V_u \geq \tilde{V}^{(1)} e^{r(u-t)}\right\}, \tag{12}$$

Let $\beta_1^{(1)} = 1 - \frac{2(\xi-\lambda)}{\eta}$. *If* $\beta_1^{(1)} > 1$ *(or equivalently* $\xi < \lambda$*), the risk-neutral value of the option to invest is*

$$p^{(1)}(v) = (\tilde{V}^{(1)} - K)\left(\frac{v}{\tilde{V}^{(1)}}\right)^{\beta_1^{(1)}} \tag{13}$$

and

$$\tilde{V}^{(1)} = \frac{\beta_1^{(1)}}{\beta_1^{(1)} - 1} K. \tag{14}$$

Investment is postponed forever if $\beta_1^{(1)} \leq 1$ *(or equivalently* $\xi \geq \lambda$*). The option value in this case is infinite.*

Proposition 7 McDonald and Siegel (1986) Valuation *Under the assumption of risk-aversion towards market risks and risk-neutrality towards idiosyncratic risks, the Bellman equation is*

$$\frac{1}{2}\eta^2 v^2 p_{vv}^{(\rho)}(v) + (\nu - r)v p_v^{(\rho)}(v) + (r - \mu^e)p^{(\rho)}(v) = 0 \tag{15}$$

with the same boundary conditions as in P6. μ_e *is the required rate of return on the investment in equilibrium. Optimal investment time* τ^* *is given by*

$$\tau^* = \inf\left\{u \geq t : V_u \geq \tilde{V}^{(\rho)} e^{r(u-t)}\right\}, \tag{16}$$

Under CAPM, the equilibrium rate of return on the project is given as $\hat{\nu} = r + \lambda\rho\eta$ *and*

$$\mu^e = r + \beta^{(\rho)}(\hat{\nu} - r) = r + \beta^{(\rho)}\lambda\rho\eta \tag{17}$$

where $\beta^{(\rho)}$ solves the quadratic

$$\frac{1}{2}\beta^{(\rho)}(\beta^{(\rho)}-1)\eta^2-\delta\beta^{(\rho)}=0$$

with $\delta=\hat{\nu}-\nu=r+\lambda\rho\eta-\nu$, the difference in the equilibrium expected rate of return and the expected return on the project. The Bellman equation becomes

$$\frac{1}{2}\eta^2 v^2 p_{vv}^{(\rho)}(v)-\delta v p_v^{(\rho)}(v)=0. \tag{18}$$

Let

$$\beta_1^{(\rho)}=1+\frac{2\delta}{\eta^2}=1-\frac{2(\xi-\lambda\rho)}{\eta}. \tag{19}$$

When $\beta_1^{(\rho)}>1$ (or equivalently, $\hat{\nu}>\nu$ or $\xi<\lambda\rho$), the value of the option to invest under the McDonald and Siegel (1986) model is

$$p^{(\rho)}(v)=(\tilde{V}^{(\rho)}-K)\left(\frac{v}{\tilde{V}^{(\rho)}}\right)^{\beta_1^{(\rho)}} \tag{20}$$

with

$$\tilde{V}^{(\rho)} = \frac{\beta_1^{(\rho)}}{\beta_1^{(\rho)} - 1} K. \tag{21}$$

Investment is postponed forever if $\beta_1^{(\rho)} \leq 1$ *(or equivalently* $\hat{\nu} \leq \nu$ *or* $\xi \geq \lambda\rho$*). The option value is infinite in this case.*

Proposition 8 *The valuation under the McDonald and Siegel (1986) model (in Proposition 7) can be re-expressed in terms of the pricing measure $\mathbb{Q}^0$ as*

$$p^{(\rho)}(v) = \sup_{t \le \tau < \infty} \mathbb{E}^{\mathbb{Q}^0} e^{-r(\tau - t)} (V_\tau - K e^{r(\tau - t)})^+.$$

Recovering the Risk Neutral and McDonald and Siegel Valuations II

Proposition 9 *Two special cases of the incomplete partial spanning model are:*
(A) **Risk-neutral:** *As* $\rho \to 1$,

$$(i)\ \beta_1^{(\rho,\gamma)} \to \beta_1^{(1)}; \qquad (ii)\ \tilde{V}^{(\rho,\gamma)} \to \tilde{V}^{(1)}; \qquad (iii)\ p^{(\rho,\gamma)}(v) \to p^{(1)}(v).$$

