

©

I) RELATIVISTIC SCATTERING
THEORY

II) GAUGE SYMMETRY BREAKING

III) THE QUANTUM HALL EFFECT

TODAY, SCATTERING THEORY

FIRST CLASSICAL,

THEN QUANTUM MECHANICAL

(1)

WE CONSIDER IN $R^{1,3}$

$$ds^2 = -dt^2 + d\vec{x}^2$$

A NONLINEAR EQUATION SUCH
AS THIS ONE

$$\square \phi - m^2 \phi - \gamma \phi^3 = 0$$

$$\square = -\frac{\partial^2}{\partial t^2} + \vec{\nabla}^2$$

IS THE LINEAR WAVE
OPERATOR

(2)

THE SPACE OF CLASSICAL
SOLUTIONS IS AN INFINITE-DIMENSIONAL
SYMPLECTIC MANIFOLD m .

FAR PAST OR FAR FUTURE - THE
WAVE DISPERSES ... EQN REDUCES
TO A LINEAR WAVE EQN THAT

$$(\square - m^2) \phi = 0$$

HAS A SPACE OF SOLUTIONS
WE CALL m_0

THE EVOLUTION FROM THE PAST

(3)

TO THE FUTURE

LINEAR DATA

/ | ,

STRONG
FIELDS

↑
TIME

/ | \

LINEAR DATA

GIVES A "SYMPLECTOMORPHISM"

FROM ω_0 TO ω_0 THAT WE CALL

S_{cl} (THE CLASSICAL LIMIT OF THE
S-MATRIX)

(4)

QUANTUM MECHANICALLY, BY

QUANTIZING THE SYMPLECTIC

MANIFOLDS m AND m_0 WE

GET HILBERT SPACES \mathcal{H} AND \mathcal{H}_0

AND S_{cl} IS REPLACED BY A

UNITARY TRANSFORMATION

$$S : \mathcal{H}_0 \rightarrow \mathcal{H}_0$$

(MY DEFINITION OF \mathcal{H}_0 IS A BIT
OVERSIMPLIFIED)

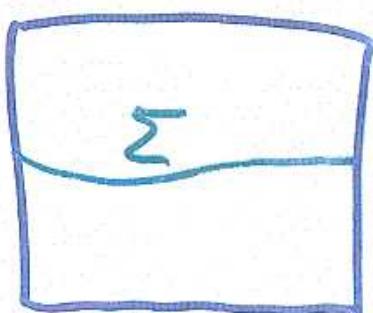
TO BEGIN TO UNDERSTAND THIS, WE
SHOULD AT LEAST QUANTIZE THE
LINEAR SPACE m_0 OF SOLUTIONS OF

$$(\square - m^2)\phi = 0$$

TO GET THE "FREE" HILBERT
SPACE \mathcal{H}_0 .

THE SYMPLECTIC STRUCTURE ON THE
LINEAR SPACE m_0 IS

$$\omega(\phi_1, \phi_2) = \int_{\Sigma} (\phi_1 * d\phi_2 - \phi_2 * d\phi_1)$$



WHERE Σ IS ANY
CAUCHY HYPERSURFACE

* = HODGE STAR

(6)

