COXETER LECTURES – FIELDS INSTITUTE, TORONTO

The Mathematics of String Theory

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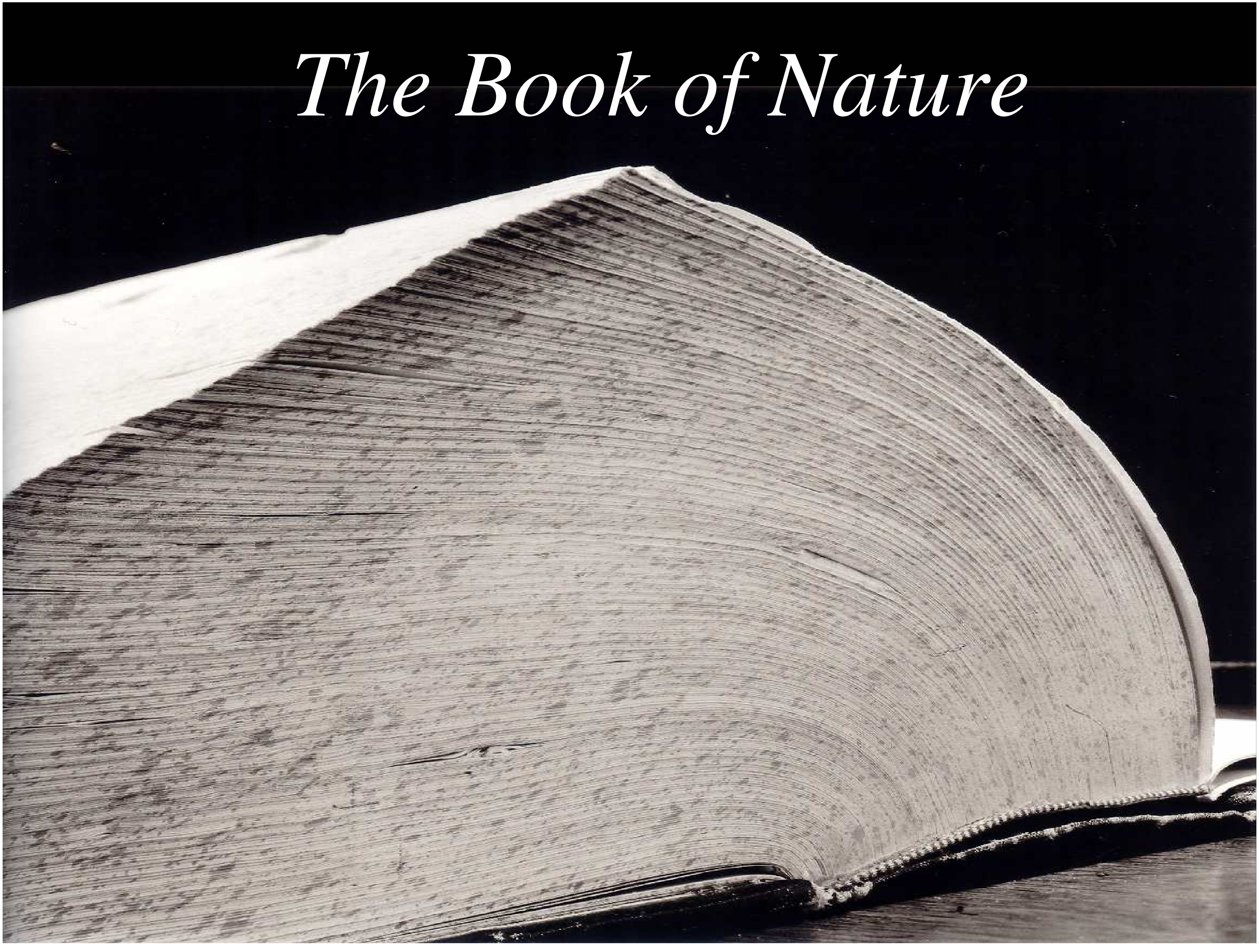
University of Amsterdam

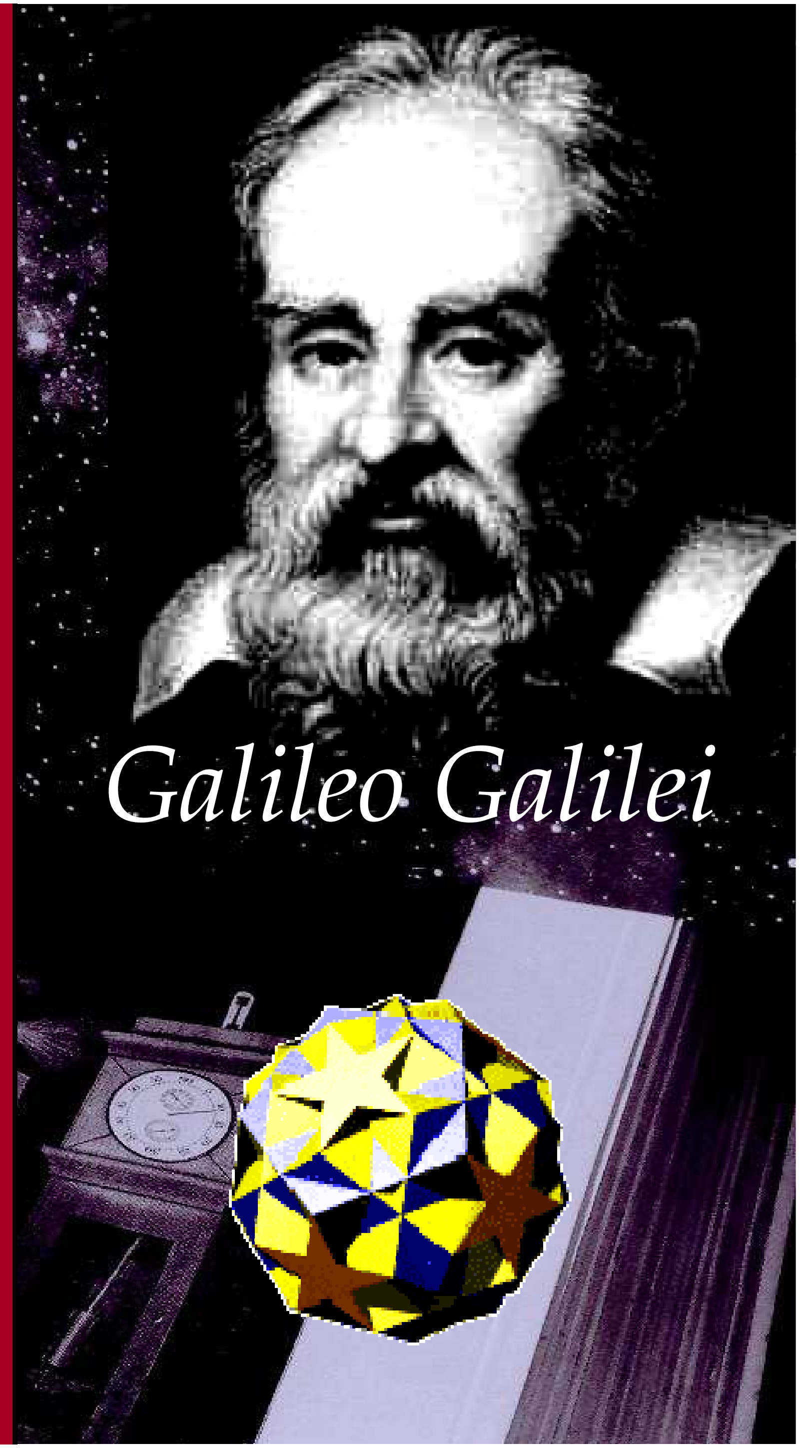
“The Unreasonable Effectiveness of Mathematics in the Natural Sciences.”


left derived
FUNCTORS
NOW
GROUPS LIE ALGEBRAS
HOMOTOPY FOR YOU!
GAUSS
TENSOR
ISOMETS OPERADS MOTIVES
MATH
CATEGORIES HERE!
MORPHISM
P-ADIC
PROBLEMS
PHYS

The Book of Nature



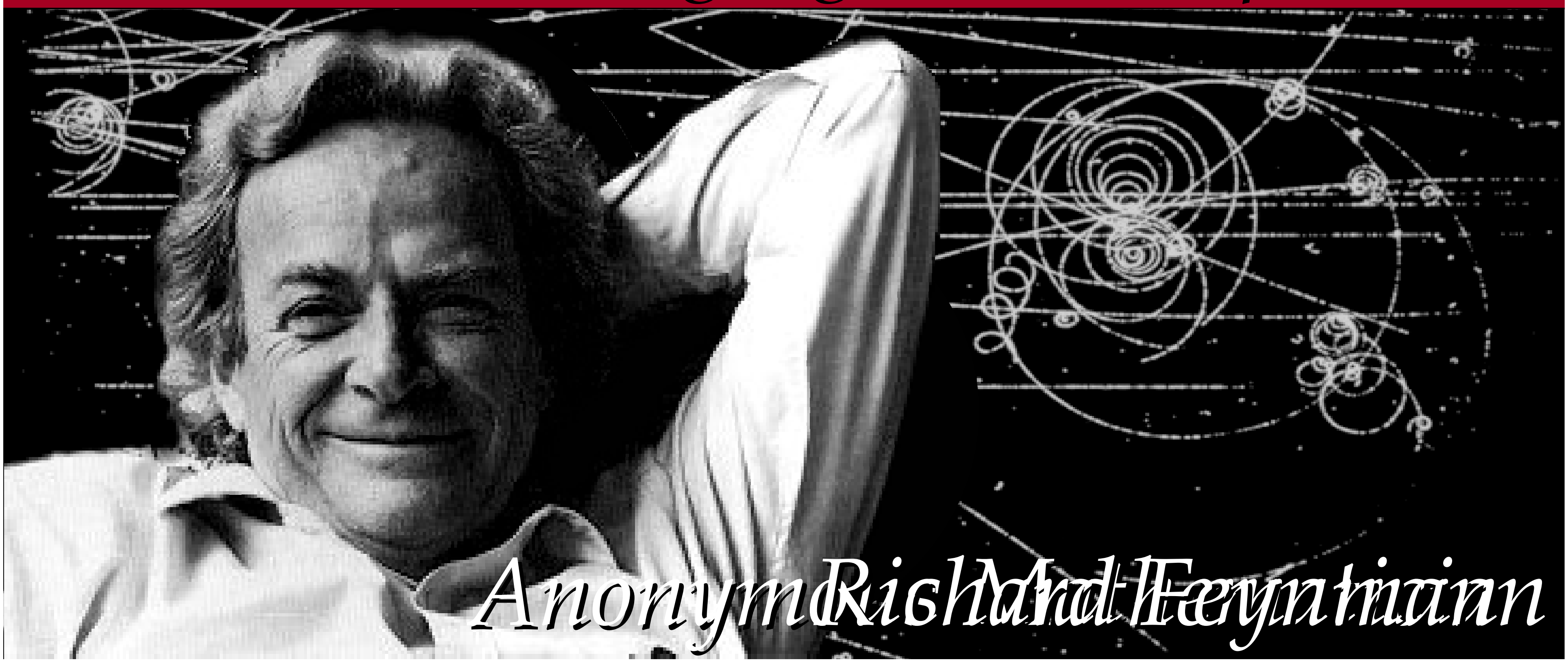
“[The universe] cannot be read until we have learned the language and become familiar with the characters in which it is written. It is written in mathematical language, and the letters are triangles, circles and other geometrical figures, without which means it is humanly impossible to comprehend a single word.”

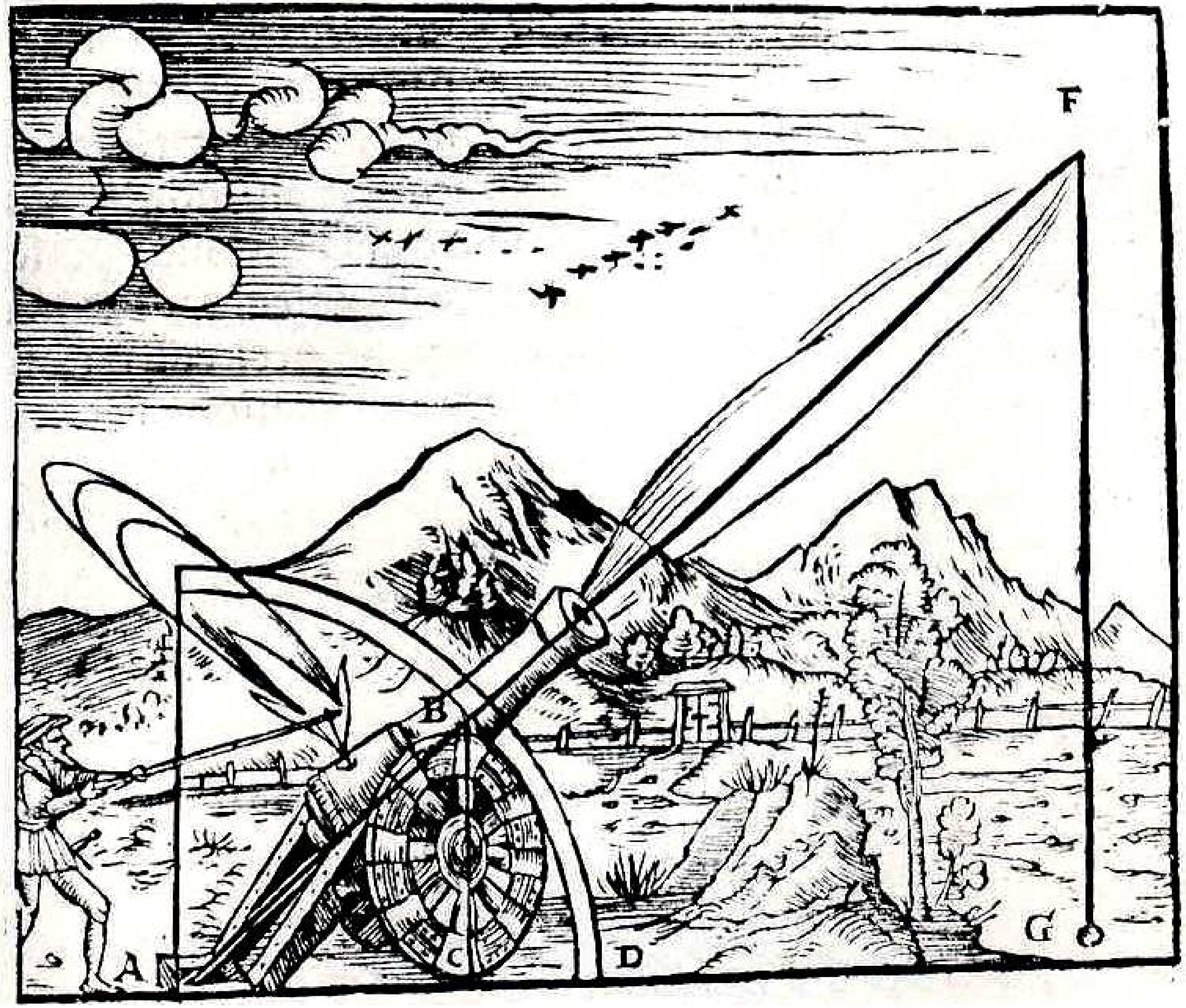


“To those who do not know mathematics it is difficult to get across a real feeling as to the beauty, the deepest beauty, of nature … If you want to learn about nature, to appreciate nature, it is necessary to understand the language that she speaks in.”

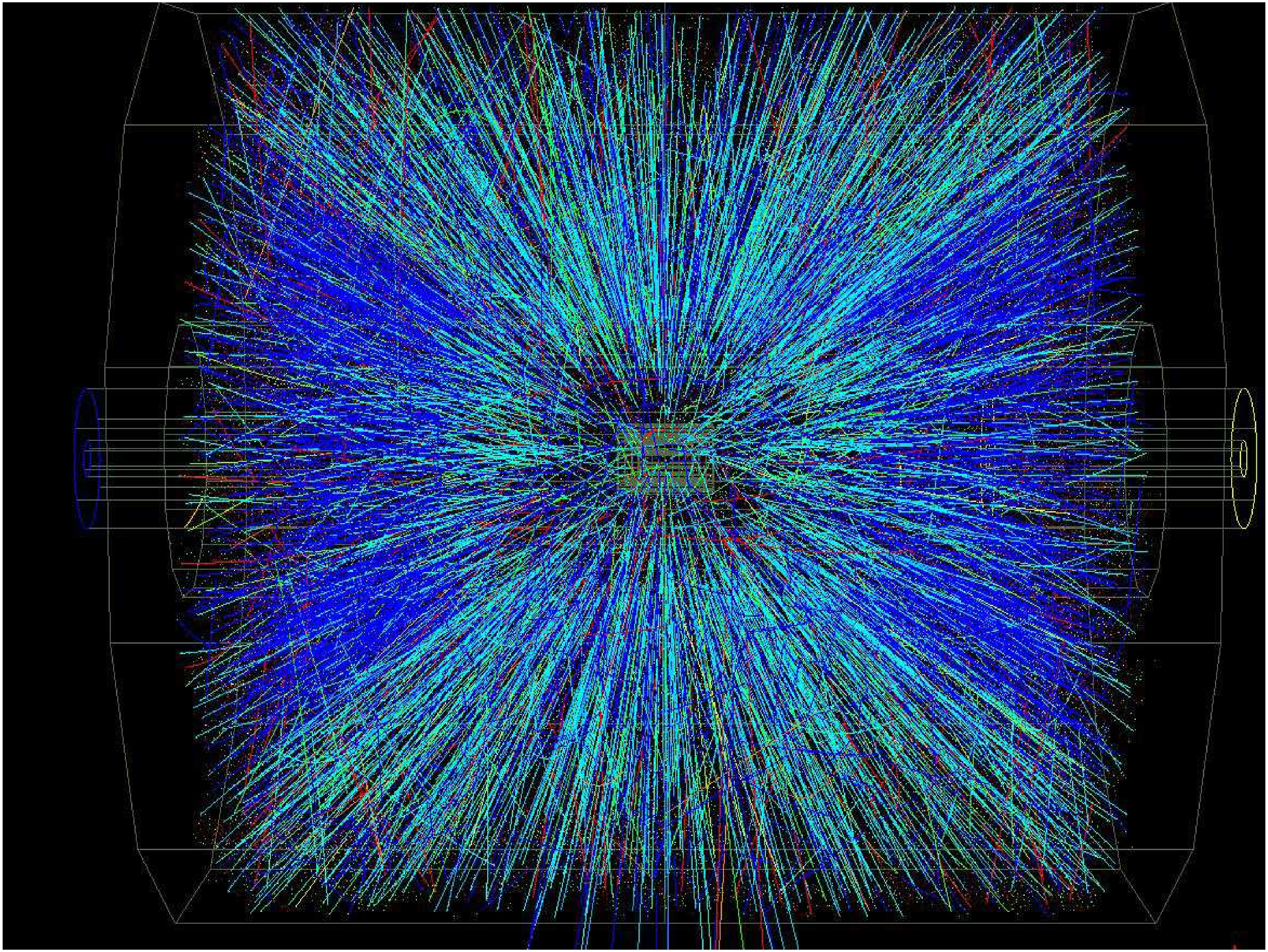
“If all mathematics disappeared today, physics would be set back exactly one week.”