(B) **McDonald and Siegel (1986):** *As* $\gamma \to 0$,

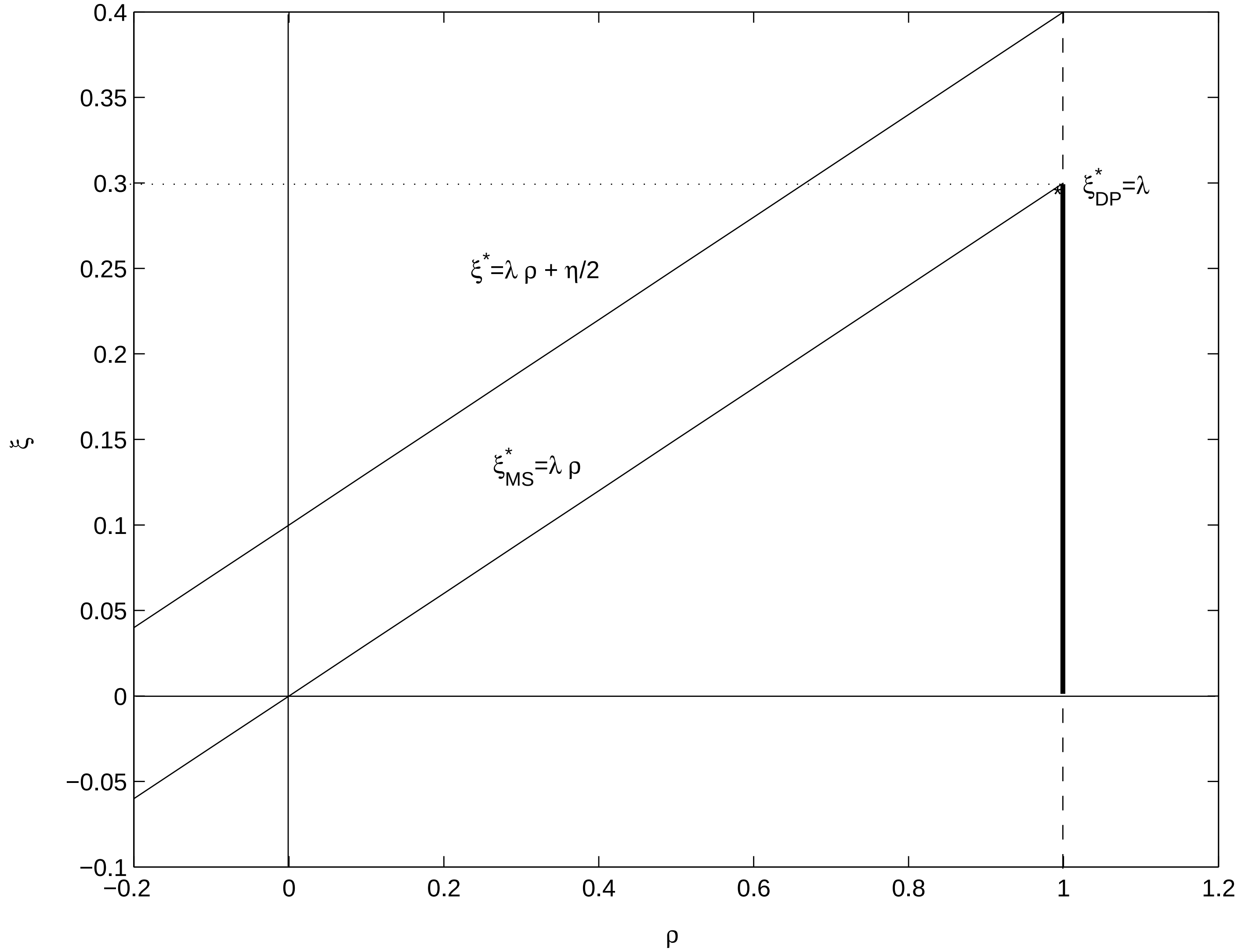
$$(i)\ \beta_1^{(\rho,\gamma)} \to \beta_1^{(\rho)}; \qquad (ii)\ \tilde{V}^{(\rho,\gamma)} \to \tilde{V}^{(\rho)}; \qquad (iii)\ p^{(\rho,\gamma)}(v) \to p^{(\rho)}(v)$$