WE CAN EXPLICITLY SOLVE THE
LINEAR WAVE EQUATION BY FOURIER
TRANSFORM

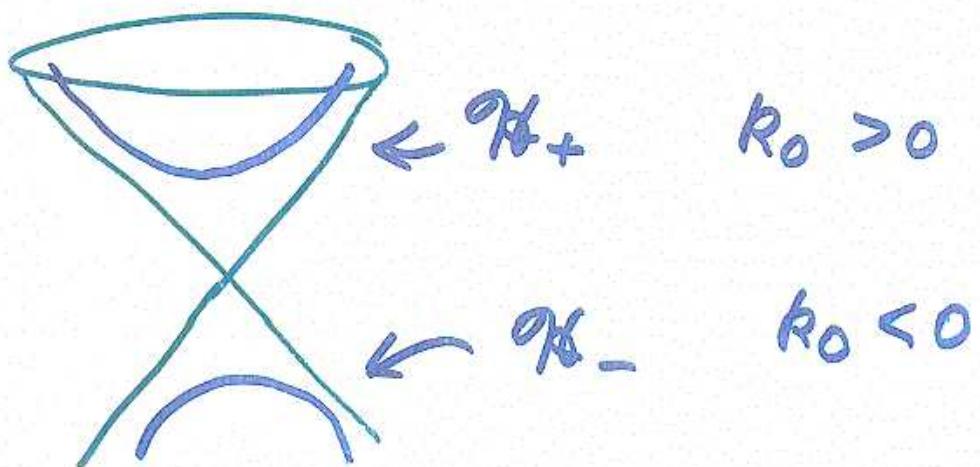
$$k \in V$$

LET $V = (R^{3/1})^*$ WITH

$$(k, k) = -k_0^2 + \vec{k}^2 \quad k = (k_0, \vec{k})$$

THE LORENTZ INVARIANT INNER
PRODUCT. IN V LET \mathcal{H} BE

THE "QUADRATIC" $(k, k) + m^2 = 0$



(7)

THE QUADRIC HAS TWO
COMPONENTS BECAUSE OF THE
LORENTZ SIGNATURE, THE

EQN.

$$(k, k) + m^2 = 0$$

BEING EXPLICITLY

$$-k_0^2 + \vec{k}^2 + m^2 = 0$$

i.e.

$$k_0 = \pm \sqrt{\vec{k}^2 + m^2}$$

(8)

THE QUADRIC HAS A NATURAL

MEASURE $d\mu$, AND THE

GENERAL SOLUTION OF THE

WAVE EQN IS $(-\square + m^2) e^{ik \cdot x}$
 $= ((k, k) + m^2) e^{ik \cdot x}$

$$\phi(x) = \int_{\mathbb{R}^n} d\mu \exp ik \cdot x a(k)$$

$$\text{WHERE } k \cdot x = k_0 t + \vec{k} \cdot \vec{x}$$

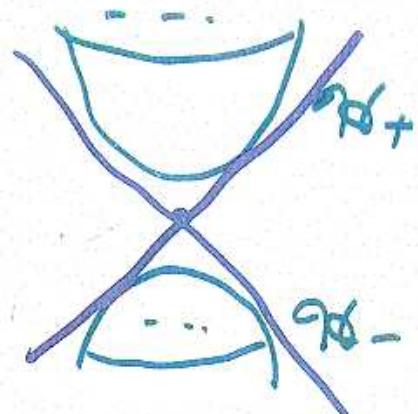
AND ϕ IS REAL IFF

$$a(k) = \bar{a}(-k)$$

(9)

WE CAN NOW SEE THAT

m_0 HAS A NATURAL
COMPLEX STRUCTURE



A GENERAL LINEAR

COMPLEX-VALUED FUNCTION ON m_0

IS

$$\int_{\mathcal{H}^+} d\mu f(k) a(k) + \int_{\mathcal{H}^-} d\mu g(k) a(k)$$

WE DEFINE THE COMPLEX STRUCTURE
ON m_0 BY DECLARING

$$\int_{\mathcal{H}^+} d\mu f(k) a(k) \quad \text{TO BE HOLOMORPHIC}$$

ON A SYMPLECTIC MANIFOLD SUCH

AS m_0 , A COMPLEX STRUCTURE

J IS SAID TO BE COMPATIBLE

WITH THE SYMPLECTIC STRUCTURE ω

IF ω IS OF TYPE $(1,1)$ AND

POSITIVE, SO m_0 , ENDOUED

WITH J AND ω , BECOMES KAHLER.

IN THE PRESENT CASE, THESE

CONDITIONS ARE OBeyed.

QUANTIZATION OF A LINEAR

(11)

KAHLER MANIFOLD m_0 IS CARRIED OUT

BY TAKING THE HILBERT SPACE

\mathcal{H}_0 TO BE THE SPACE OF

HOLOMORPHIC FUNCTIONS ON m_0

(ACTUALLY - HOLOMORPHIC SECTIONS OF A

CERTAIN TRIVIAL HOLOMORPHIC BUNDLE THAT

ARE L^2 IN A CERTAIN SENSE)

IN PRACTICE, WE CAN JUST USE

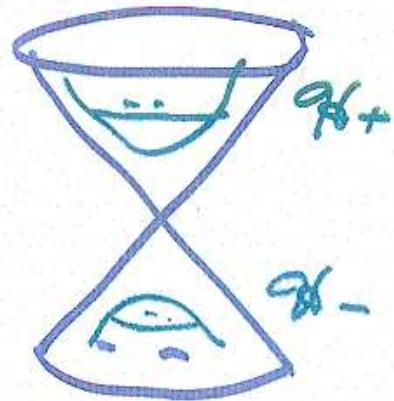
POLYNOMIAL FUNCTIONS.