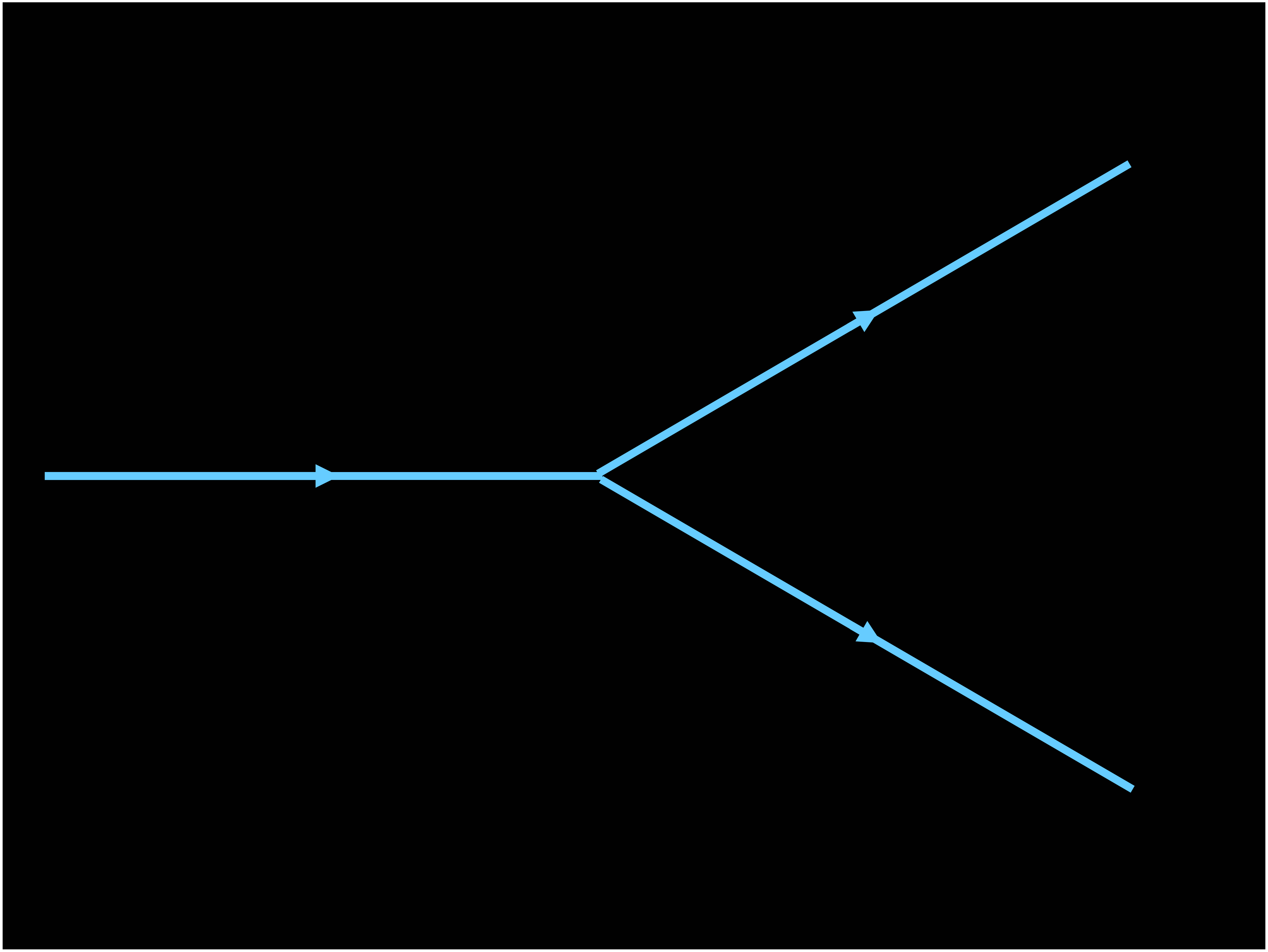
“That was the week that God …”

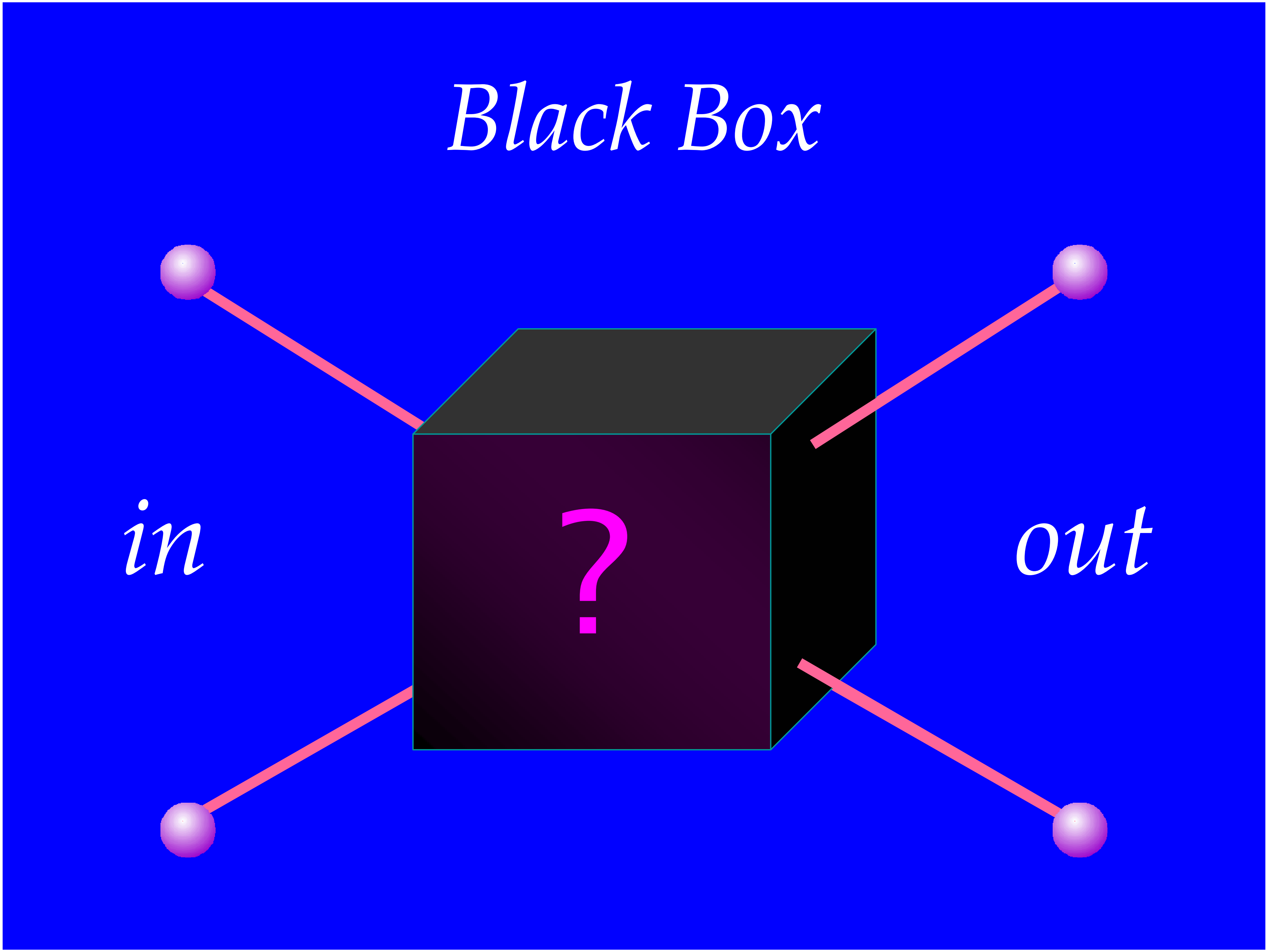



F
B
A
C
D
G

Quantum world

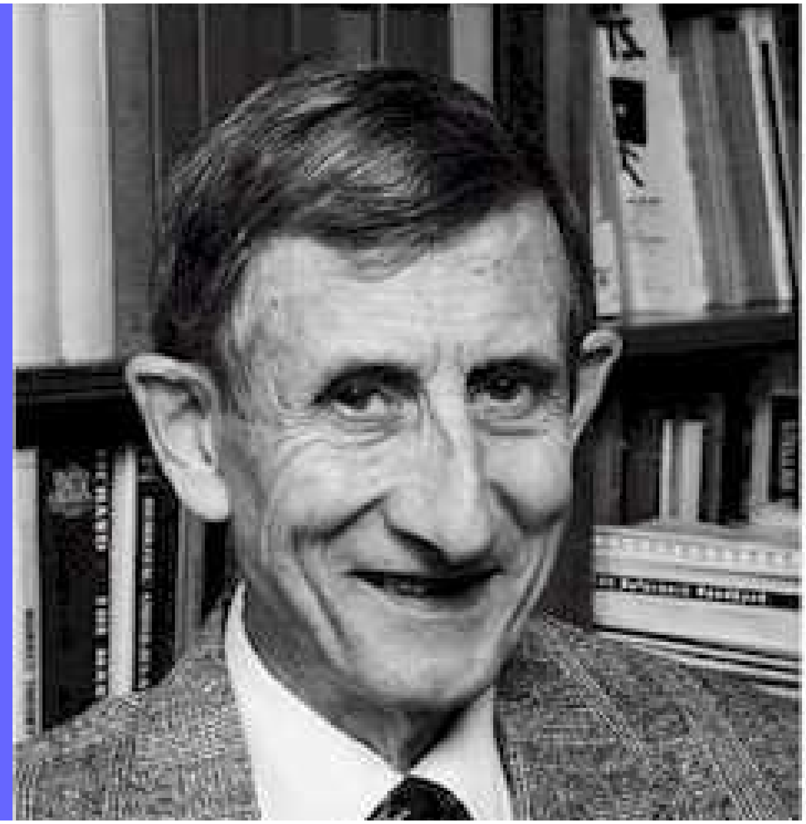





Black Box
in
?
out

Freeman Dyson

(Gibbs Lecture, 1972)



"I am acutely aware of the fact that the marriage between mathematics and physics, which was so enormously fruitful in past centuries, has recently ended in divorce."

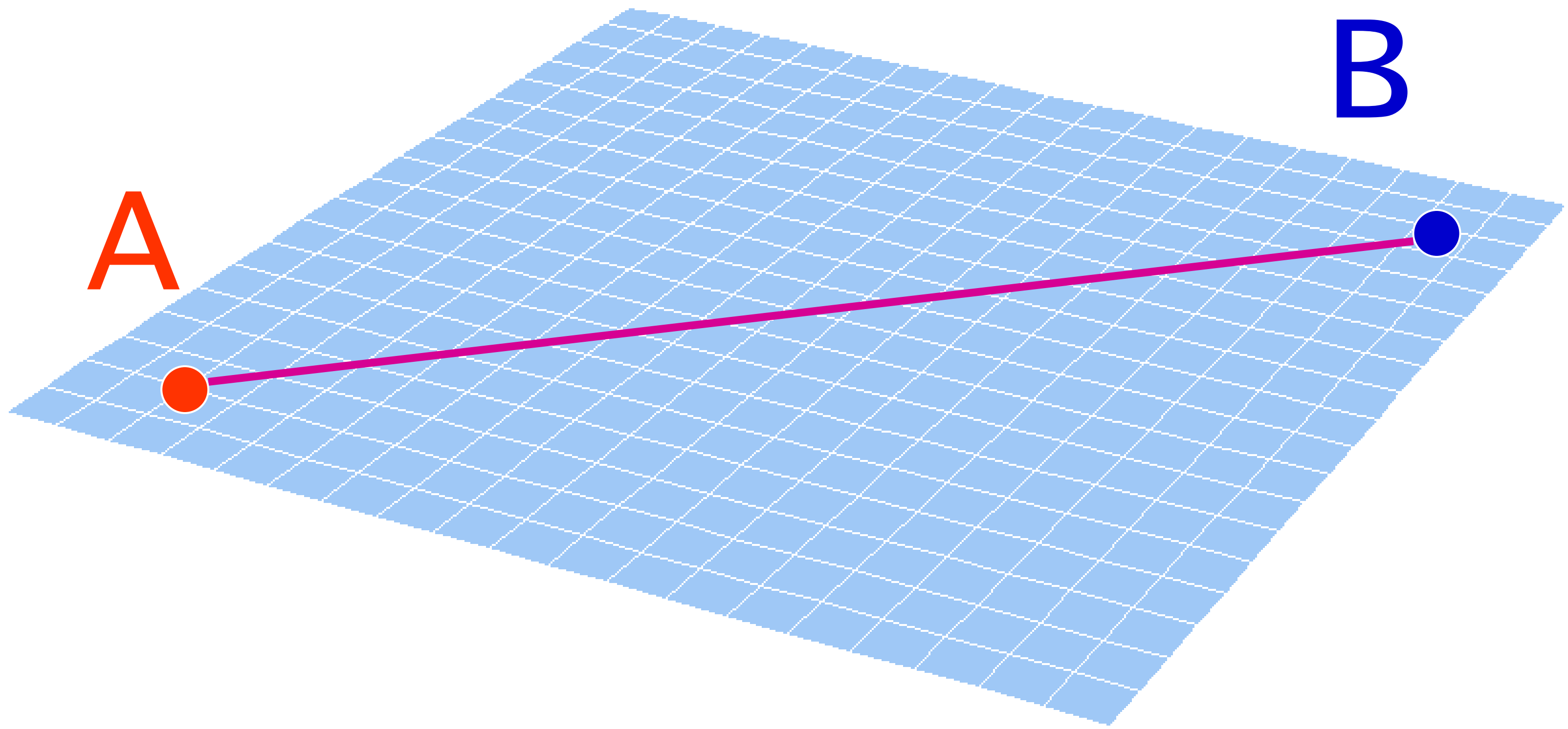

$M \heartsuit \Phi$

Standard Model

$$L = F^2 + \psi\left(i\not{D} + \varphi\right)\psi + \left|D\varphi\right|^2 + \left(|\varphi|^2 - 1\right)^2$$

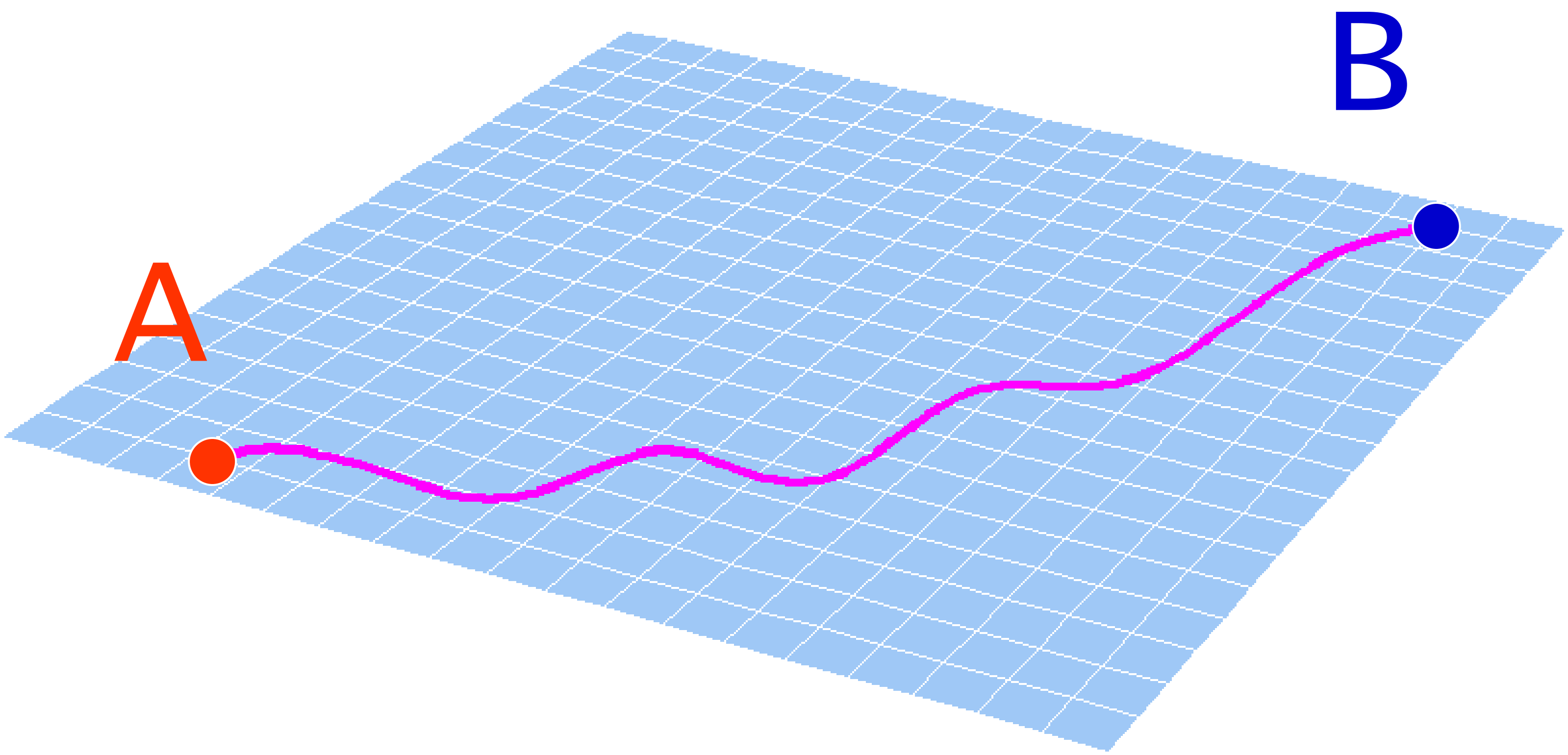
ABC of Quantum Physics

Classical Mechanics



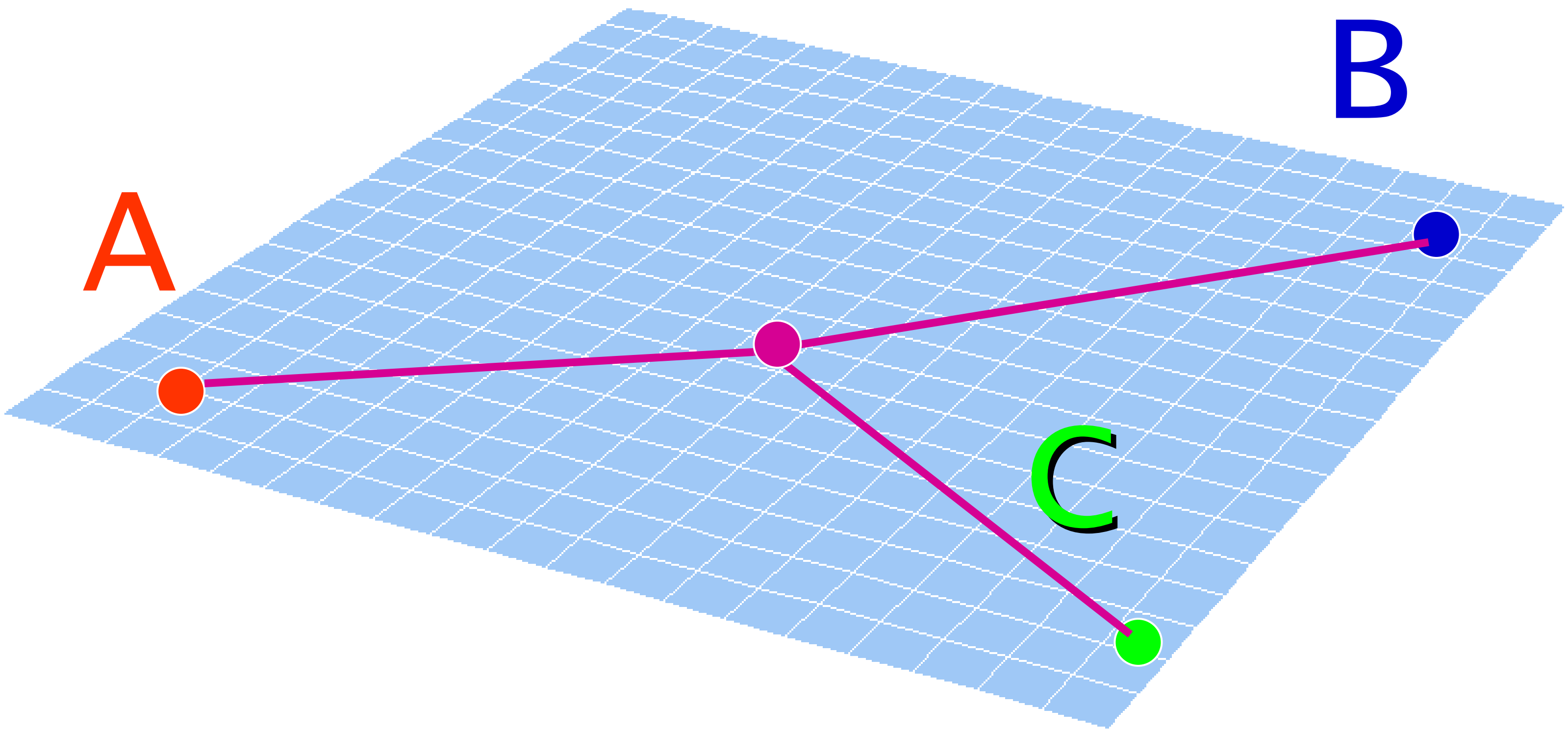
calculus, differential geometry, dynamical systems,…