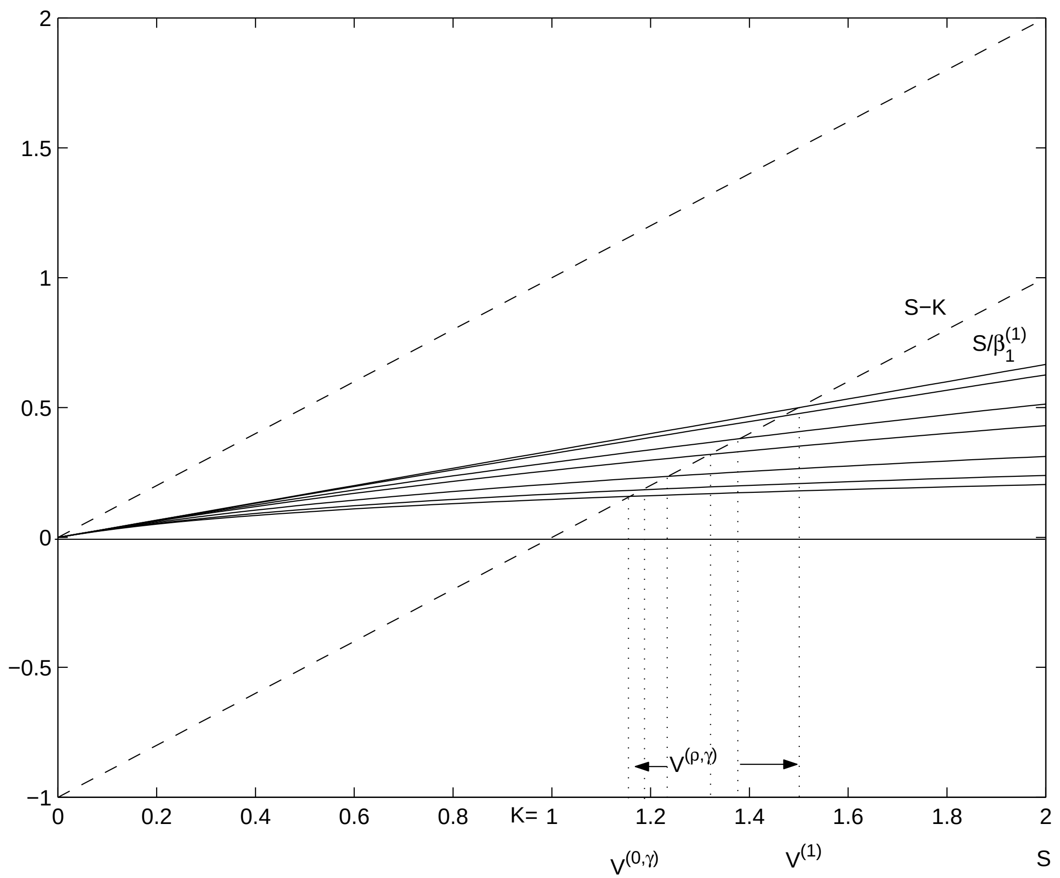
Qualitative Differences due to Incomplete Markets

Fix r, λ and η. Let $\xi^* = \xi^*(\rho, \gamma)$ be the largest value of the project's Sharpe ratio in the partial spanning model, given values of ρ and γ, for which there is a finite investment trigger, and for which the value of the option to invest is finite. Then $\xi^* = \lambda\rho + \frac{\eta}{2}$. Similarly, for the perfect spanning model, $\xi^*_{RN} = \xi^*_{RN}(1, \gamma) = \lambda$ and for the McDonald and Siegel model, $\xi^*_{MS} = \xi^*_{MS}(\rho, 0) = \lambda\rho$.