(12)

THE LINEAR HOLOMORPHIC
 FUNCTIONS ON \mathfrak{M}_0 ARE OUR
 FRIENDS

$$\int_{\mathcal{H}^+} d\mu f(k) \alpha(k)$$

WHERE f IS ANY
 COMPLEX-VALUED
 FUNCTION ON \mathcal{H}^+ .



THE HILBERT SPACE STRUCTURE IS
 WHAT YOU MIGHT GUESS; IF ψ_f
 IS SUCH A STATE THEN

$$\langle \psi_f, \psi_f \rangle = \int_{\mathcal{H}^+} d\mu \bar{f} f$$

WE WRITE $|f\rangle$ FOR ψ_f .

(13)

THE SPACE OF POLYNOMIALS

ON m_0 IS

$$\mathcal{H}_0 = \mathbb{C} \cdot 1 \oplus \mathcal{V} \oplus \text{Sym}^2 \mathcal{V} \oplus \text{Sym}^3 \mathcal{V} \oplus \dots$$

φ constant functions	φ linear functions	φ quadratic functions	φ cubic functions
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WHICH IS OFTEN ABBREVIATED

$$\mathcal{H}_0 = \text{Sym}^* \mathcal{V}$$

WE CALL "1" THE VACUUM,
 \mathcal{V} THE SPACE OF ONE-PARTICLE STATES,
 $\text{Sym}^2 \mathcal{V}$ THE SPACE OF TWO-PARTICLE
 STATES, ETC.

THE GROUP OF TRANSLATIONS
 $x \in \mathbb{R}^{1,3}$

$$x^{\mu} \rightarrow x^{\mu} + a^{\mu}$$

(WHERE $x^{\mu} = (x^0, \vec{x}) = (t, \vec{x})$)

ACTS IN AN OBVIOUS WAY ON

THE SPACE m_0 OF SOLUTIONS OF

$$(\square - m^2)\phi = 0$$

PRESERVING THE COMPLEX STRUCTURE
 OF m_0 AND HENCE MAPPING
 THE SPACE OF k^{th} ORDER
 HOMOGENEOUS HOLOMORPHIC POLYNOMIAL
 FUNCTIONS TO ITSELF.

(15)

THE CONSTANT FUNCTION 1 ON
 \mathcal{M}_0 IS OF COURSE INVARIANT.

LINEAR COMPLEX FUNCTIONS

$$\psi_f \leftrightarrow \int_{\mathcal{H}^+} d\mu f(k) a(k)$$

WHERE

$$\phi(x) = \int_{\mathcal{H}} d\mu a(k) e^{ik \cdot x}$$

$$e^{-ik \cdot a}$$

$$a \rightarrow a e^{-ik \cdot a}$$

TRANSFORM UNDER $x \rightarrow x+a$

BY $f(k) \rightarrow f(k) e^{ik \cdot a}$

THIS DETERMINES THE ACTION ON HIGHER
 POLYNOMIALS

(16)

TO DIAGONALIZE THE ACTION OF
 THE TRANSLATIONS ON ONE-PARTICLE
 STATES, WE JUST TAKE $f(k) = \delta(k, k^*)$
 FOR SOME $k^* \in \mathbb{H}_+$

SO $f(k) \rightarrow f(k) e^{ik^* \cdot a}$

THE CORRESPONDING STATE

$$\psi_{k^*} \leftrightarrow f = \delta(k, k^*)$$

TRANSFORMS LIKE

$$\psi_{k^*} \rightarrow \psi_{k^*} e^{ik^* \cdot a}$$

(17)

THE "ENERGY-MOMENTUM OPERATORS"
 ARE THE LIE ALGEBRA OF
 THE TRANSLATION GROUP

$$P_m = -i\hbar \frac{\partial}{\partial x_m}$$

⇒

$$P_m \psi_{k^*} = \hbar k_m \psi_{k^*}$$

i.e.

$$\boxed{E = \hbar \omega}$$

WHERE E IS THE ENERGY AND
 $\omega = k_0$ IS THE "FREQUENCY"

(18)

WE HAVE GOTTEN PARTICLES, OF

ENERGY

$$A e^{i\omega t}$$

$$E = \hbar\omega$$

$$A \cos\omega t$$

$$E = A^2\omega^2$$

FROM WAVES OF FREQUENCY ω

VIA QUANTIZATION. NOTE THAT

$\omega > 0$ AS WE ARE ON \mathbb{R}^+ .