Quantum Mechanics



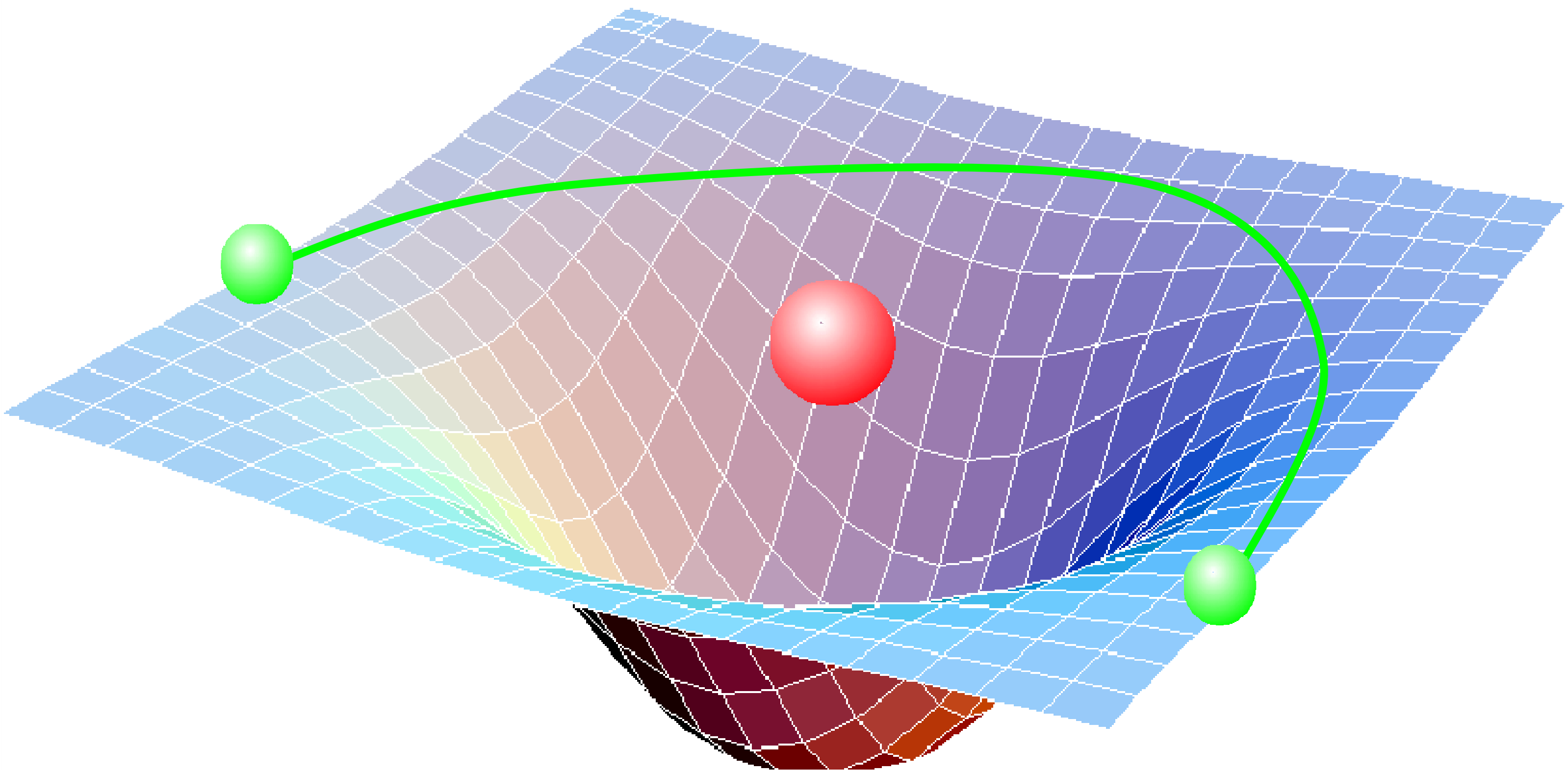
functional analysis, operator algebra, index theory, K-theory,…

Quantum Field Theory



knot theory, differential topology,
3- & 4-manifold invariants

Quantum Gravity & String Theory

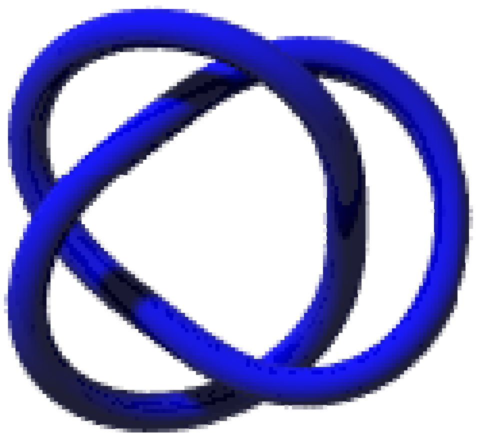


Gromov-Witten invariants,mirror symmetry, non-commutative geom., ...

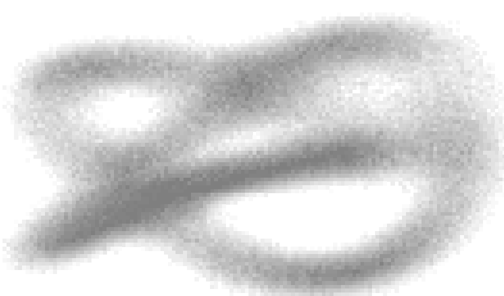
“The Unreasonable Effectiveness of Quantum Physics in Mathematics.”

Quantum Theory

geometric object



knot K

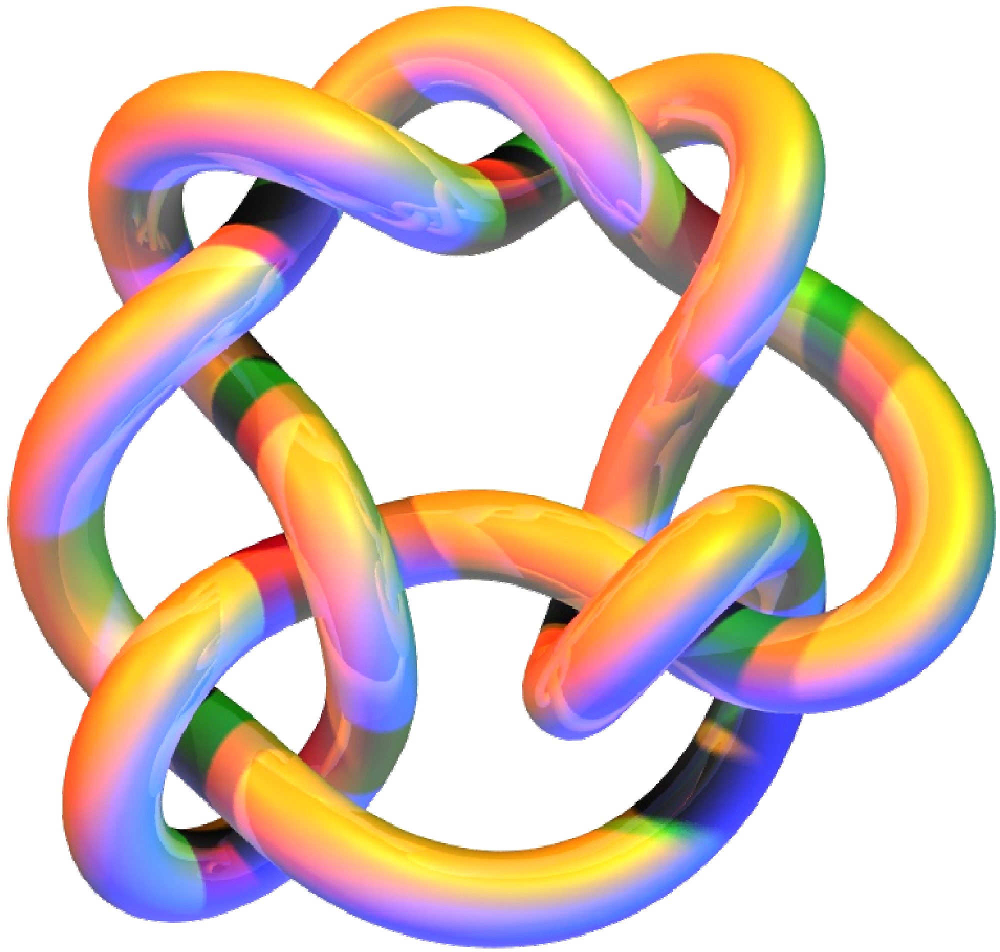


[*Geometry*]

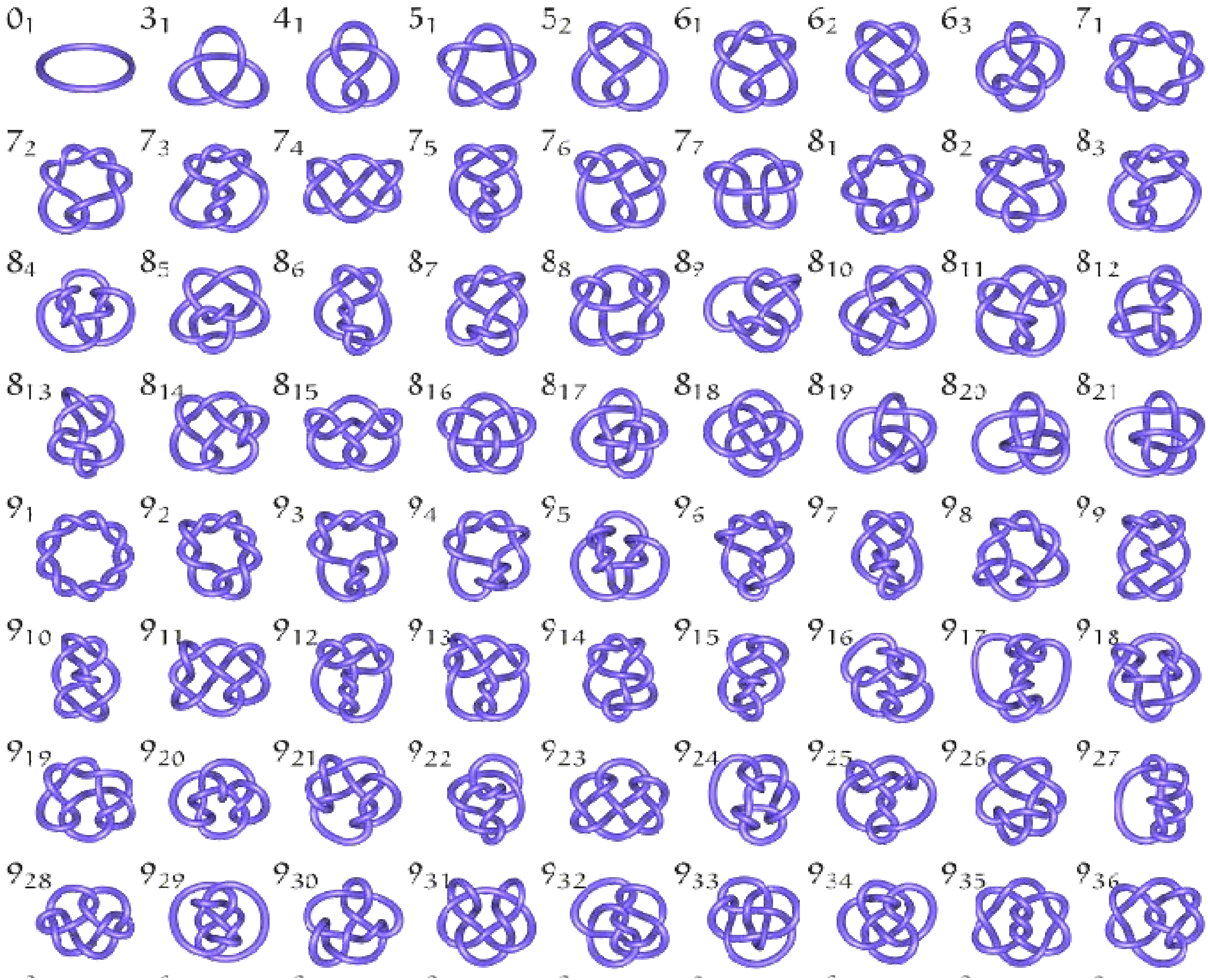
quantum invariant

$$Z(K) \in \square$$

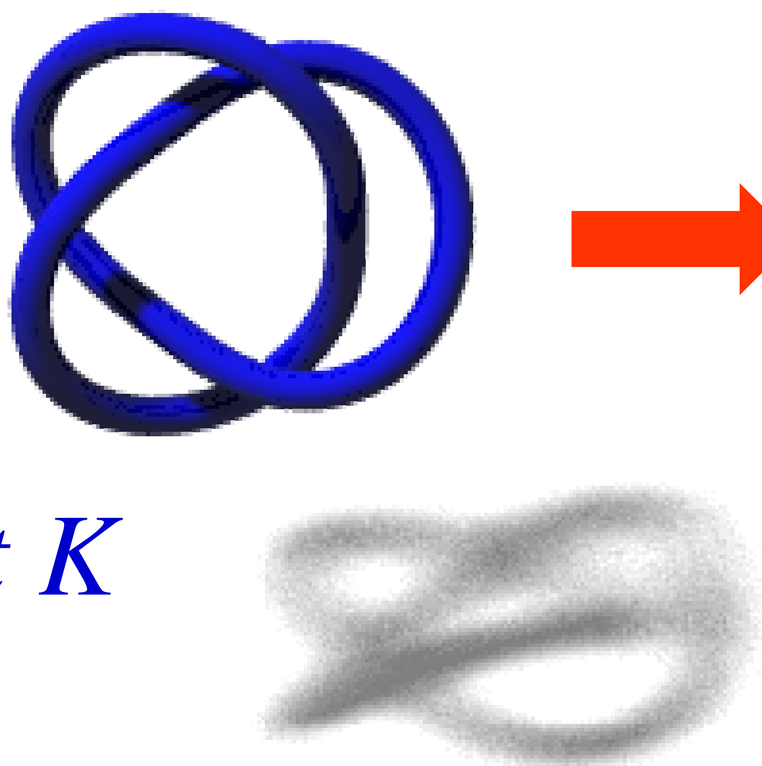
[*Algebra*]



Mathematical knot


0_1 3_1 4_1 5_1 5_2 6_1 6_2 6_3 7_1
7_2 7_3 7_4 7_5 7_6 7_7 8_1 8_2 8_3
8_4 8_5 8_6 8_7 8_8 8_9 8_{10} 8_{11} 8_{12}
8_{13} 8_{14} 8_{15} 8_{16} 8_{17} 8_{18} 8_{19} 8_{20} 8_{21}
9_1 9_2 9_3 9_4 9_5 9_6 9_7 9_8 9_9
9_{10} 9_{11} 9_{12} 9_{13} 9_{14} 9_{15} 9_{16} 9_{17} 9_{18}
9_{19} 9_{20} 9_{21} 9_{22} 9_{23} 9_{24} 9_{25} 9_{26} 9_{27}
9_{28} 9_{29} 9_{30} 9_{31} 9_{32} 9_{33} 9_{34} 9_{35} 9_{36}