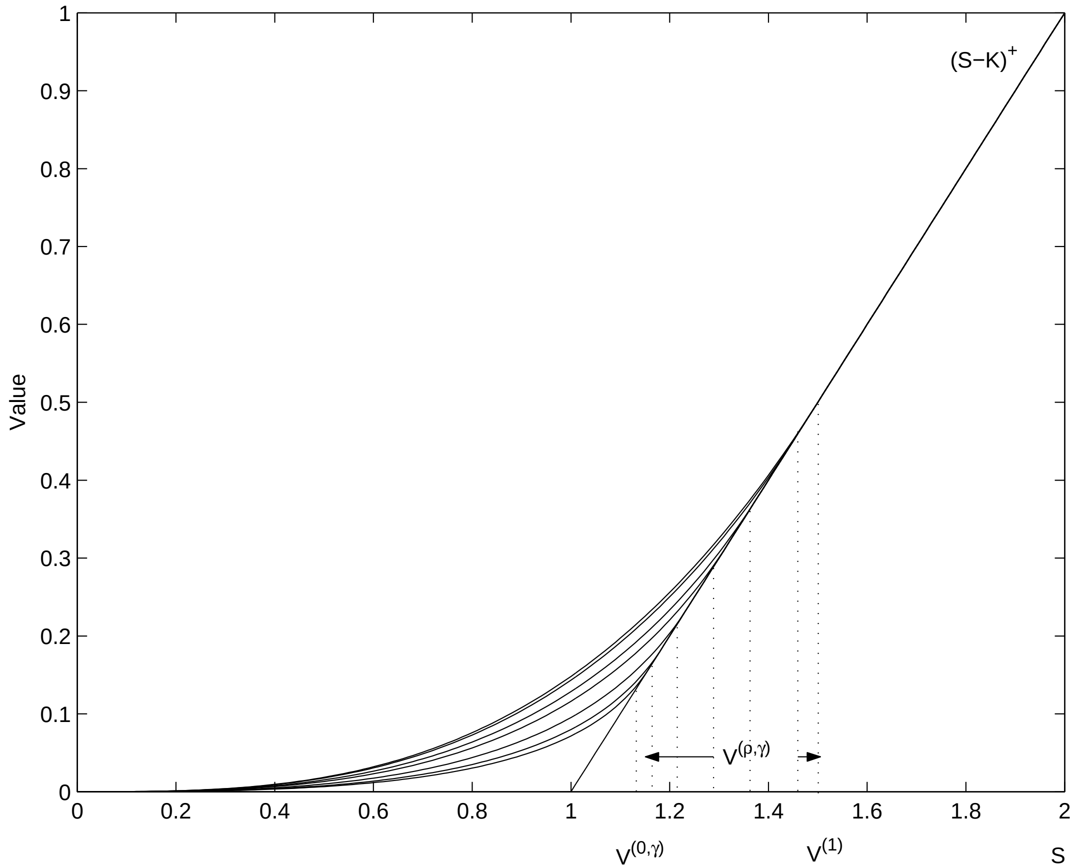
Theorem 10

(i) For $\eta > 0$, $\xi^*(\rho, \gamma)$ does not tend to $\xi^*_{RN}(1, \gamma)$ as $\rho \to 1$;
(ii) For $\eta > 0$, $\xi^*(\rho, \gamma)$ does not tend to $\xi^*_{MS}(\rho, 0)$ as $\gamma \to 0$.

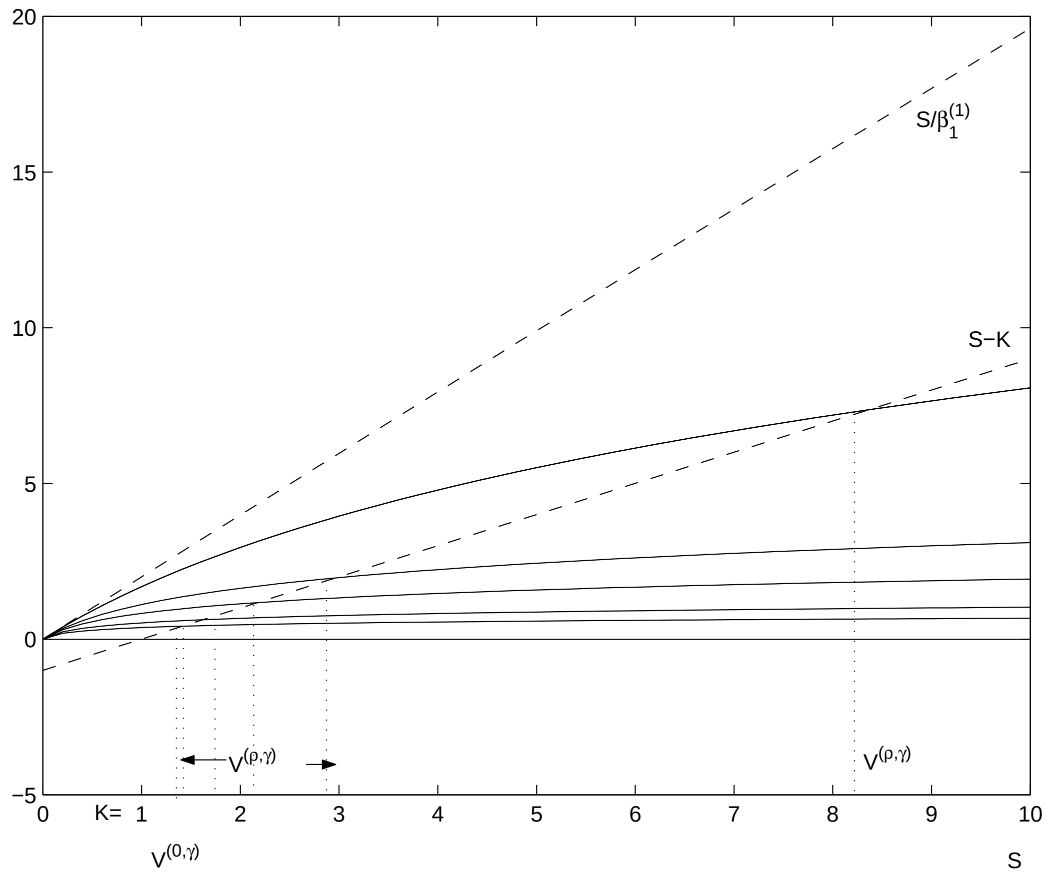

0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
−0.05
−0.1
ξ
ξ*=λ ρ + η/2
ξ*MS=λ ρ
ξ*DP=λ
−0.2
0
0.2
0.4
0.6
0.8
1
1.2
ρ