THE "VACUUM" (\leftrightarrow CONSTANT FUNCTION
1 ON \mathbb{R}^0)

HAS ZERO ENERGY, AND OTHER STATES

HAVE POSITIVE ENERGY.

IF WE WOULD REPEAT THIS
EXERCISE FOR THE ELECTROMAGNETIC
FIELD, WE WOULD LEARN HOW
THAT FIELD IS REINTERPRETED
QUANTUM MECHANICALLY IN TERMS
OF PHOTONS.

(19)

(10)

A k -PARTICLE STATE IS

AN ELEMENT OF

Sym^{k^n}

$V: f: \mathcal{H}_+ \rightarrow \mathbb{C}$

WHICH WE CAN INTERPRET AS THE

SPACE OF FUNCTIONS

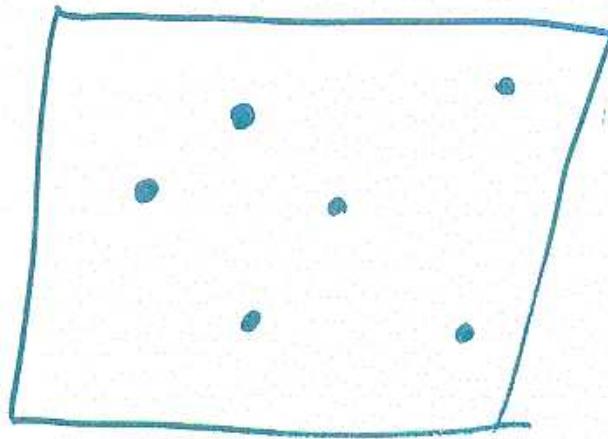
$f(k_1, \dots, k_n) : \text{Sym}^{k^n} \mathcal{H}_+ \rightarrow \mathbb{C}$

f IS SYMMETRIC IN ITS ARGUMENTS

\Leftrightarrow WE CAN SAY WHAT MOMENTA
THE PARTICLES HAVE (OR WHERE THEY
ARE IN SPACE) BUT WE CANNOT SAY
WHICH PARTICLE HAS WHICH
MOMENTA.

(21)

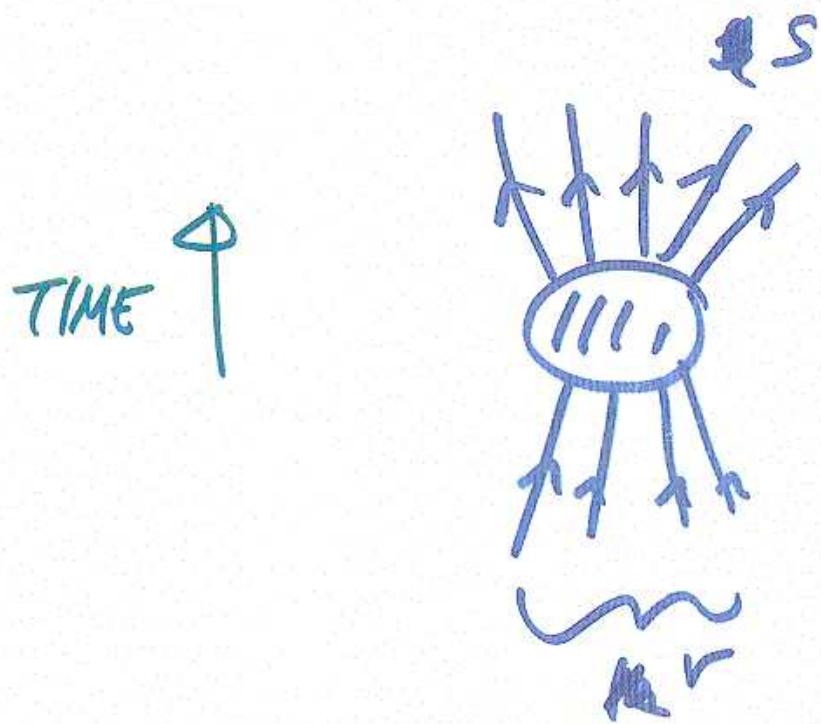
THE PARTICLES ARE "INDISTINGUISHABLE"
IN A VERY STRONG SENSE



A FACT THAT HAD BEEN FIRST
DISCOVERED IN THE LATE 19TH
CENTURY WHEN STATISTICAL MECHANICS WAS
APPLIED TO CHEMICAL REACTIONS.