Knot Invariant

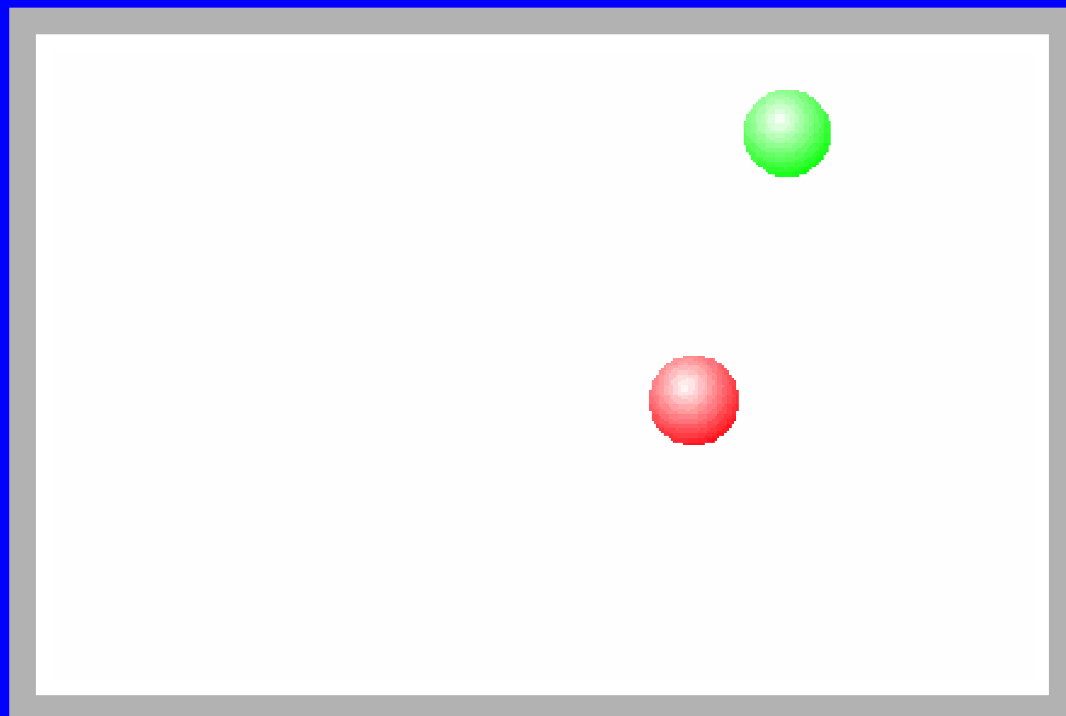


$$Z(K) \in \square$$

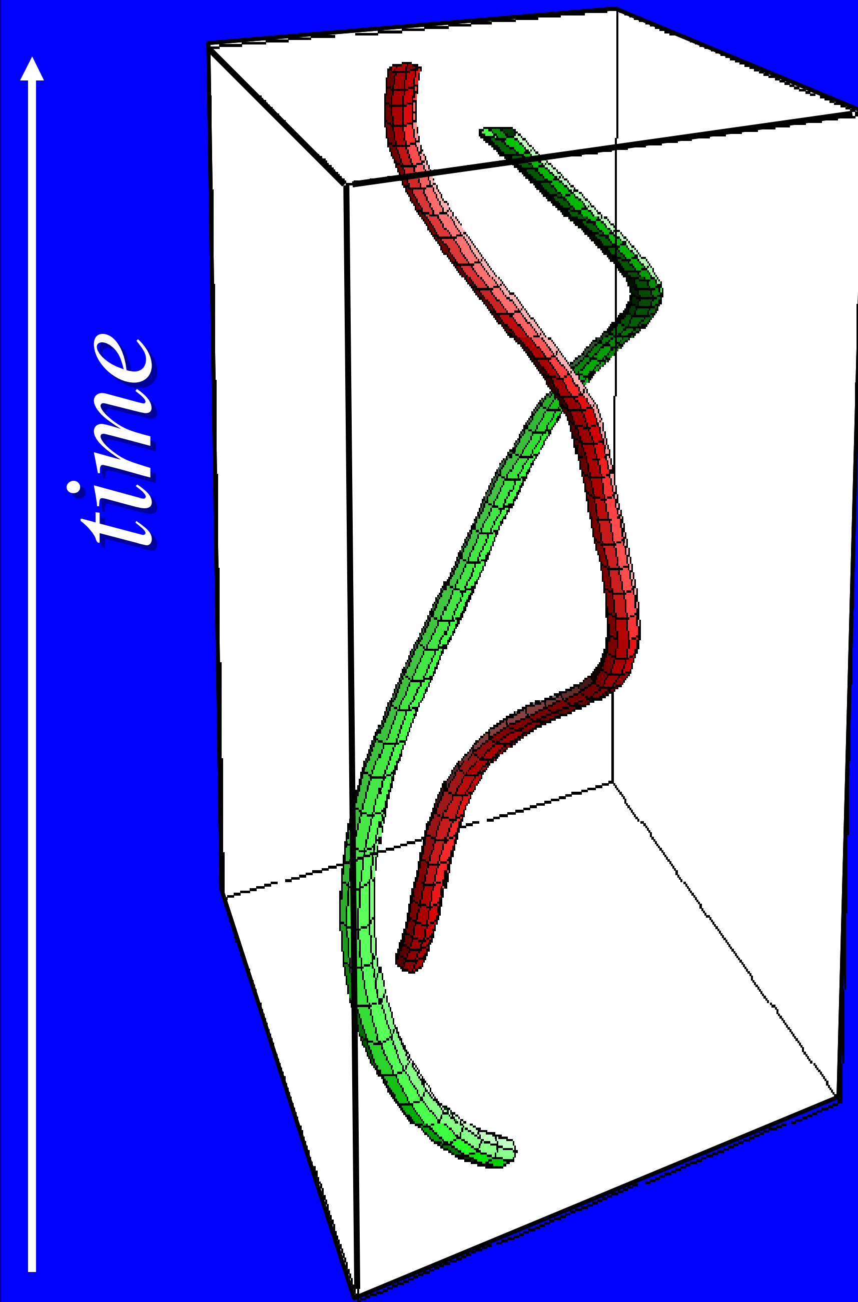
knot K

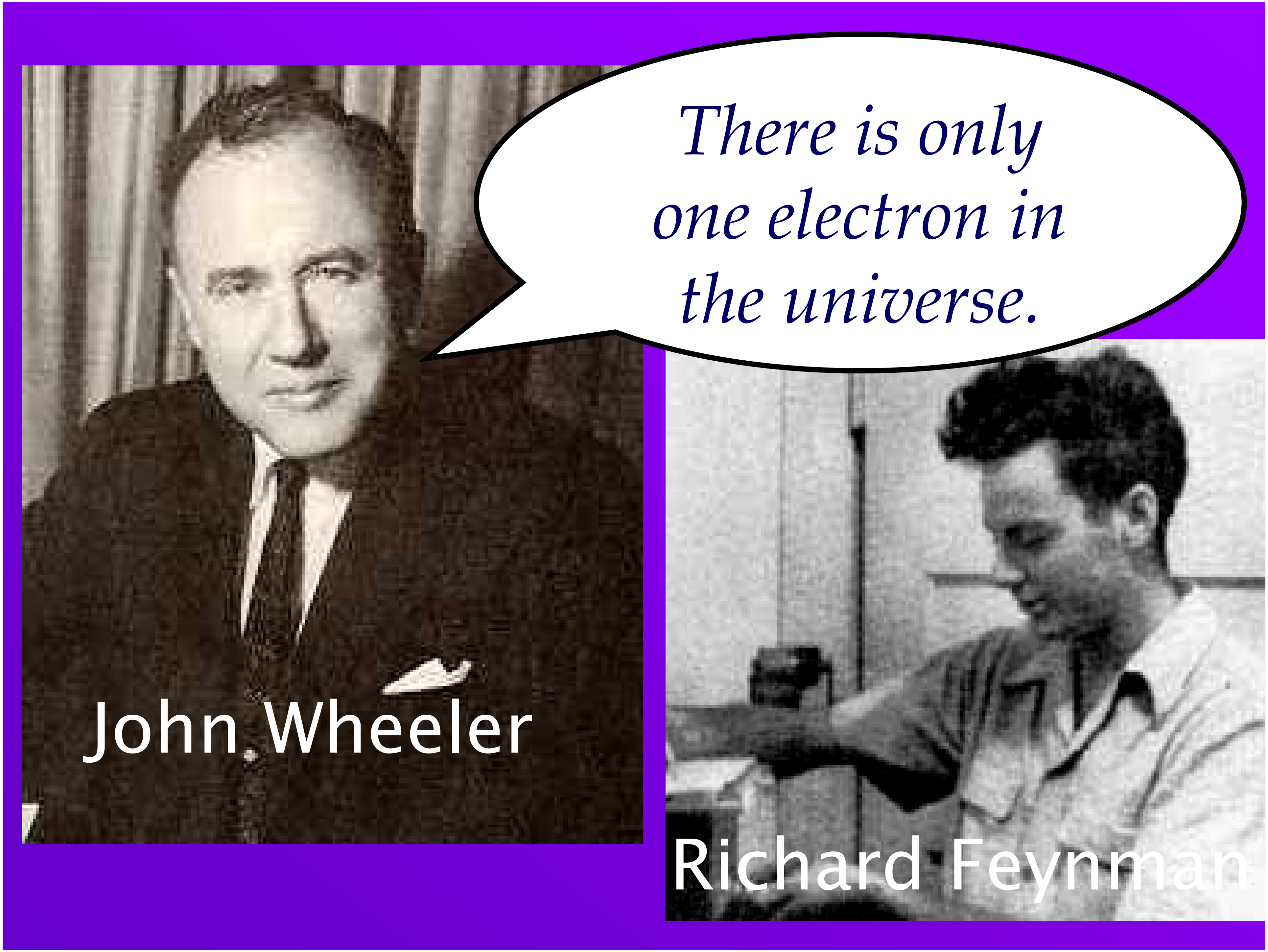
algebra

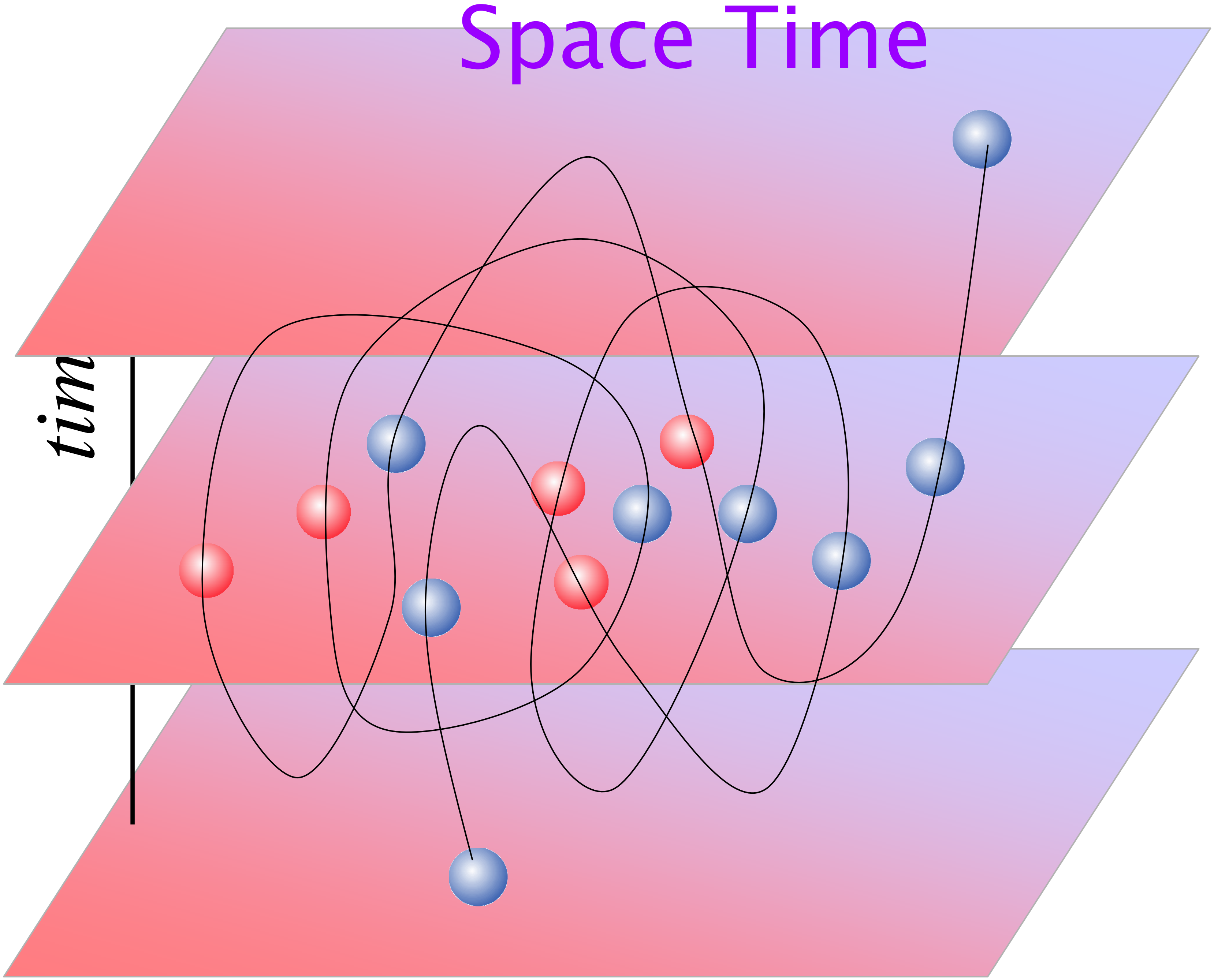
2 dimensions

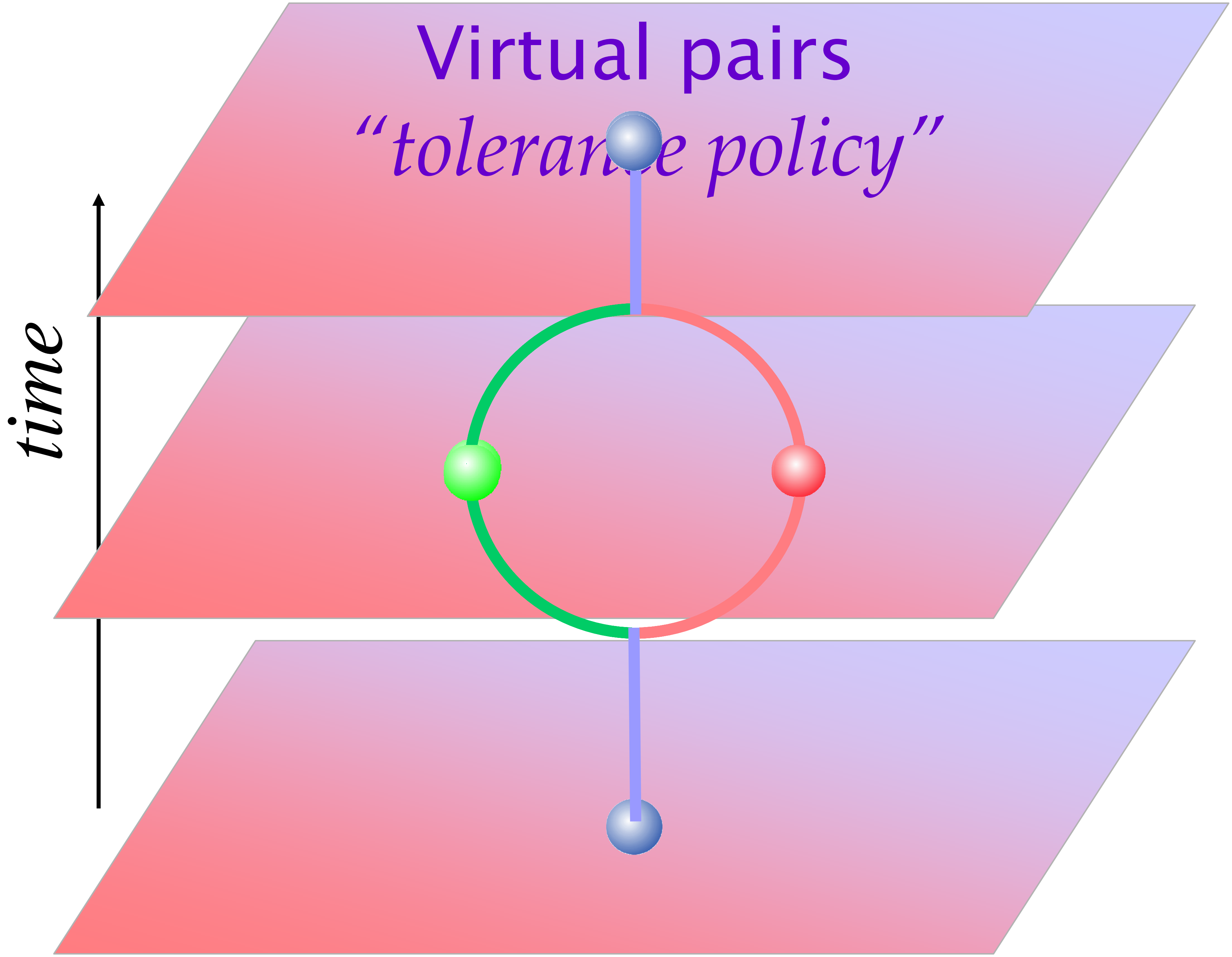


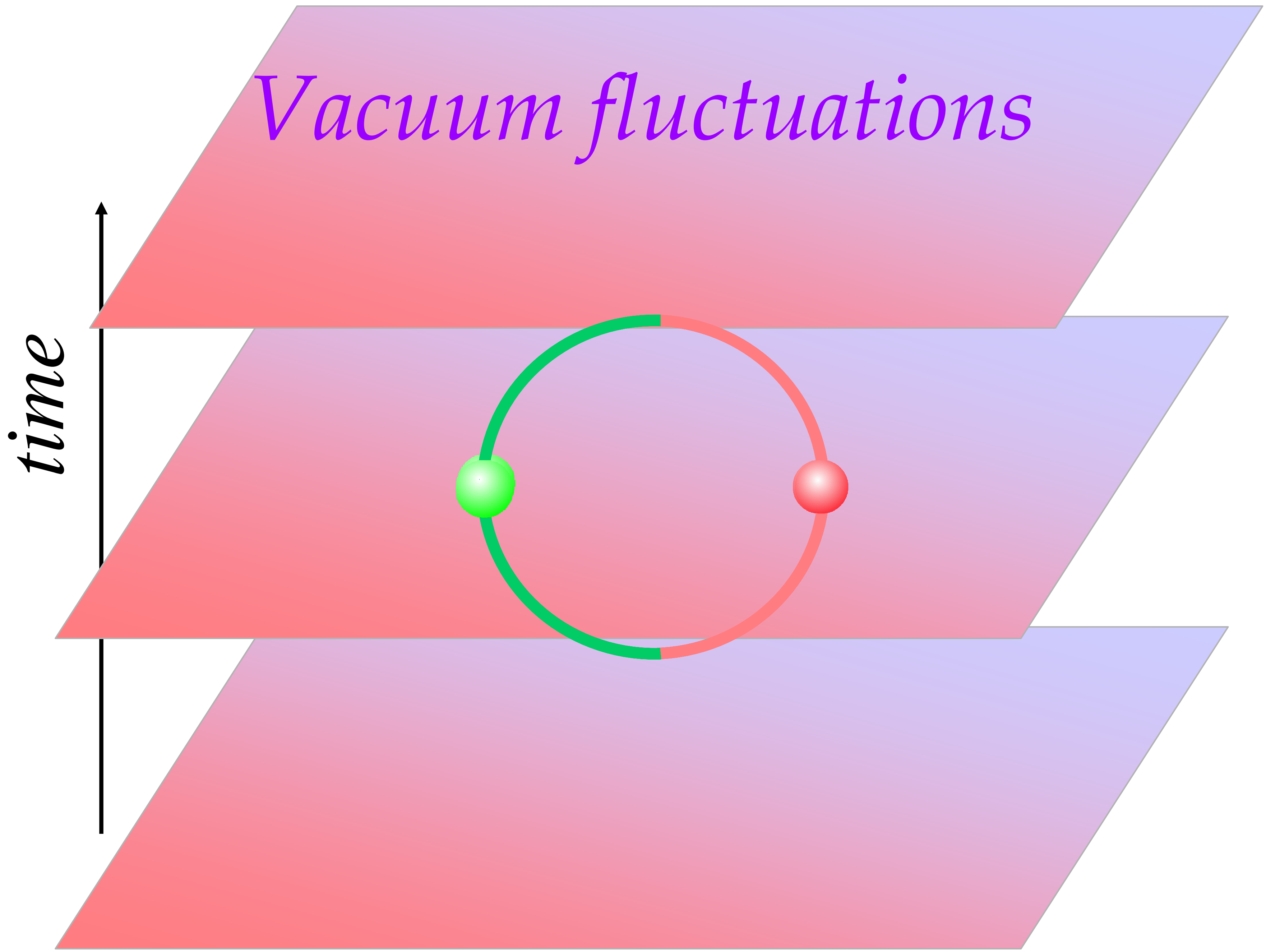
3rd dimension = *time*


time

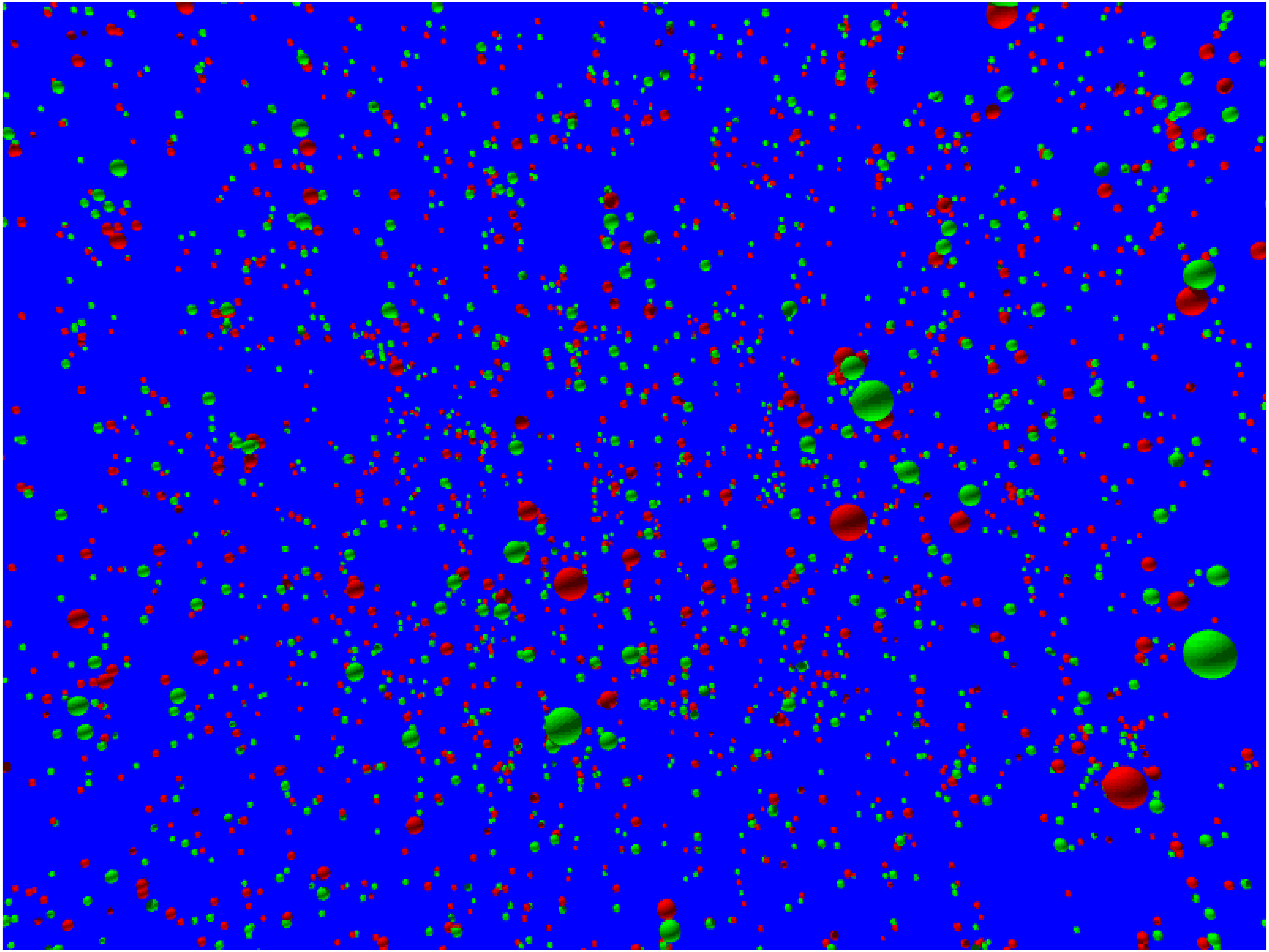

There is only one electron in the universe.
John Wheeler
Richard Feynman

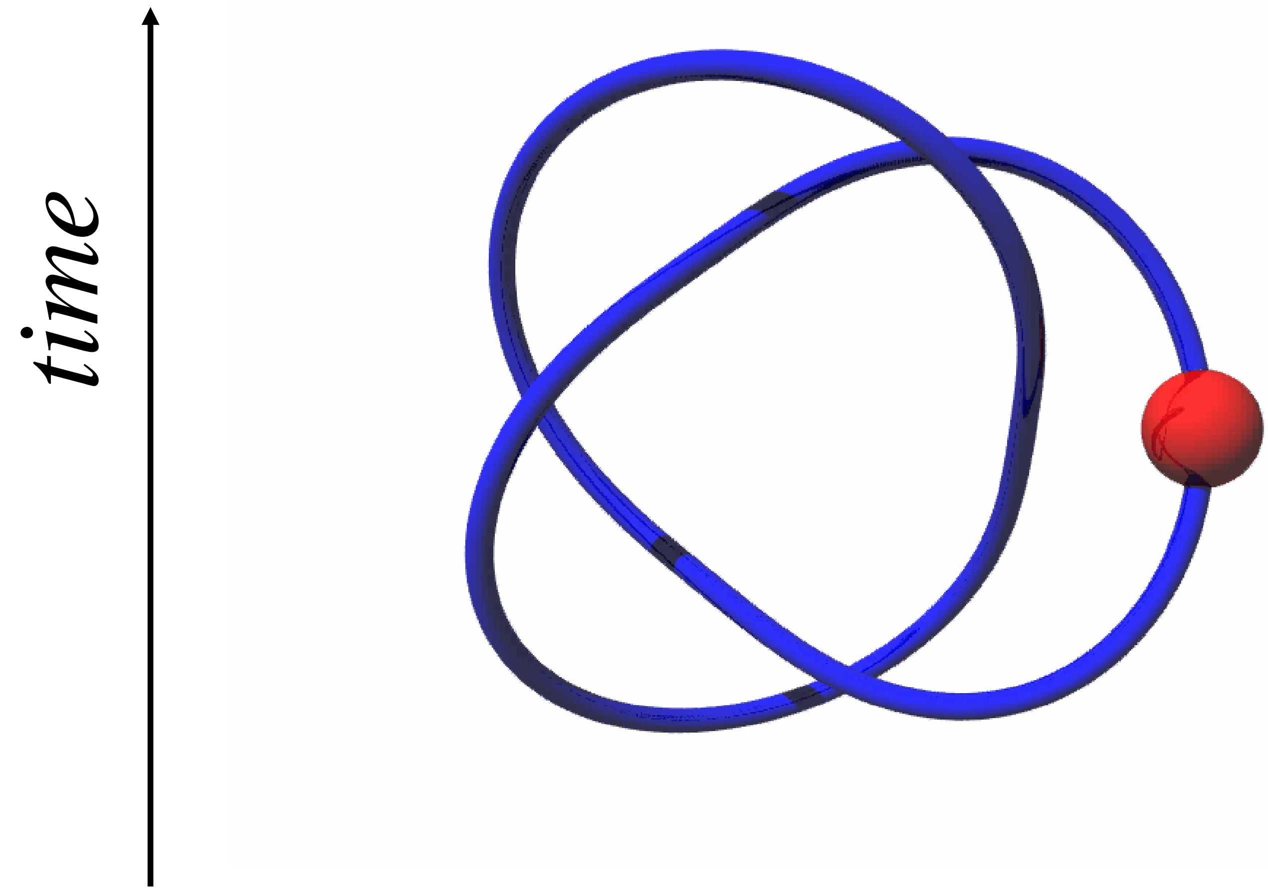

Space Time
tim


Virtual pairs
“tolerance policy”