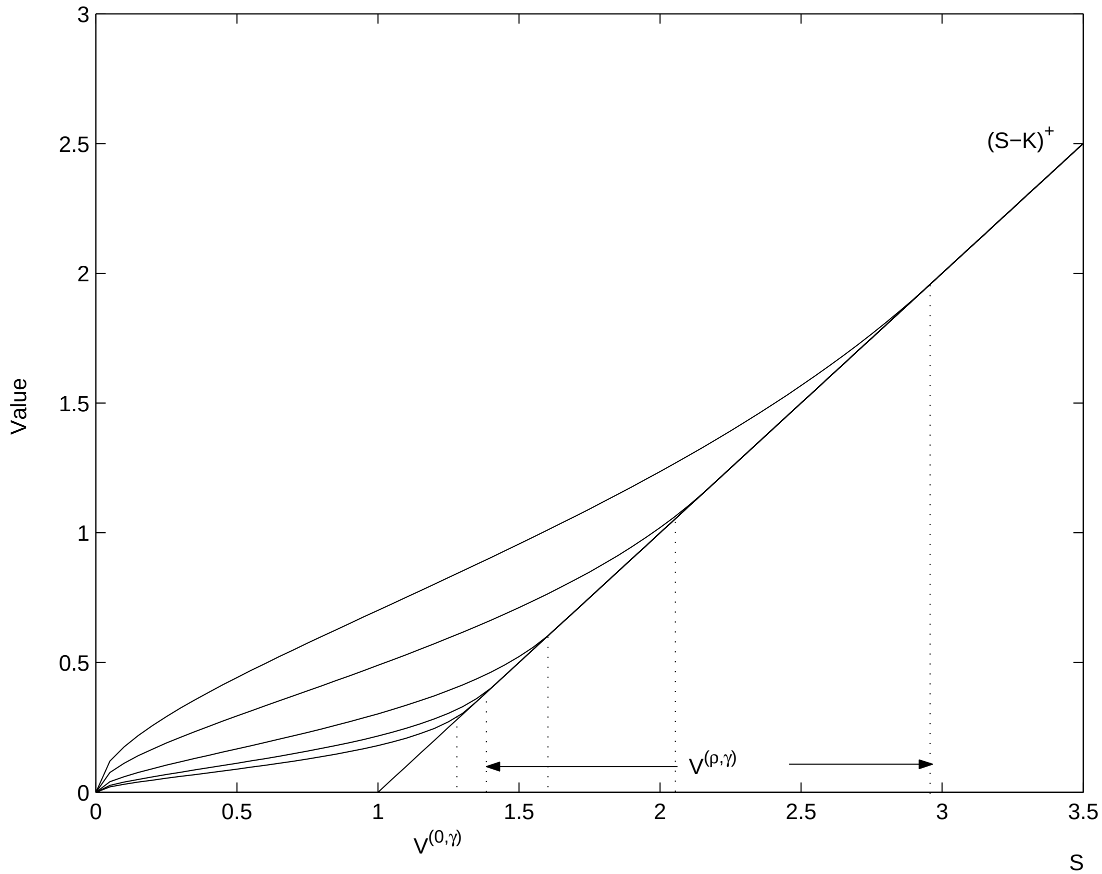
Case $\beta_1^{(\rho,\gamma)} > 1$ (or $\xi < \xi^*_{MS}$). The figure shows the investment trigger $\tilde{V}^{(\rho,\gamma)}$ for a range of correlations.