(22)

NOW WE WANT TO DISCUSS
SCATTERING



r PARTICLES IN, s PARTICLES OUT

INITIAL STATE ψ_f FOR SOME

$$f(k_1, k_2, \dots, k_r) \quad k_i \in \mathbb{N}_+$$

FINAL STATE ψ_g FOR SOME

$$g(k_1, \dots, k_s) \quad k_i \in \mathbb{N}_+$$

SCATTERING "MATRIX" S

(23)

THE MATRIX ELEMENT OF THE

S-MATRIX IS

$$\langle \psi_g, S \psi_f \rangle$$

IT MUST BE A LINEAR FUNCTION

OF $f : \text{Sym}^r \mathcal{H}_+ \rightarrow \mathbb{C}$

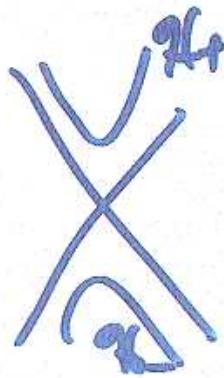
AND AN ANTILINEAR FUNCTION

OF $g : \text{Sym}^s \mathcal{H}_+ \rightarrow \mathbb{C}$

IT IS INCONVENIENT TO
DEAL WITH AN ANTILINEAR
FUNCTION OF g . ONE USUALLY
AVOIDS THIS BY DEFINING g

ON $\text{Sym}^s \mathcal{X}_-$ BY

$$g(k_1, \dots, k_s) = \bar{g}(-k_1, \dots, -k_s)$$



SO NOW WE HAVE

$$\langle \psi_g, S \psi_f \rangle$$

$$= \int d\mu \quad \int d\mu$$

$$\text{Sym}^r \mathcal{H}^+ + \text{Sym}^s \mathcal{H}^-$$

$$f(k_1 \dots k_r) \quad g(k'_1 \dots k'_s)$$

$$S(k_1 \dots k_r; k'_1 \dots k'_s)$$

WHERE S IS A FUNCTION ON

$$\text{Sym}^r \mathcal{H}^+ \times \text{Sym}^s \mathcal{H}^-$$

NOW I AM GOING TO EXPLAIN
SOME THING THAT I HOPE
YOU'LL FIND REMARKABLE -
IT IS REALLY THE REASON THAT
I HAVE CHOSEN TO GIVE THIS
LECTURE.

(27)

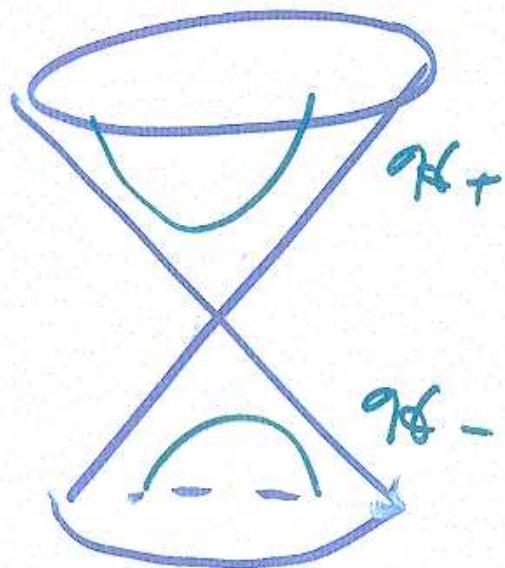
THOUGH THE REAL QUADRIC

$$q_k: (k, k) + m^2 = 0$$

HAS TWO

COMPONENTS q_{k+}

AND q_{k-} ,



THE CORRESPONDING COMPLEX

QUADRIC - DEFINED BY THE SAME

EQUATION FOR A COMPLEX VARIABLE k

- IS CONNECTED AND IRREDUCIBLE

THE FUNCTION

$$S(k_1 \dots k_r; k'_1 \dots k'_s)$$

CAN BE ANALYTICALLY CONTINUED

IN ALL VARIABLES TO A

FUNCTION ON

$$\mathcal{H}_C \times \mathcal{H}_C \times \dots \times \mathcal{H}_C$$

$\underbrace{\hspace{10em}}$
 $r+s$

AND THIS FUNCTION IS

SYMMETRIC IN ALL VARIABLES

(29)