time

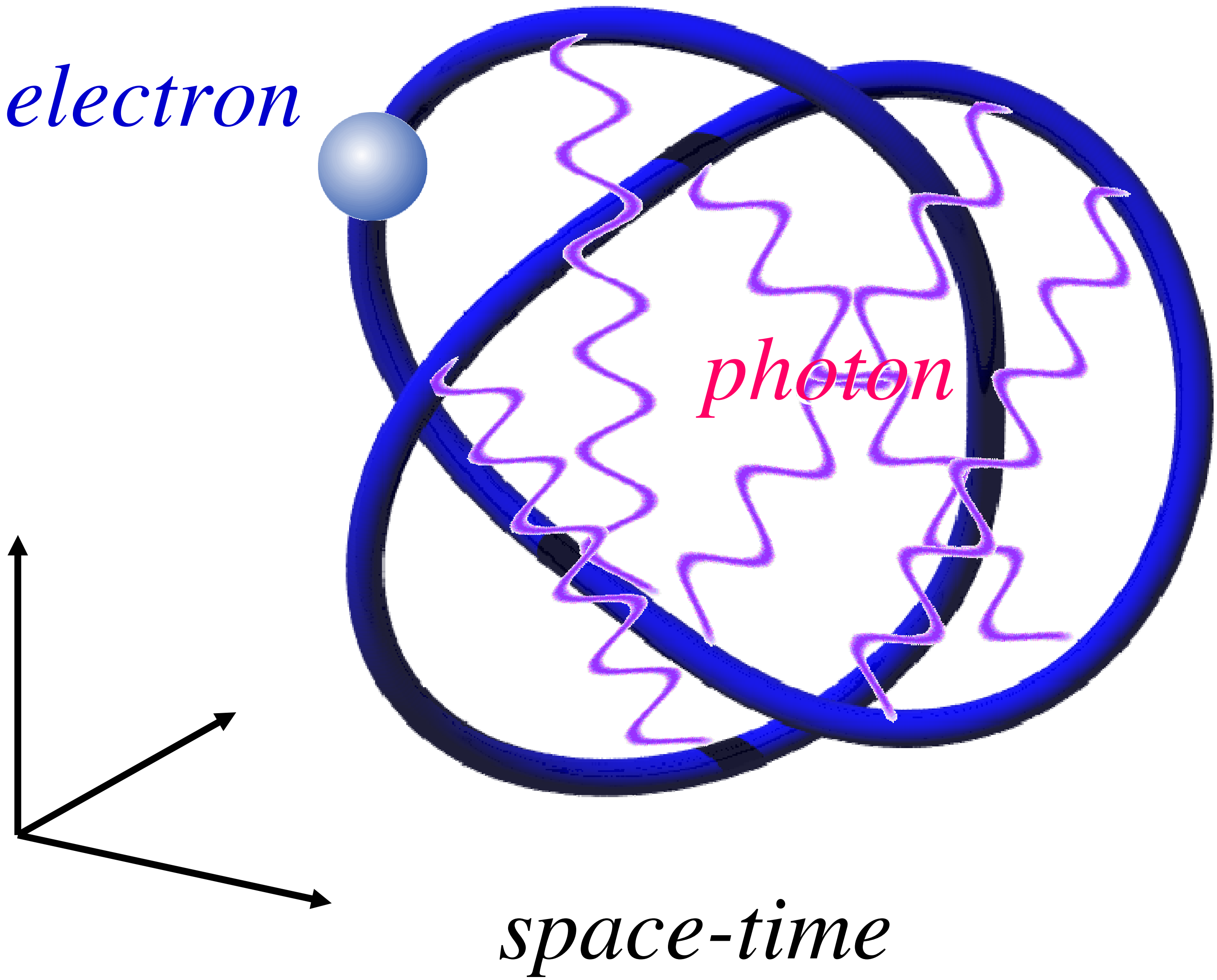

Vacuum fluctuations
time

Vacuum




time

Quantum Amplitude



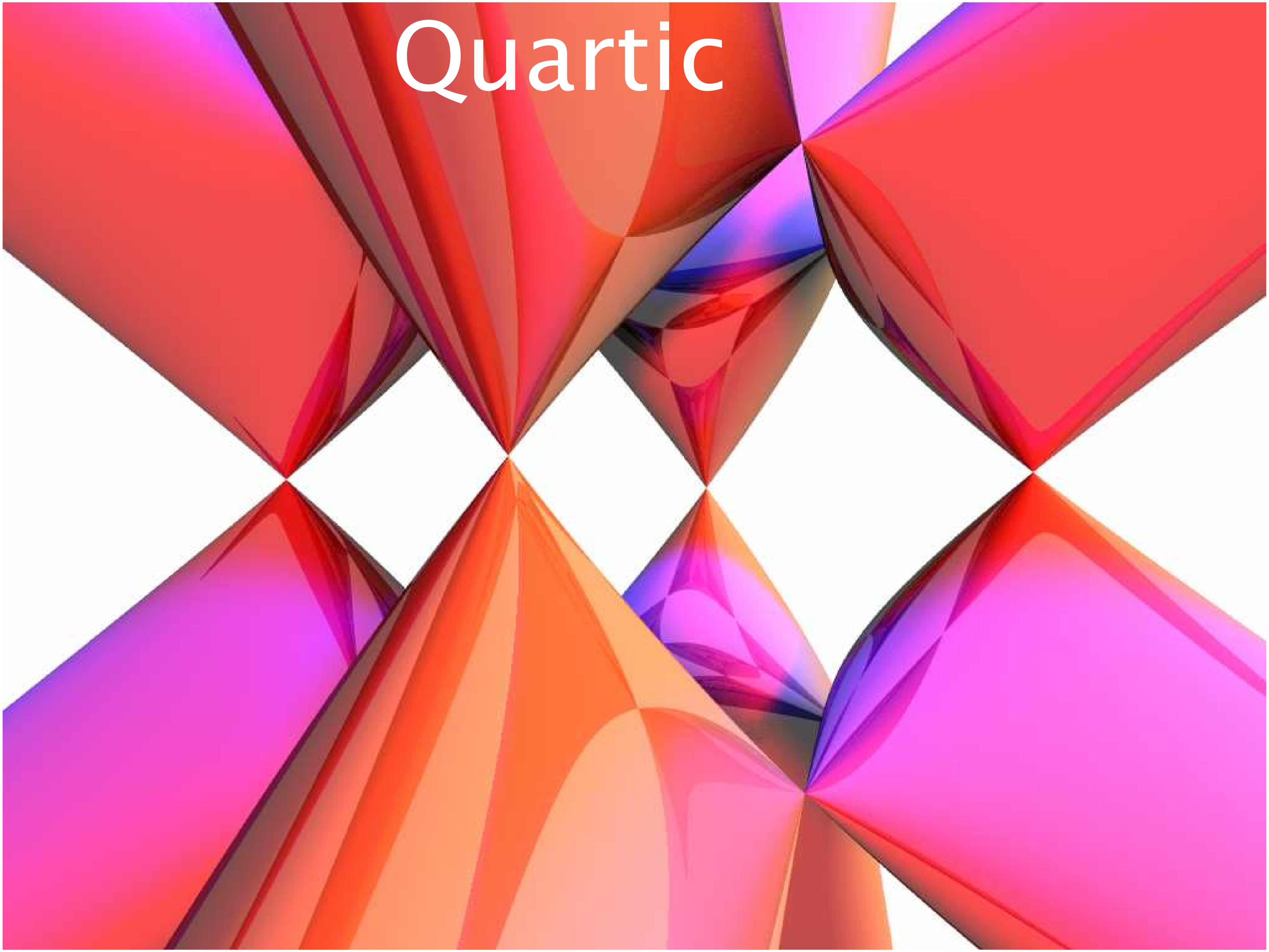
Algebraic Geometry

"The Quintic"

$$x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 = 0$$

Calabi-Yau Threefold

Quartic



"Counting Curves"

$$x_1(z) = a_{1,d} z^d + a_{1,d-1} z^{d-1} + \ldots + a_{1,1} z^1 + a_{1,0}$$

$$\cdots = \quad \cdots\cdots\cdots\cdots$$

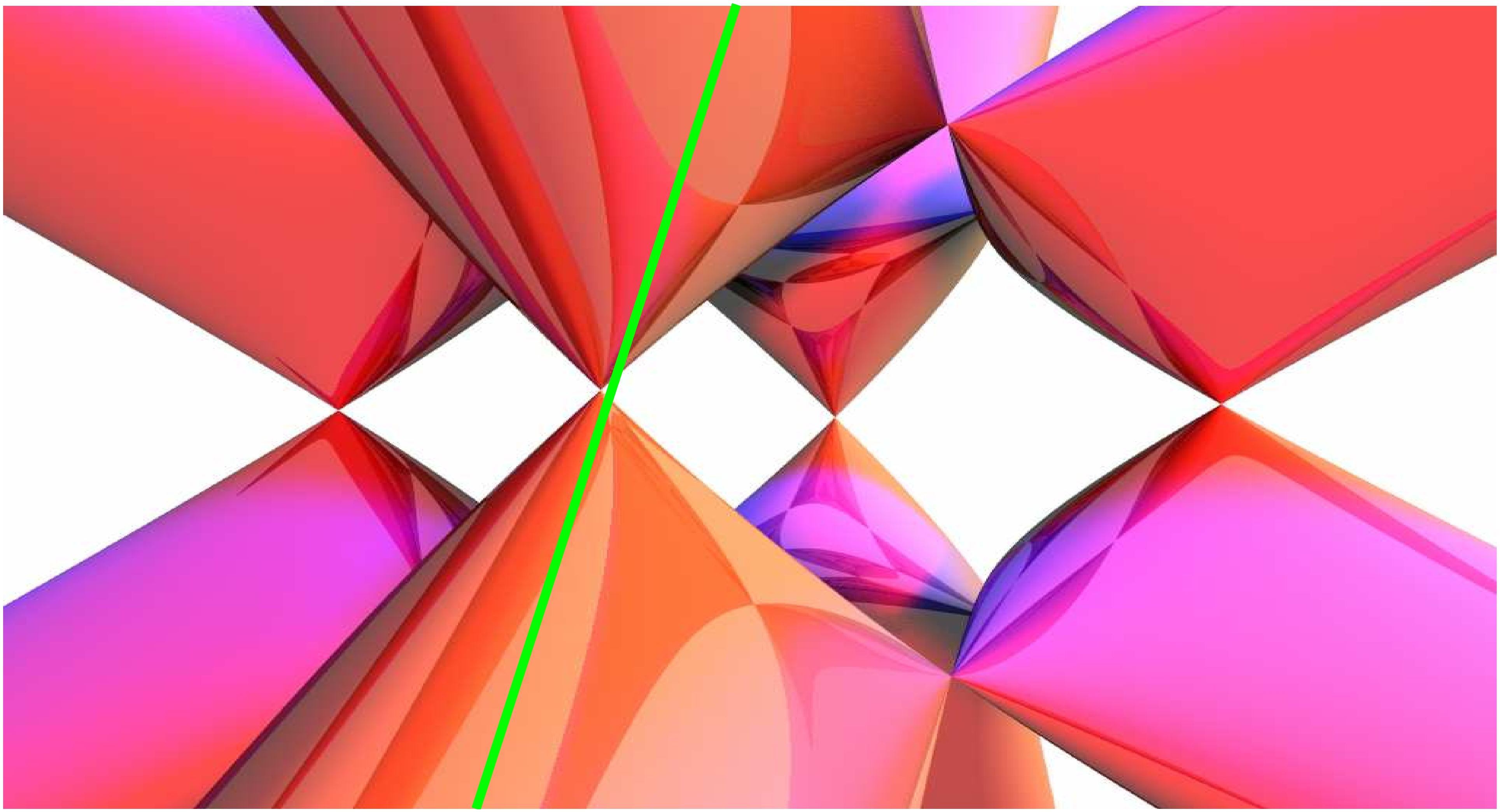
$$x_5(z) = a_{5,d} z^d + a_{5,d-1} z^{d-1} + \ldots + a_{5,1} z^1 + a_{5,0}$$