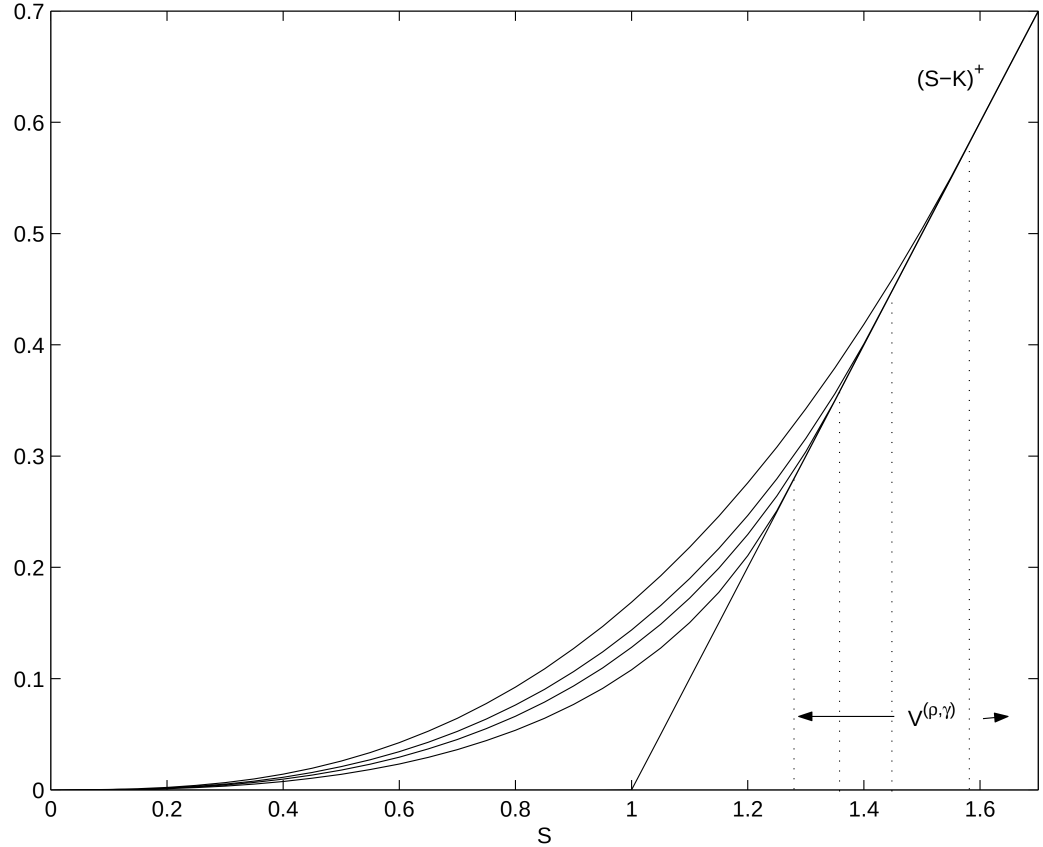
Case $\beta_1^{(\rho,\gamma)} > 1$ (or $\xi < \xi^*_{MS}$). The figure shows the value of the option to invest for a range of correlations.



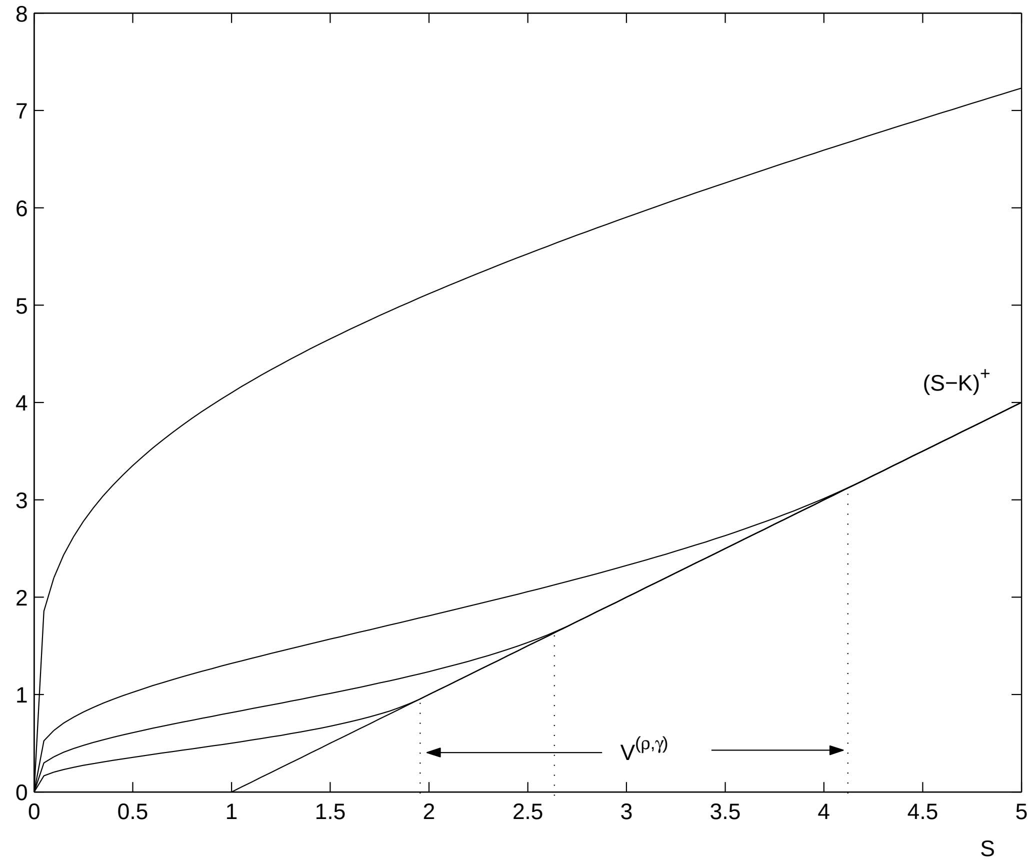
Case $0 < \beta_1^{(\rho,\gamma)} \leq 1$ (or $\xi^*_{MS} \leq \xi < \xi^*$). The figure shows the investment trigger $\tilde{V}^{(\rho,\gamma)}$ for a range of correlations.