THUS THERE IS A HOLOMORPHIC
FUNCTION

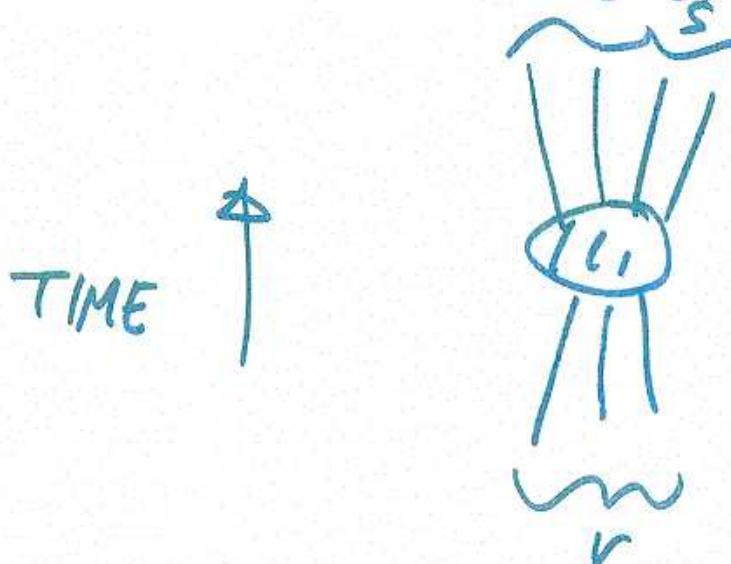
$$S(k_1, k_2, \dots, k_{r+s})$$

ON $\text{Sym}^{r+s} \mathcal{H}_C$

WHOSE RESTRICTION TO

$$\text{Sym}^r \mathcal{H}_+ \times \text{Sym}^s \mathcal{H}_-$$

DESCRIBES OUR REACTION



THE SAME ANALYTIC FUNCTION

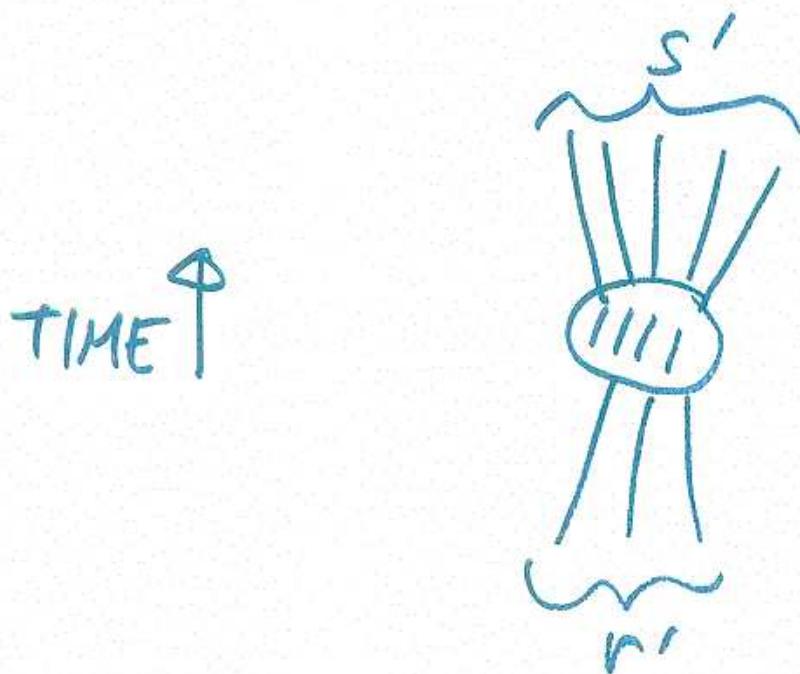
$S(k_1, k_2, \dots, k_m)$ CAN ALSO

BE RESTRICTED TO

$$\text{Sym}^{r'} \mathcal{H}_+ \times \text{Sym}^{s'} \mathcal{H}_-$$

WHERE $r' + s' = r+s$ AND

DESCRIBES OTHER REACTIONS



(31)

WE CAN ANALYTICALLY CONTINUE
FROM THE PAST TO THE FUTURE

AND BACK, A PROCESS THAT

PHYSICISTS CALL "CROSSING SYMMETRY"

(ca. 1950)

ONE HOLOMORPHIC FUNCTION

$S(k_1, \dots, k_n)$ DESCRIBES

MANY REACTIONS.