Polynomials of degree d

$$N_d = \#\,\text{solutions}\{a_{i,j}\}$$

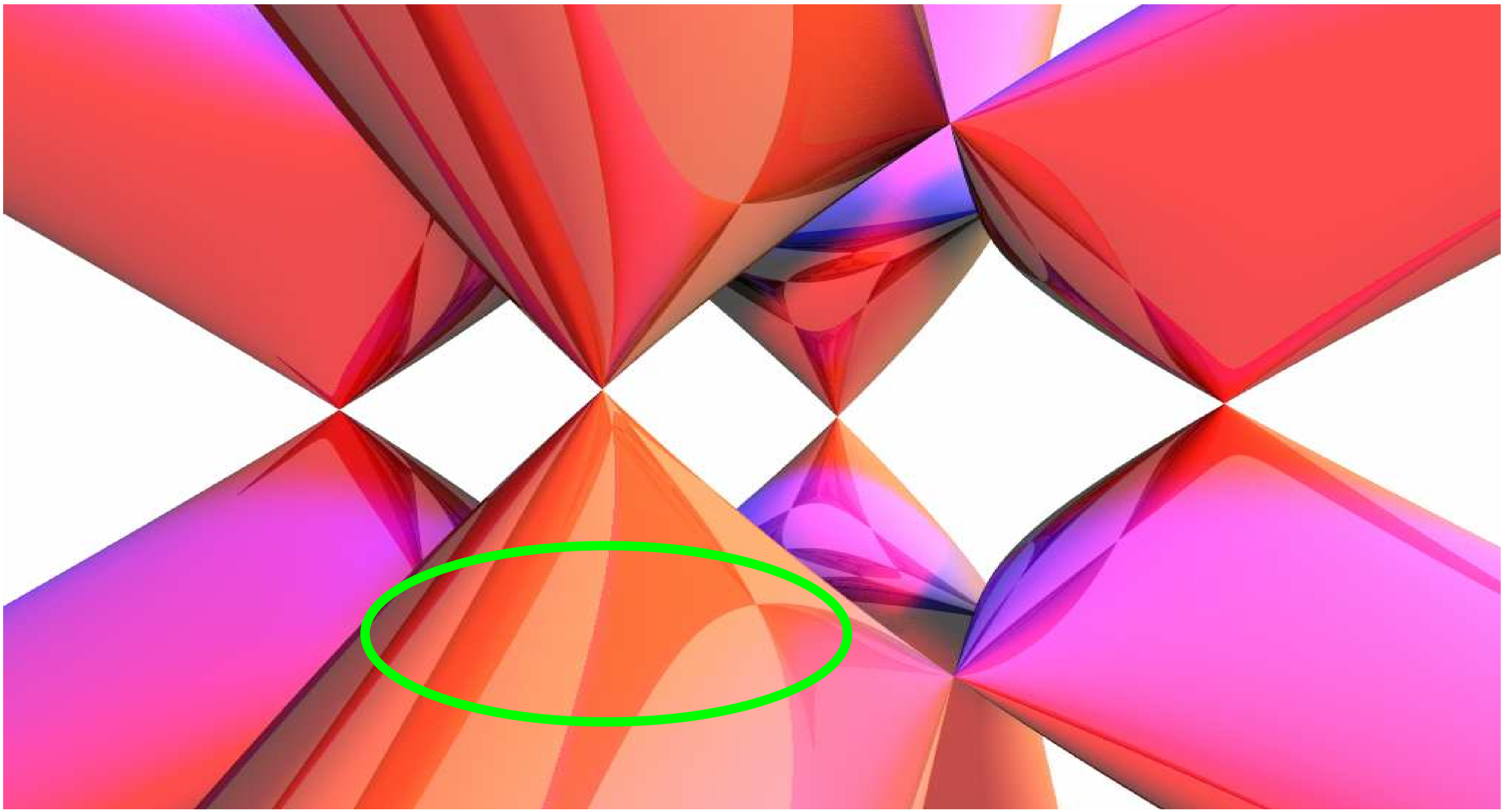
$d=1$ Lines

$N_1=2875$



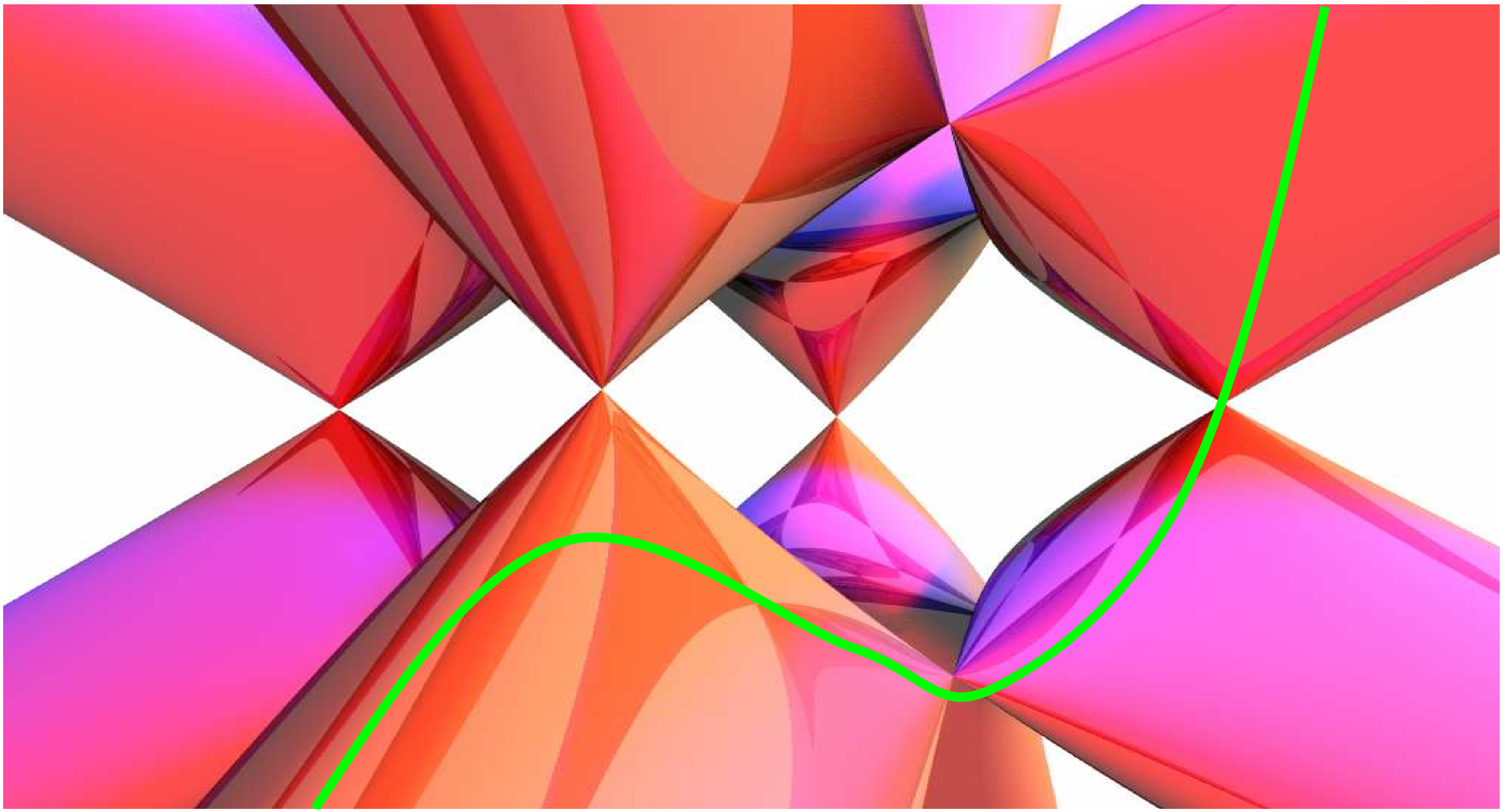
$d=2$ Conics

$N_2=609250$



$d=3$ Cubics

$N_3=317206375$



$N_1 = 2875$

$N_2 = 609250$

$N_3 = 317206375$

$N_4 = 242467530000$

$N_5 = 229305888887625$

$N_6 = 248249742118022000$

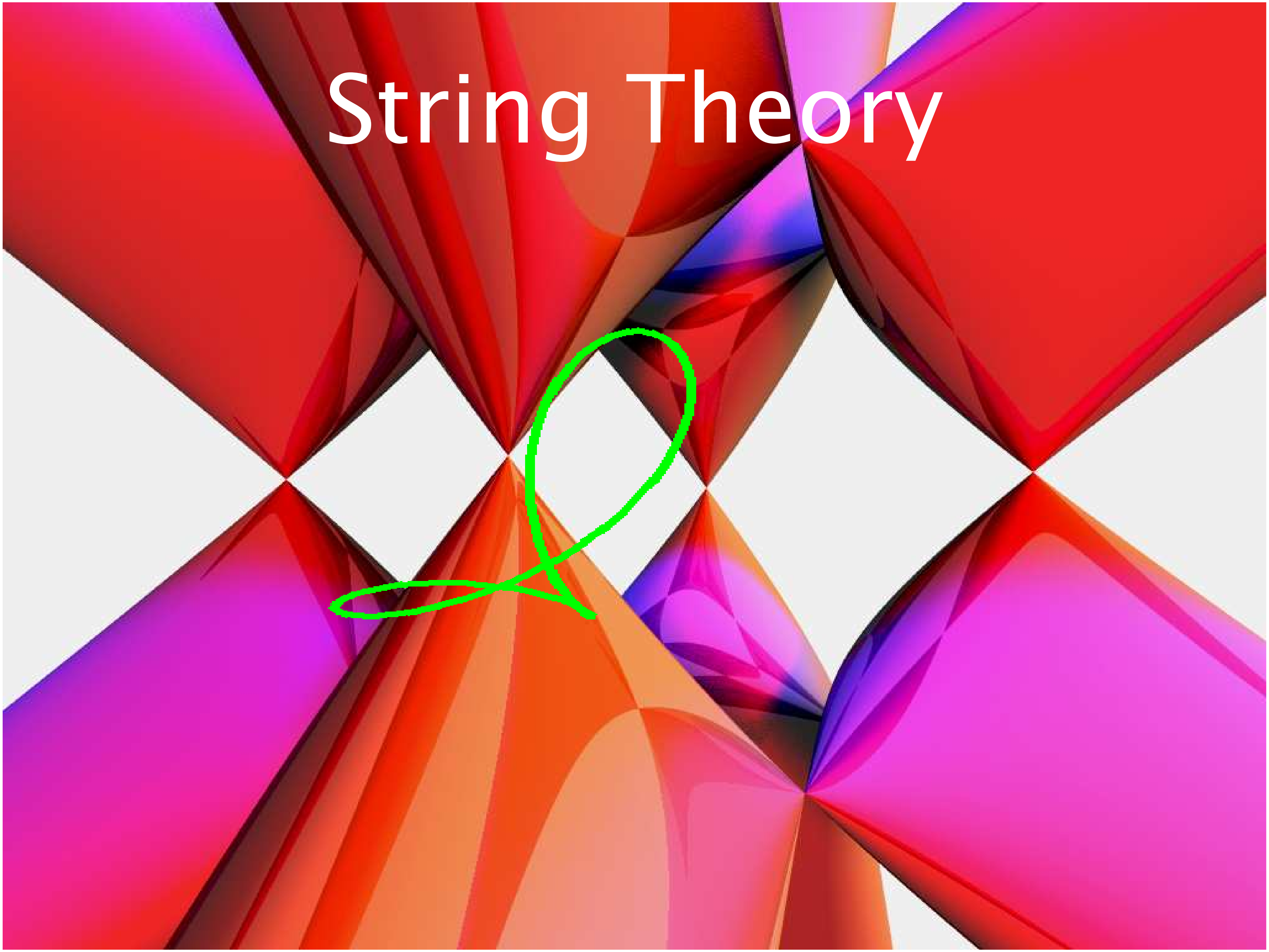
$N_7 = 295091050570845659250$

$N_8 = 375632160937476603550000$

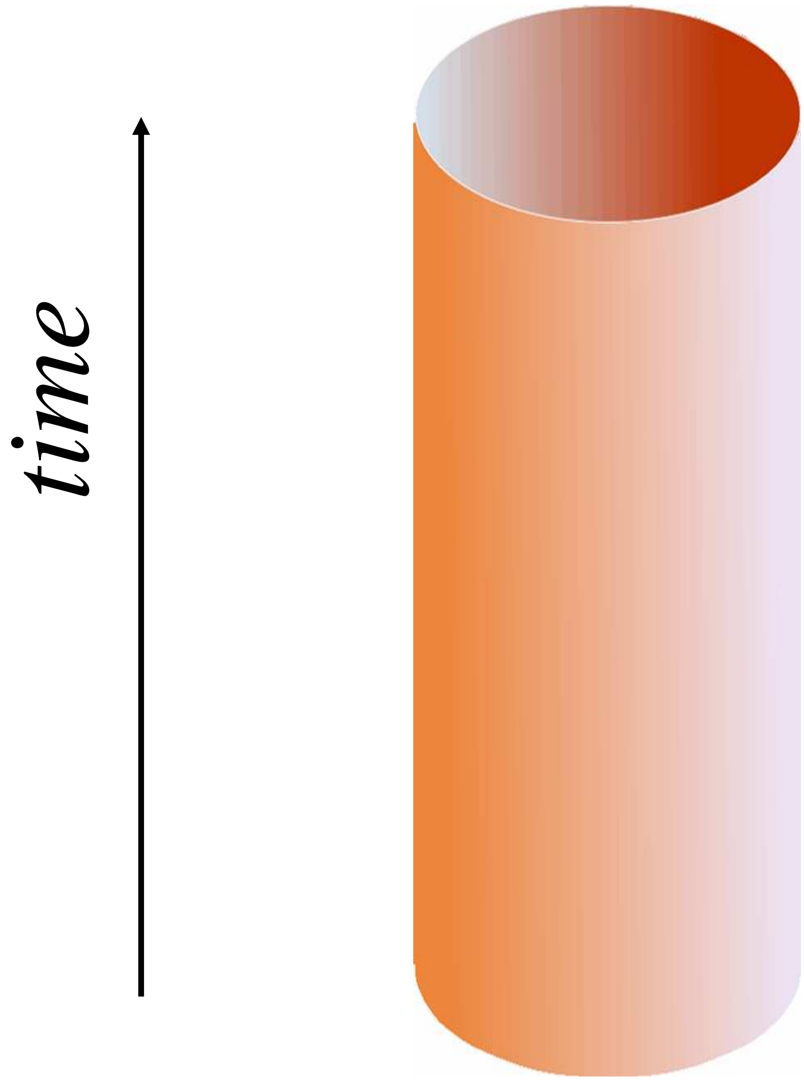
$N_9 = 503840510416985243645106250$

$N_{10} = 704288164978454686113488249750$

String Theory



Riemann surface *(complex curve)*

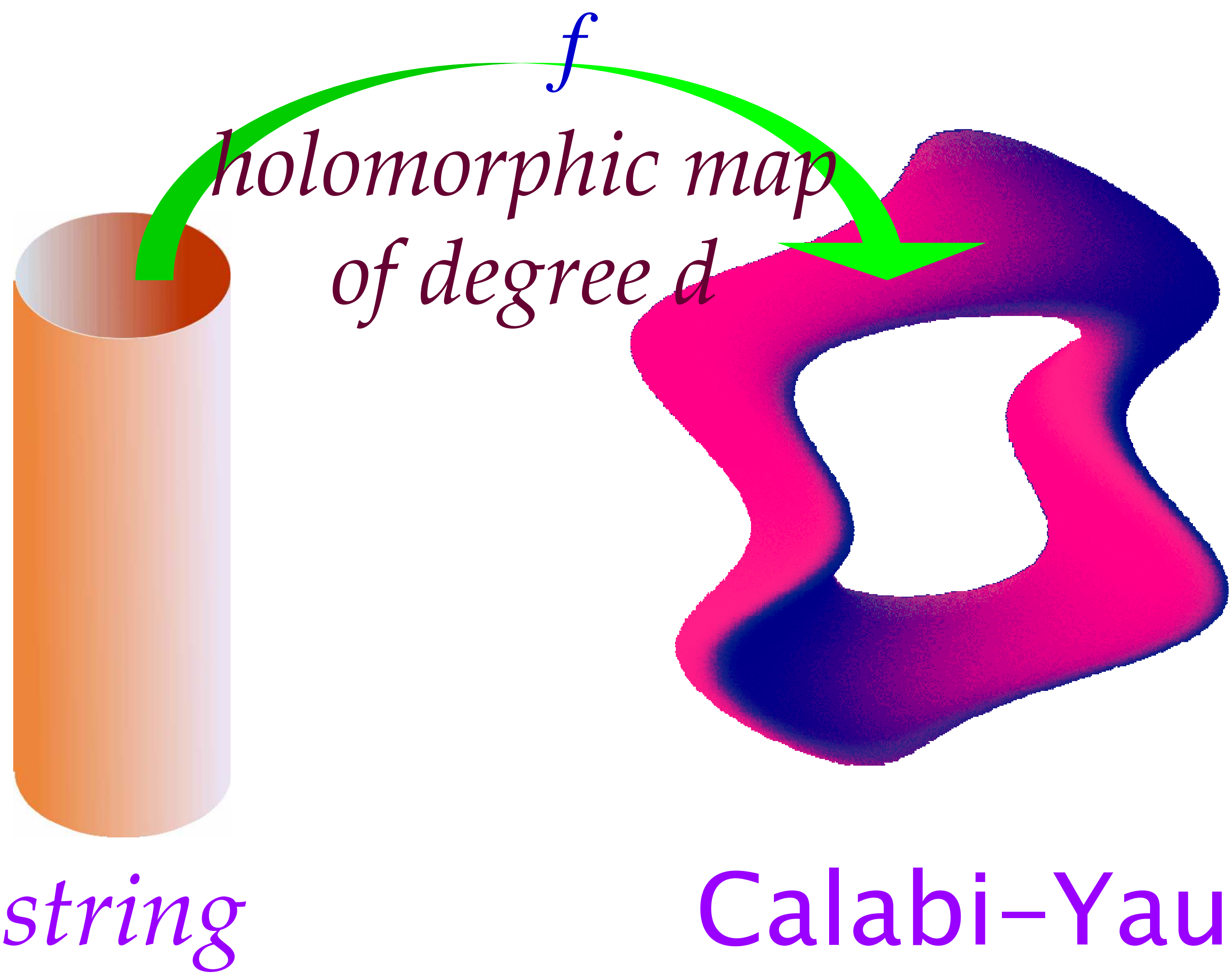


string

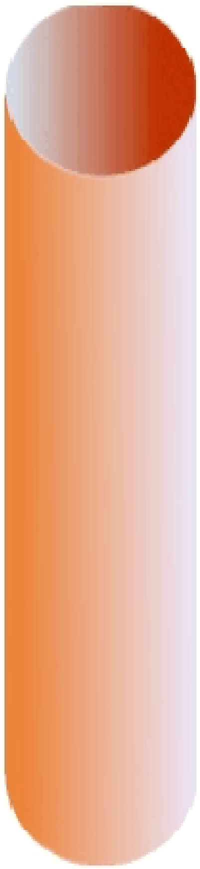
quantization

Conformal Field Theory

Counting curves

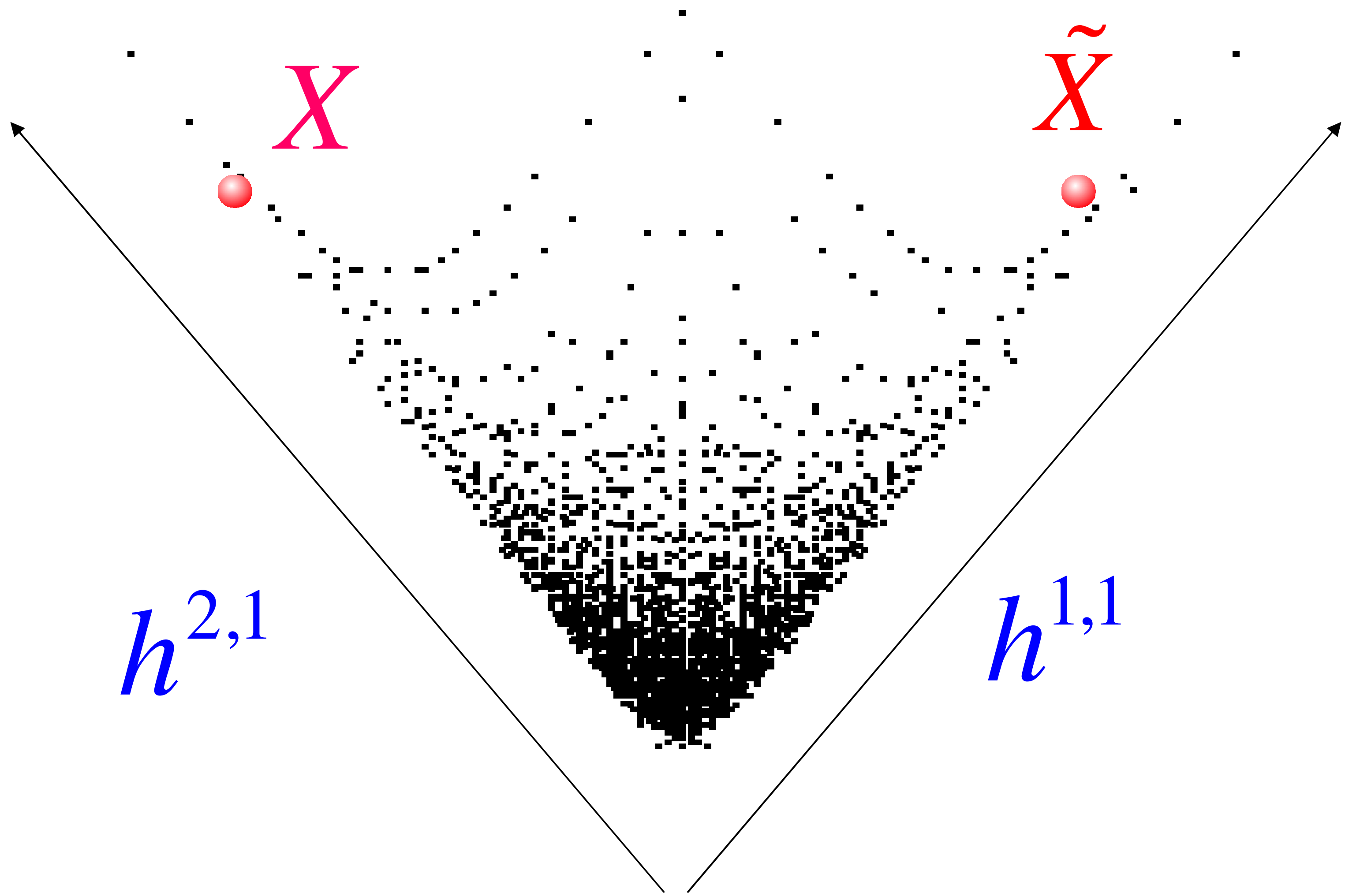


Instanton Sum



$$F(t) = \sum_{d \geq 0} N_d e^{-dt/\hbar}$$

Calabi–Yau manifolds

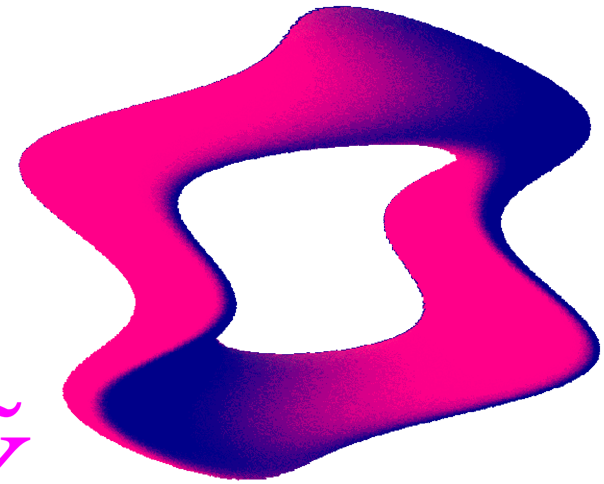


Mirror Symmetry

X

...