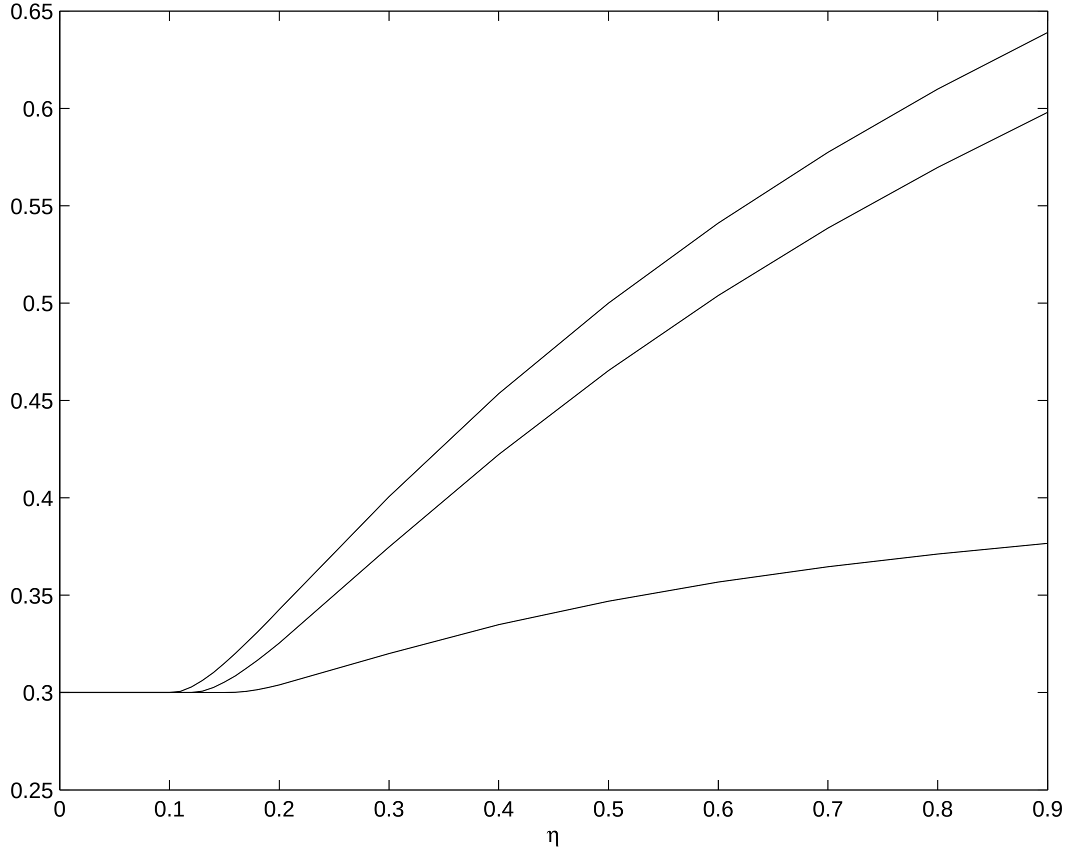
Case $0 < \beta_1^{(\rho,\gamma)} \leq 1$ (or $\xi^*_{MS} \leq \xi < \xi^*$). The figure shows the value of the option to invest for a range of correlation values.