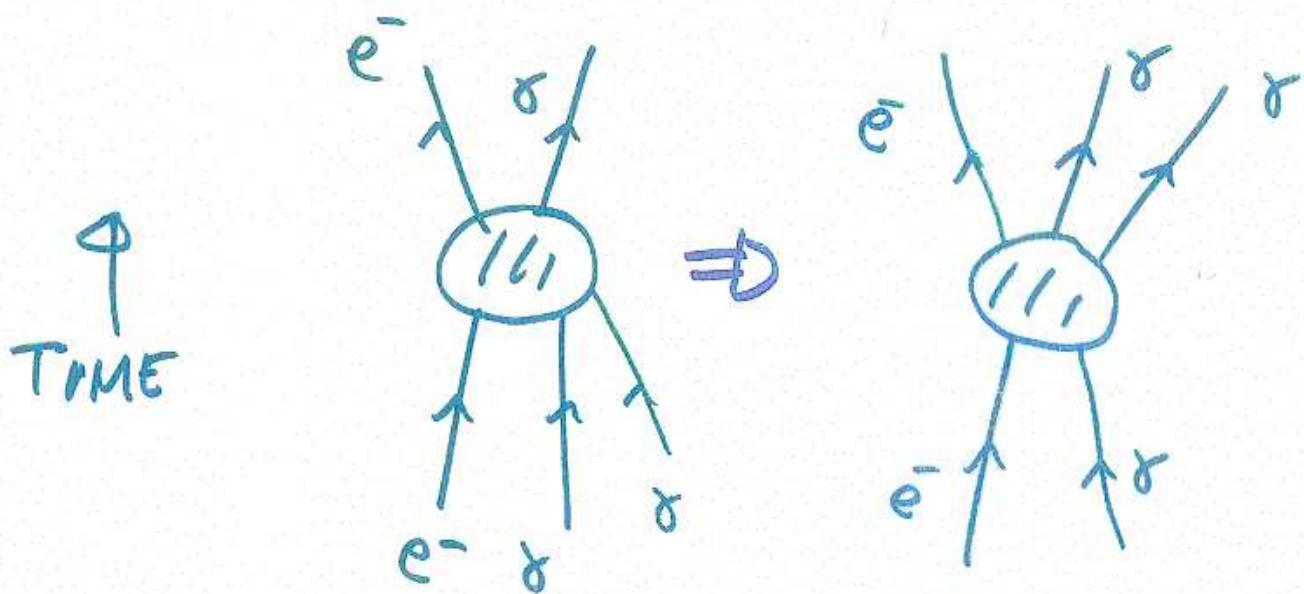
FROM THIS WE CAN DEDUCE
ONE OF THE MOST STARTLING
INSIGHTS OF THE TWENTIETH
CENTURY --- ACTUALLY ACHIEVED
AROUND 1930 IN A DIFFERENT WAY.

WE APPLY CROSSING SYMMETRY

(33)