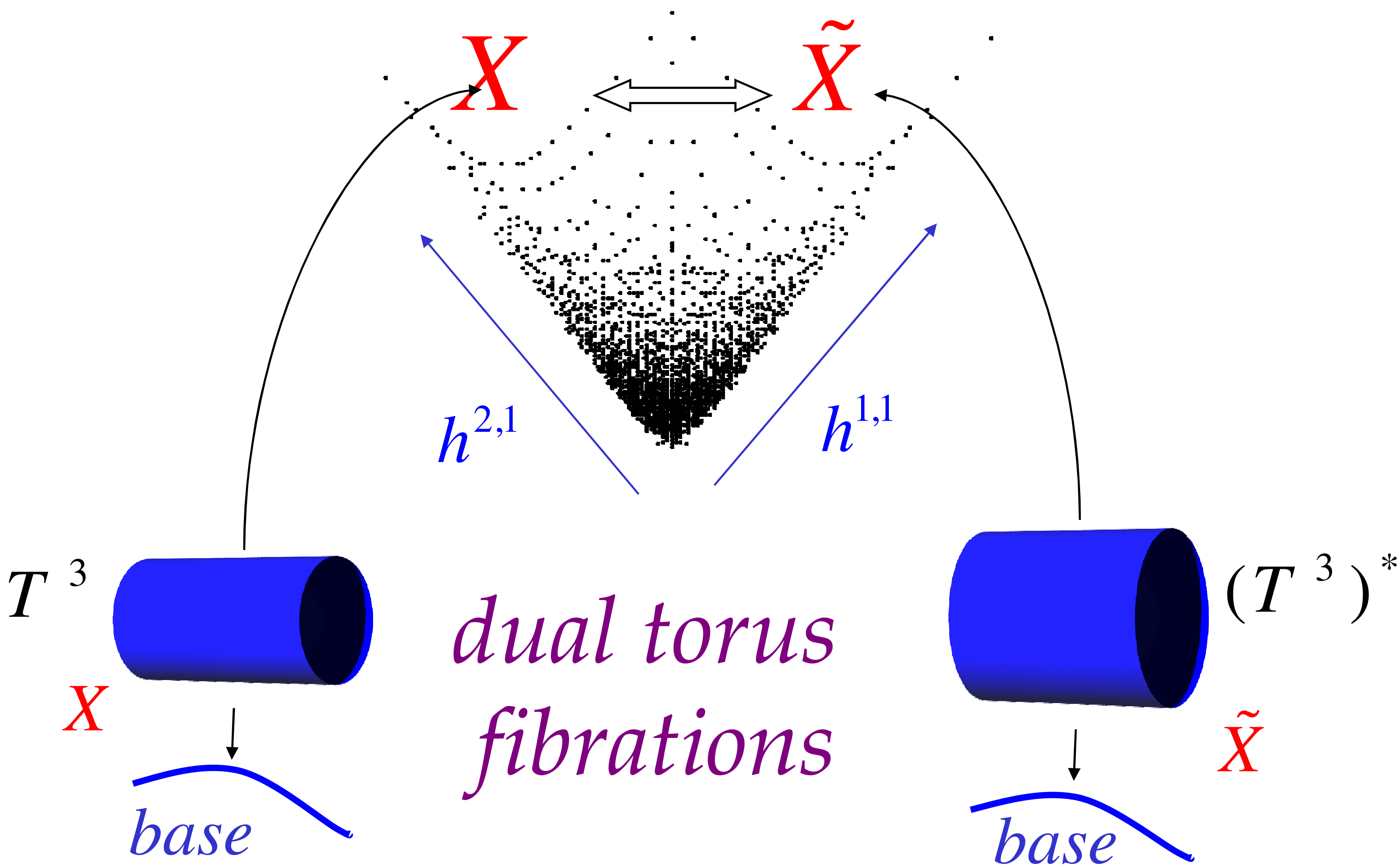
$\tilde{X}$



$$F(t) = \sum_{d \geq 0} N_d e^{-td/\hbar}$$

Picard-Fuchs differential eqns

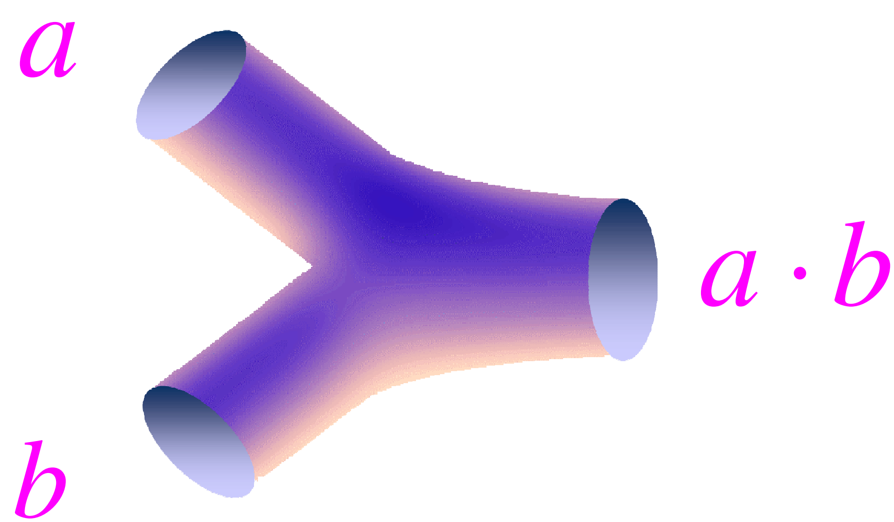
Fiberwise Duality



String Interactions

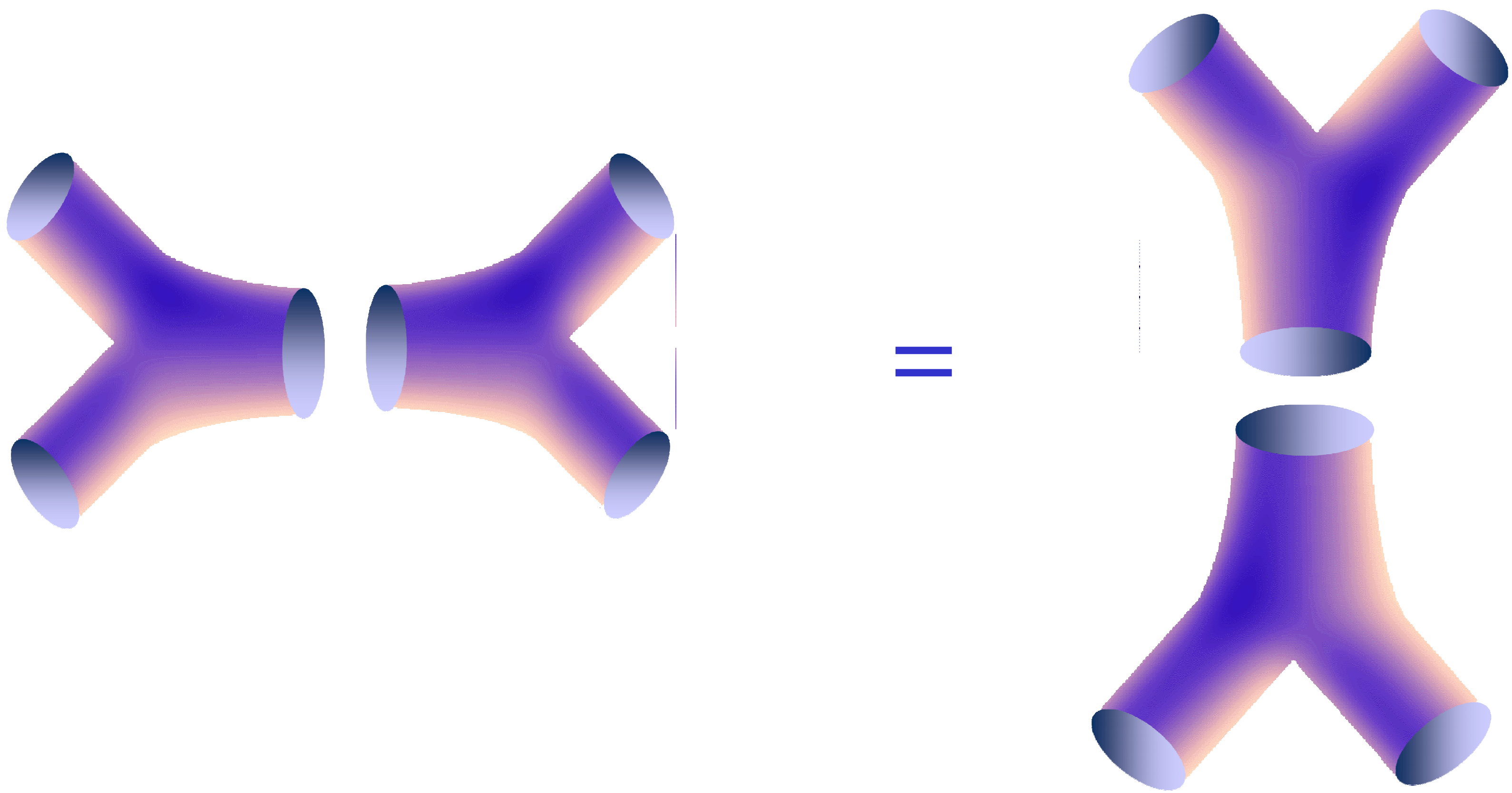
Frobenius algebra

$$\boxed{\frac{\partial^3 F}{\partial t^3}}$$



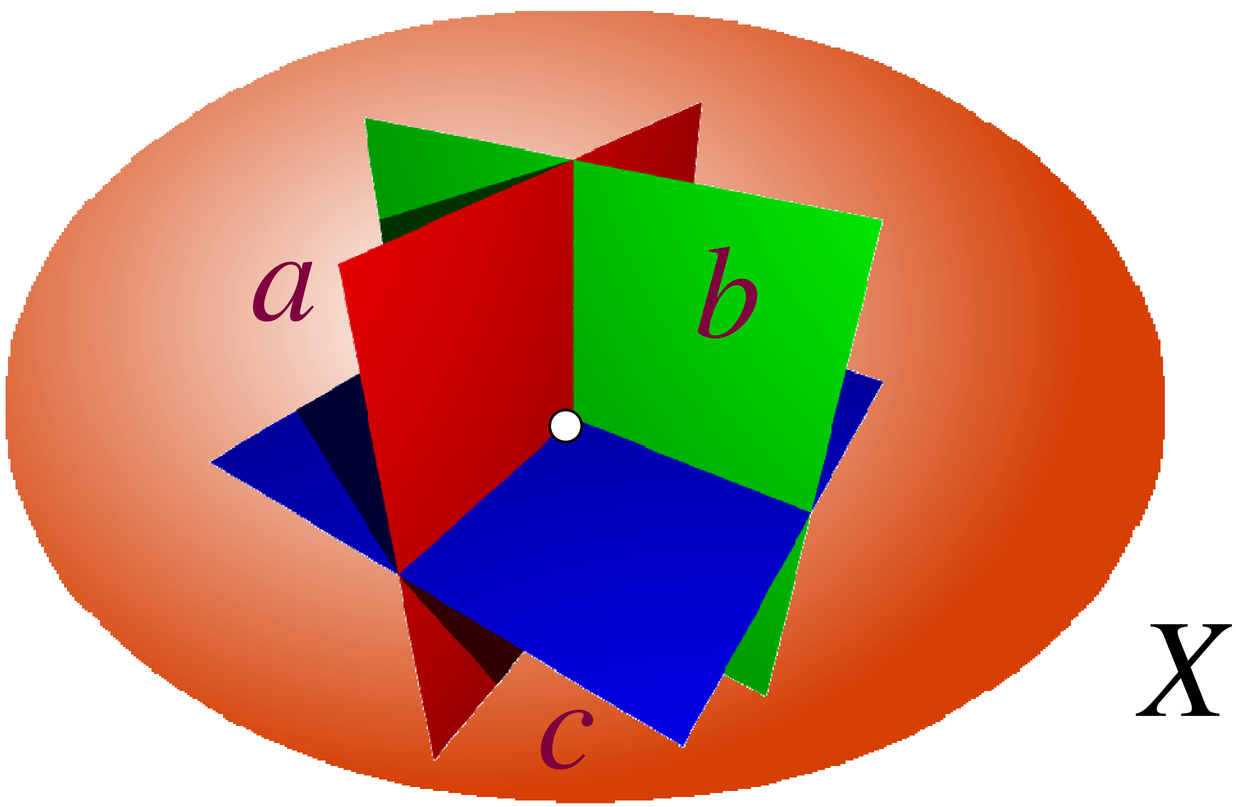
$\Rightarrow$ *integrable systems*

Commutative, Associative



Intersection Product

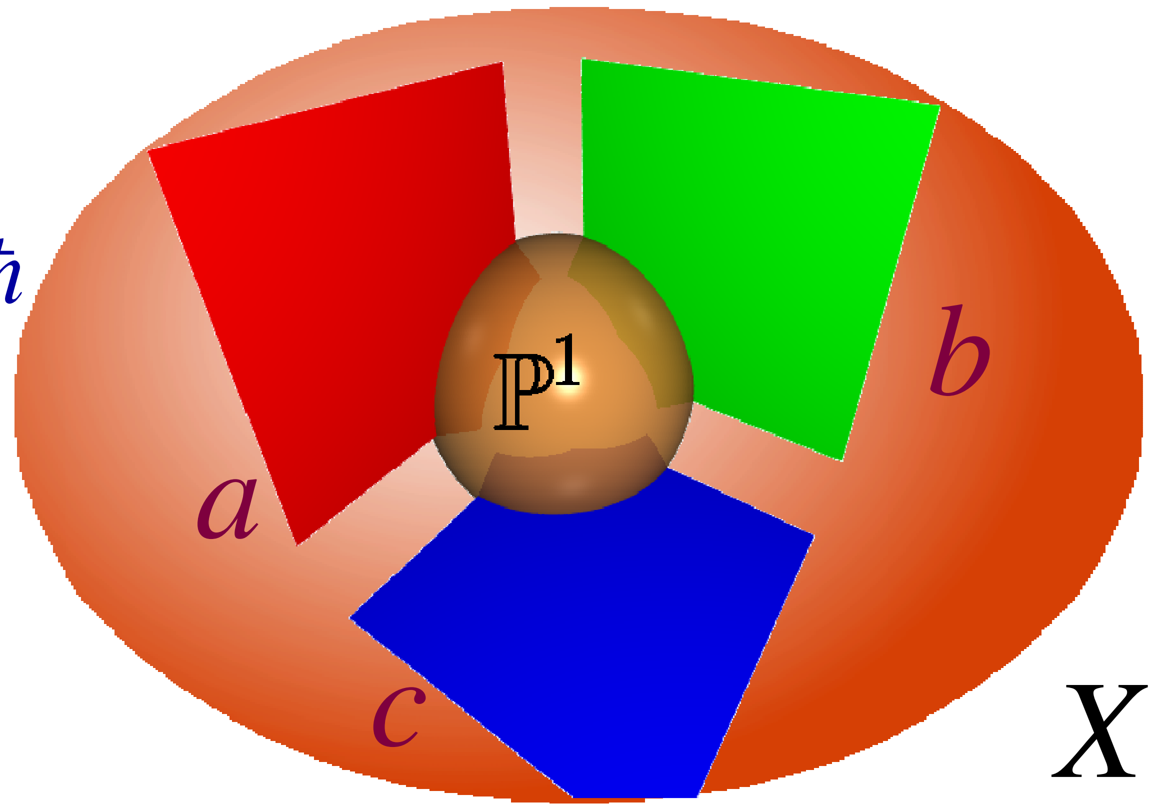
$a \cap b \cap c$



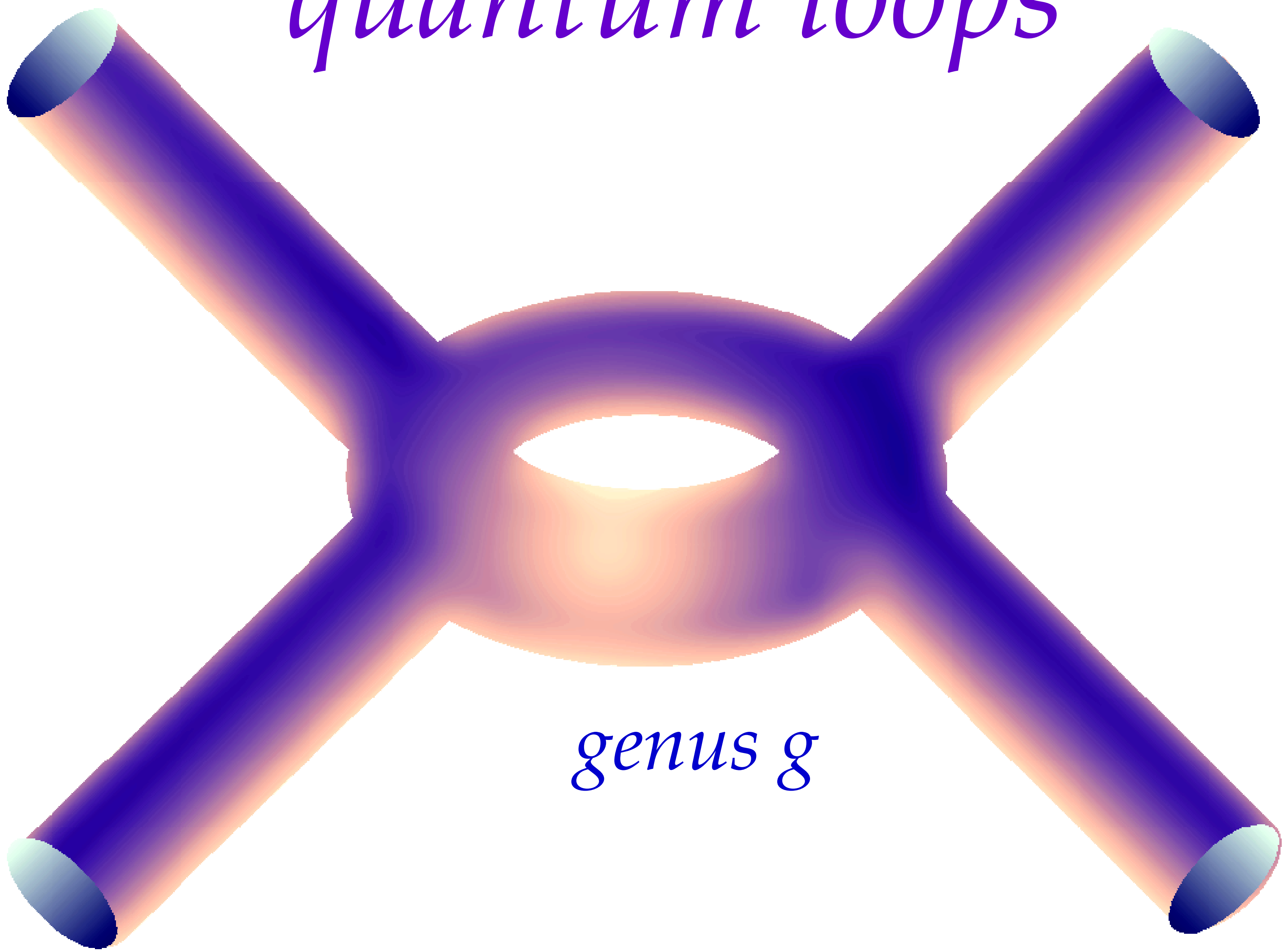
$a, b, c \in H_*(X)$

Quantum Cohomology

$$\sum_{\text{degree } d} e^{-dt/\hbar}$$

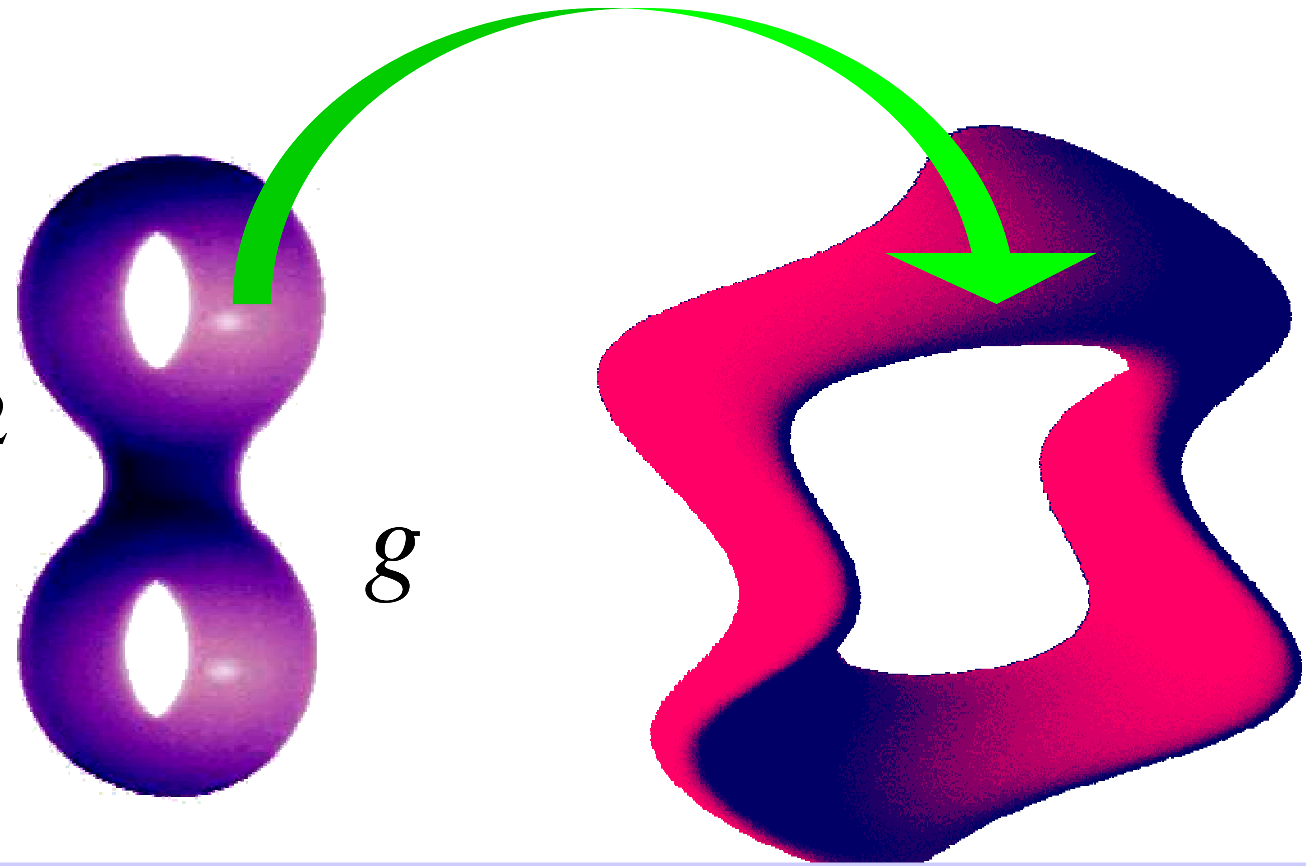