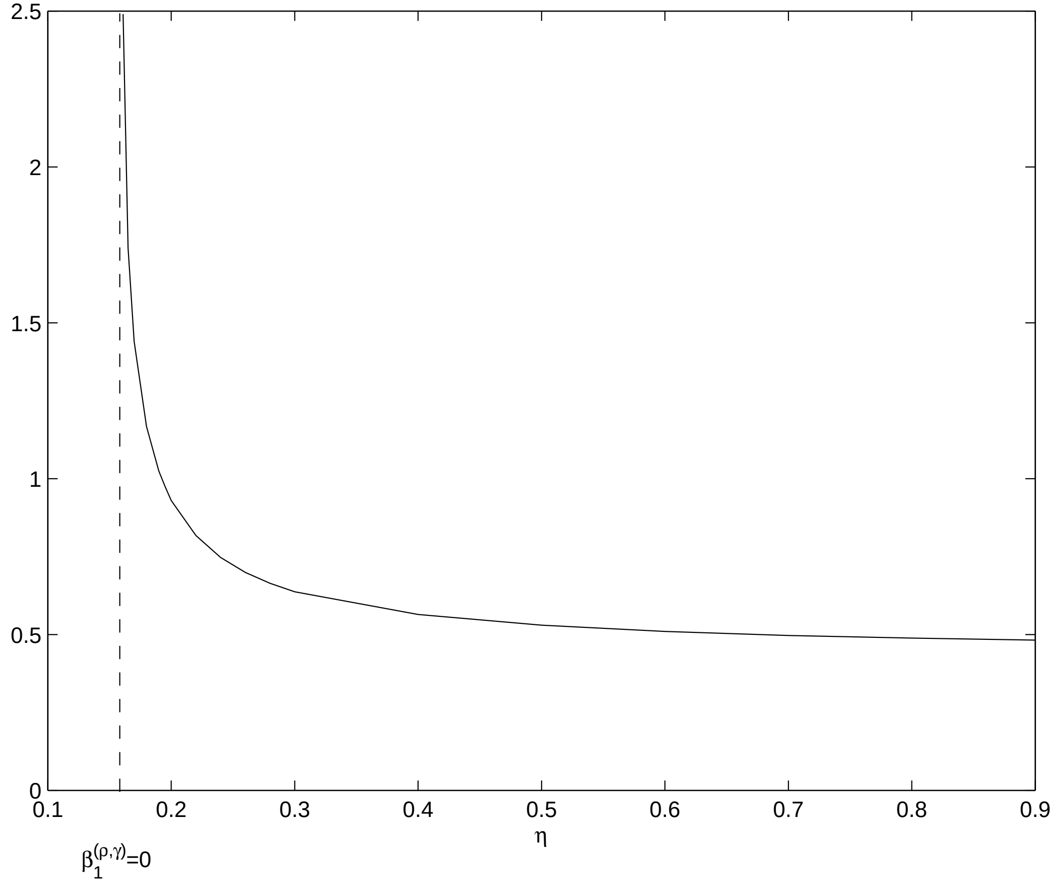
Case $\beta_1^{(\rho,\gamma)} > 1$. The figure shows the value of the option to invest for a range of γ against the discounted project value for a fixed correlation $\rho = 0.9$.



Case $0 < \beta_1^{(\rho,\gamma)} \leq 1$. The figure shows the value of the option to invest for a range of γ against the discounted project value for a fixed correlation $\rho = 0.9$



Case $\beta_1^{(\rho,\gamma)} > 1$. The figure shows the value of the option to invest for a range of η for correlation $\rho = 0.9$



Case $0 < \beta_1^{(\rho,\gamma)} \leq 1$. The figure shows the value of the option to invest for a range of η for correlation $\rho = 0.9$

Convexity of Option Value

Proposition 11 *(i) If* $\beta_1^{(\rho,\gamma)} \geq 1$*, or equivalently* $\xi \leq \xi^*_{MS}$*,* $\frac{\partial^2}{\partial v^2} p^{(\rho,\gamma)}(v) > 0$ *and the value of the option is convex in* v*.*
(ii) If $0 < \beta_1^{(\rho,\gamma)} < 1$*, or equivalently* $\xi^*_{MS} < \xi < \xi^*$*, the value of the option may be convex or concave depending on the value of* v*.*

- Mixed effect - usual convexity from option payoff but also concavity from manager's utility function.
- Case (i): convexity dominates
- Case (ii): Convex near where value and payoff meet due to value matching and smooth pasting condition. But for low V, utility function has larger proportional effect for low option values, and concavity dominates

Conclusions and Further Research

• The partial spanning asset extends the classic models: the complete model and the McDonald and Siegel (1986) model. Both are recovered as limiting cases.

• Classic models are overstating the worth of the option to invest and recommending a firm waits too long to invest

• Approximating investment decisions with classic models can lead to the wrong decision.

• Widely held belief that a complete model is a good approximation in an "almost complete" situation is incorrect.

Conclusions and Further Research II

- Other Utilities ? Less tractable. Power utility + when to buy/sell asset (with J Evans and D Hobson)
- Can be extended to: mean-reverting project value, option to abandon etc, finite horizon
- Corporate finance applications ? Over-investment problem (with Pierre Mella-Barral)
- Empirical Testing ?
- Competition ?