TO THE REAL WORLD OF

ELECTRONS AND PHOTONS

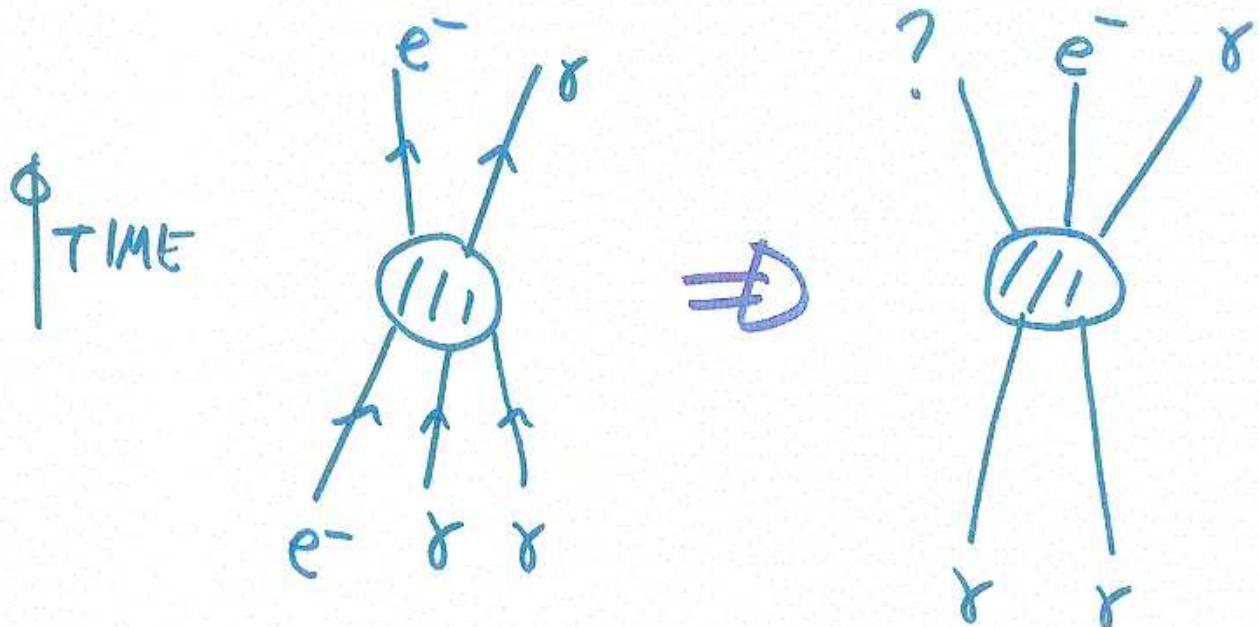


ANALYTIC CONTINUATION OF A
PHOTON FROM THE PAST TO
THE FUTURE

(34)

NOW TRY IT FOR AN

ELECTRON:



THE OBJECT LABELED ?

MUST HAVE POSITIVE ELECTRIC

CHARGE. ANALYTIC CONTINUATION

FROM PAST TO FUTURE REVERSES

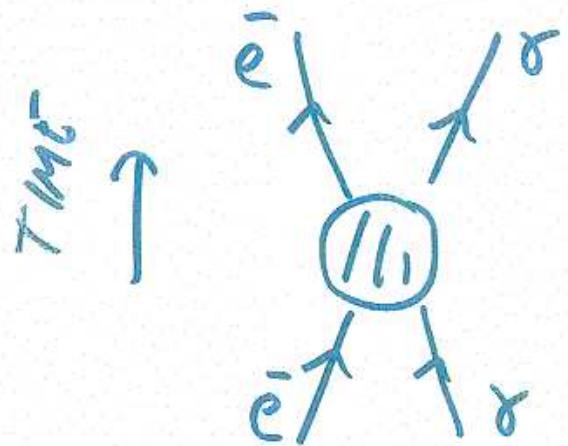
THE SIGN OF THE CHARGE.

THIS MYSTERY OBJECT IS

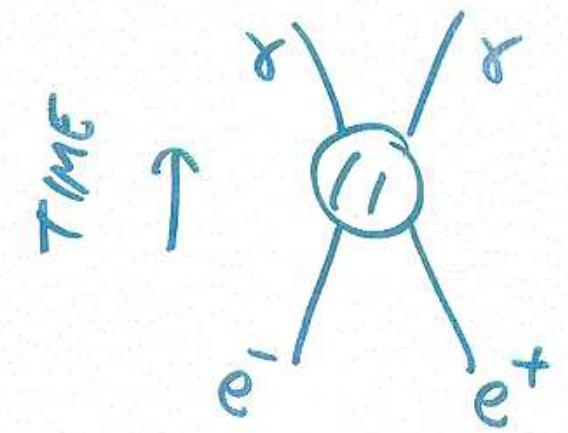
e^+ , THE POSITRON

OR ANTI-MATTER COUNTER

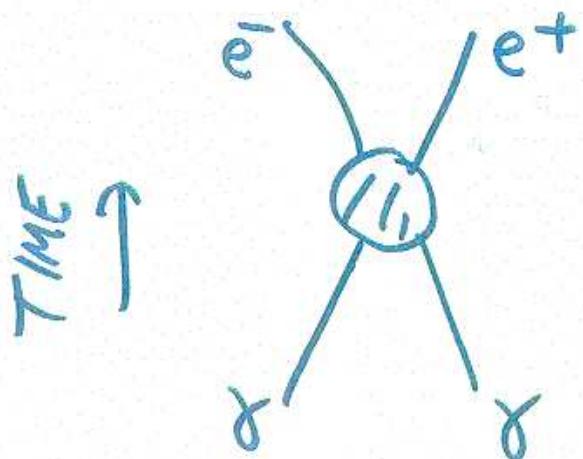
OF THE ELECTRON.



ELECTRON-PHOTON
SCATTERING



MATTER-ANTIMATTER
ANNIHILATION
TO RADIATION



CREATION OF
MATTER AND
ANTIMATTER
FROM RADIATION