quantum loops



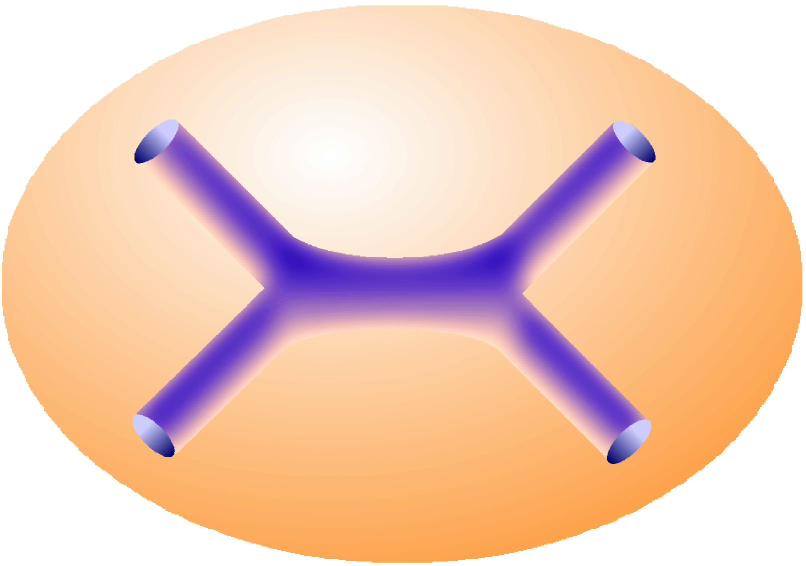
Gromov-Witten theory

$$\sum_g \lambda^{2g-2}$$

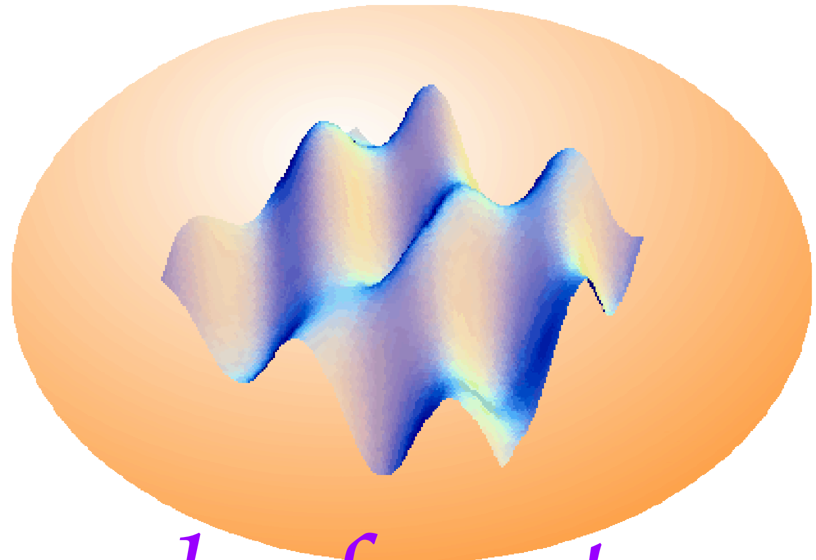


$$Z(\lambda, t) = \exp \sum_{g,d} \lambda^{2g-2} e^{-dt} N_{g,d}$$

strings



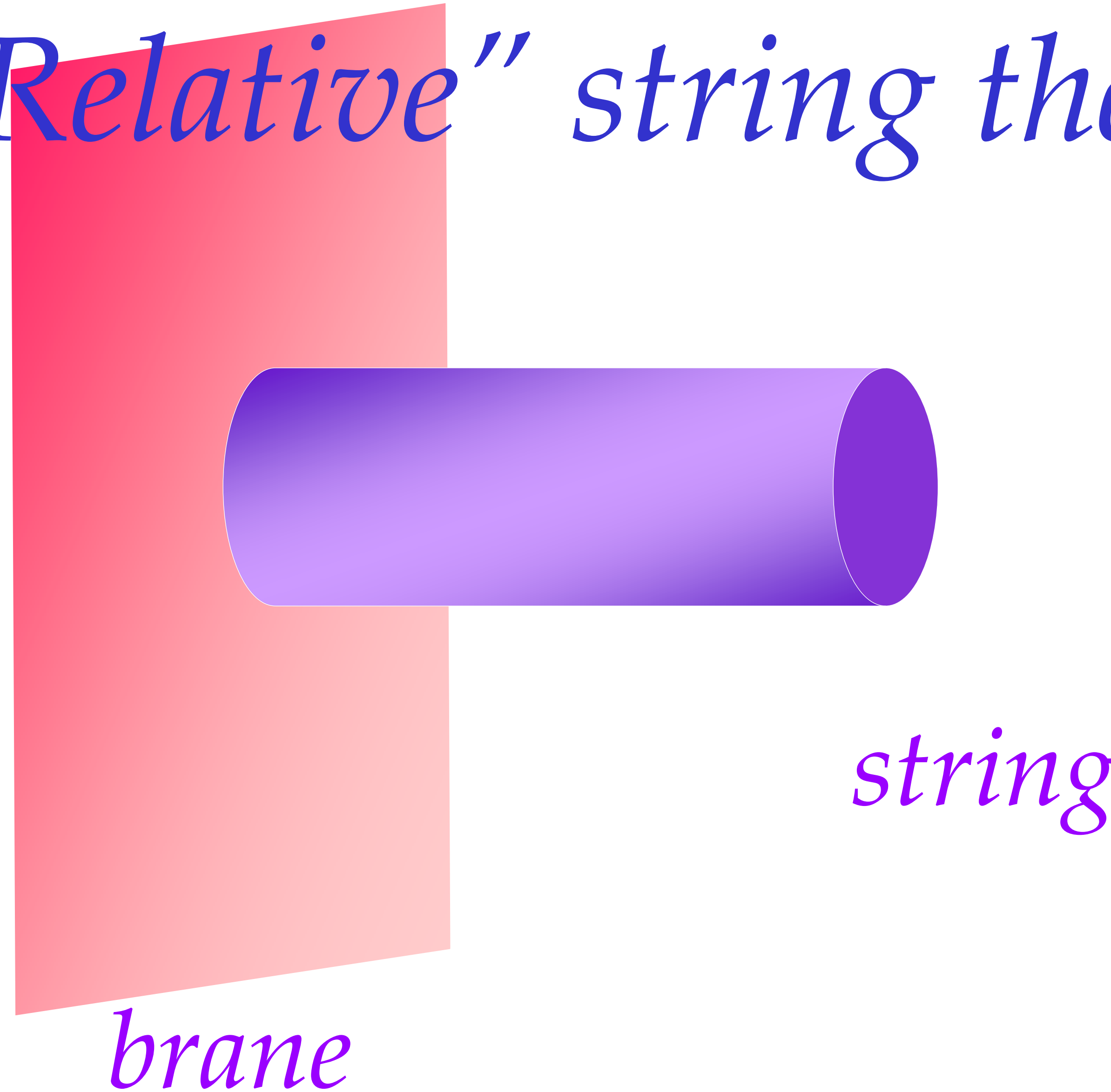
branes



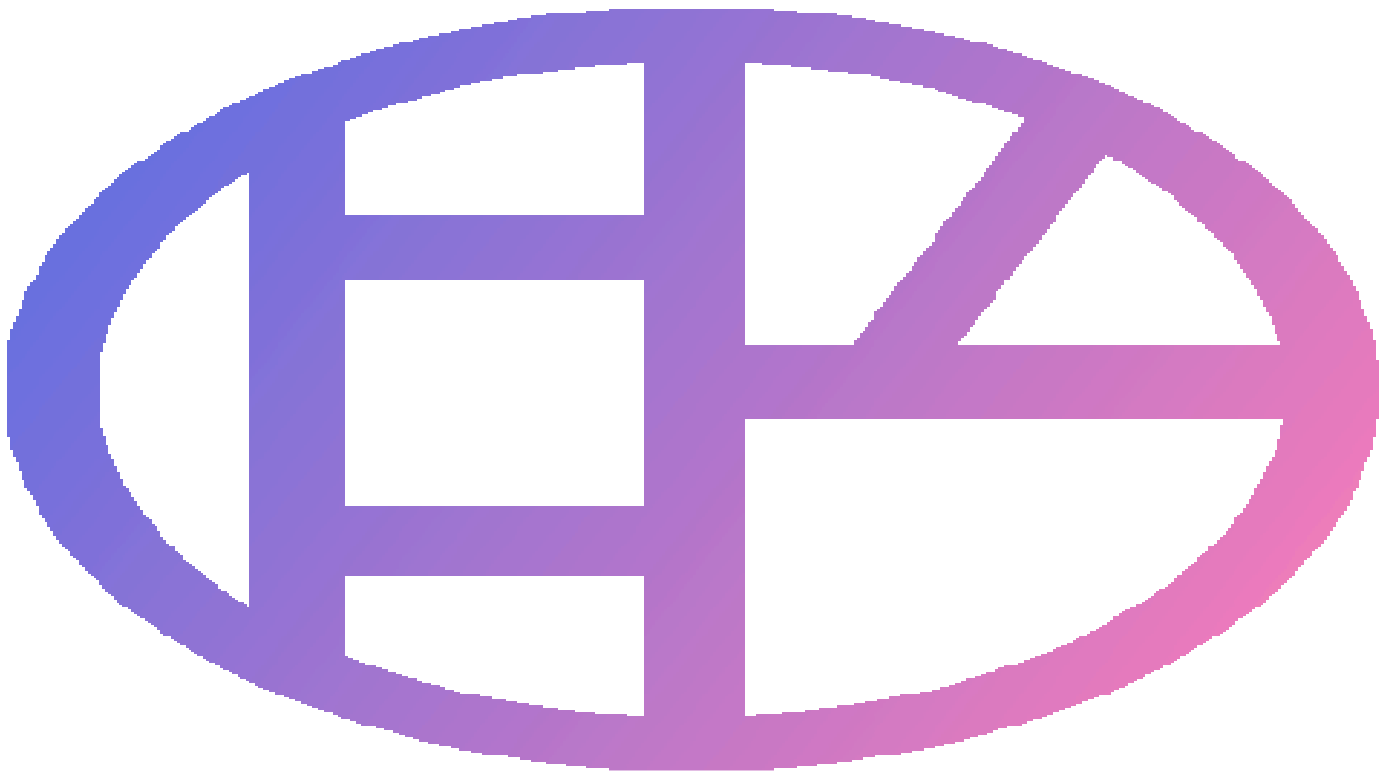
sheaf, vector bundle

$\lambda = 0$ $\lambda = \infty$

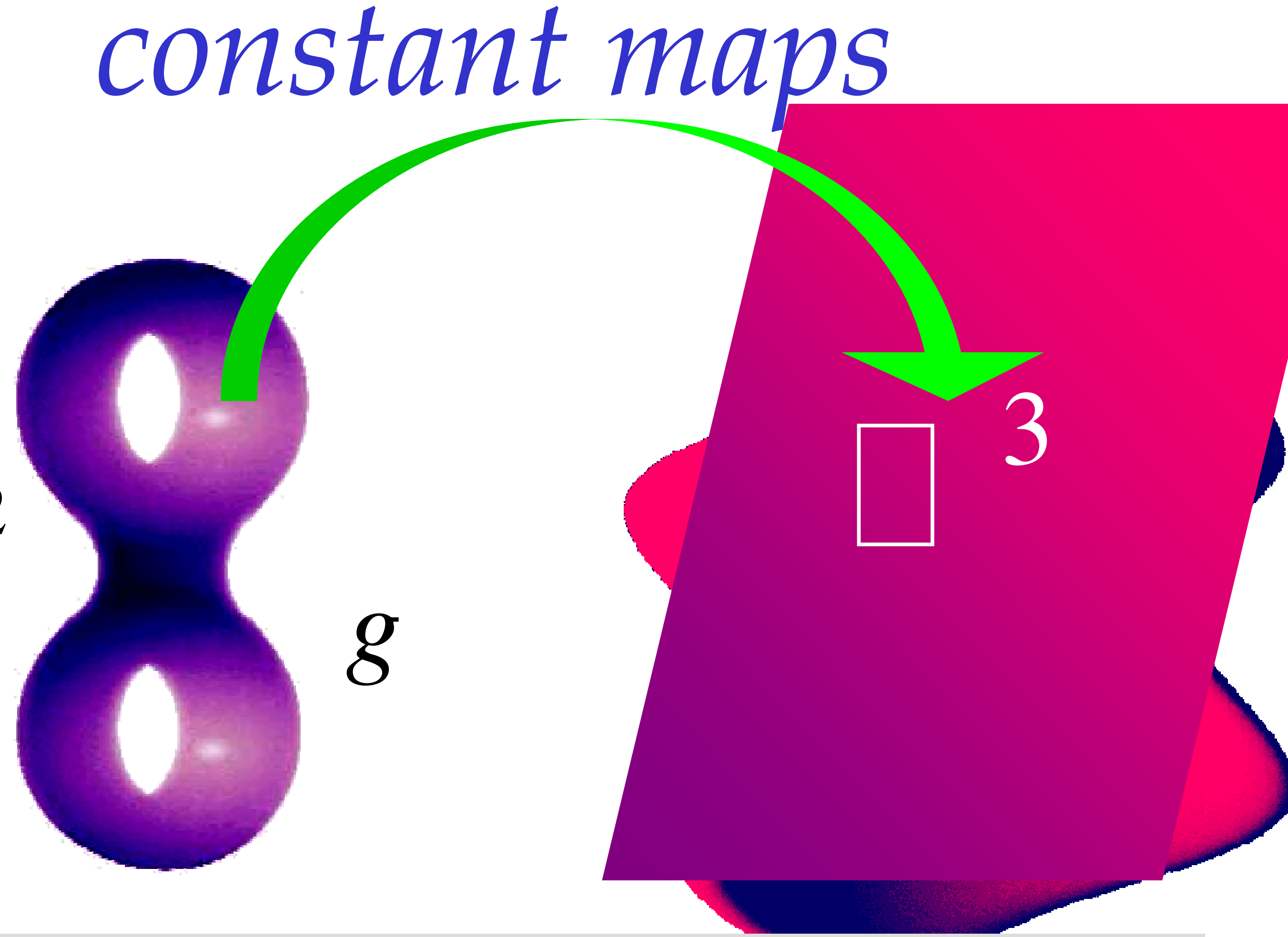
“Relative” string theory



open string diagrams



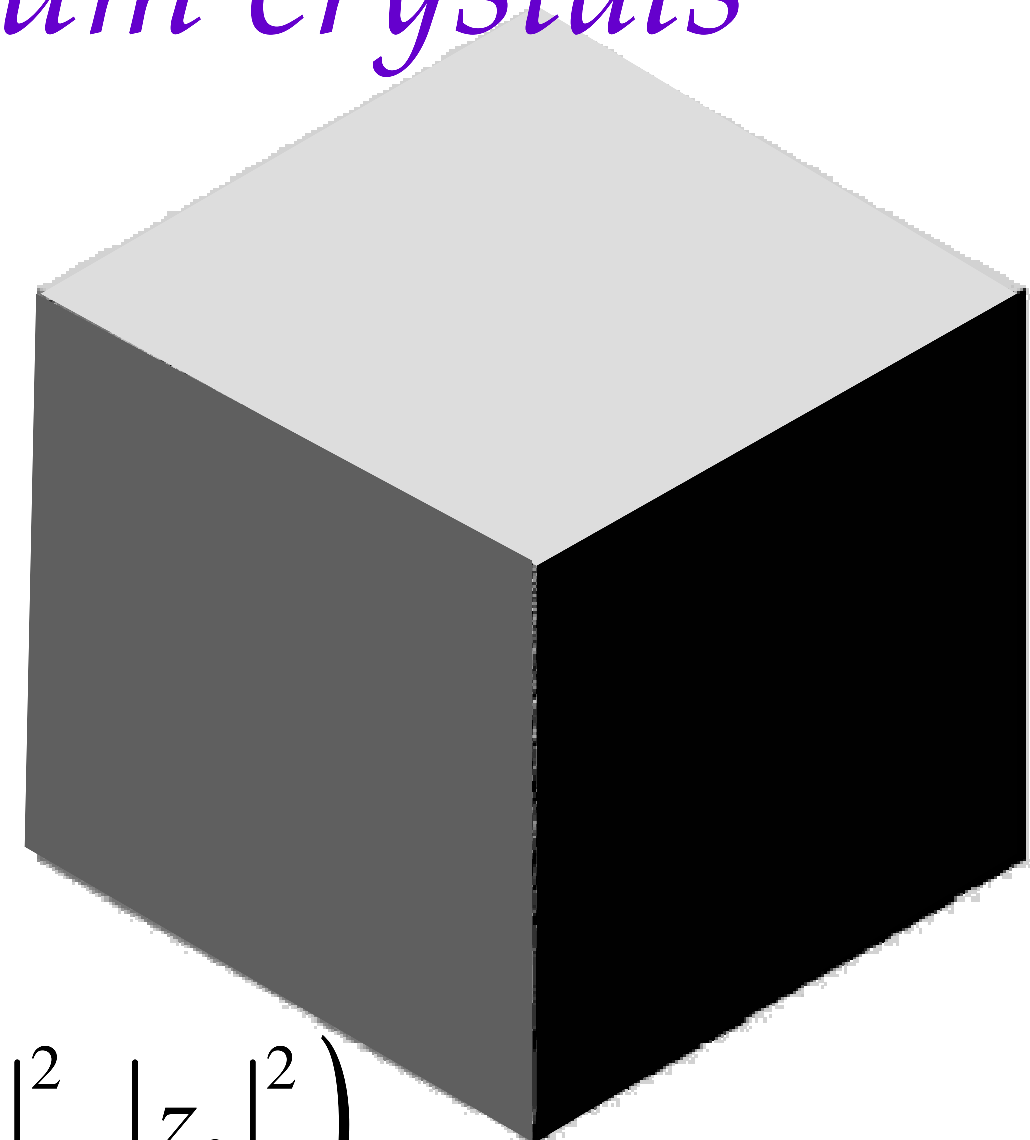
$$\sum_g \lambda^{2g-2}$$



$$N_{g,d=0} = \int_{\overline{M}_g} \lambda_{g-1}^3 = \frac{B_{2g} B_{2g-2}}{2g(2g-2)(2g-2)!}$$

Quantum crystals

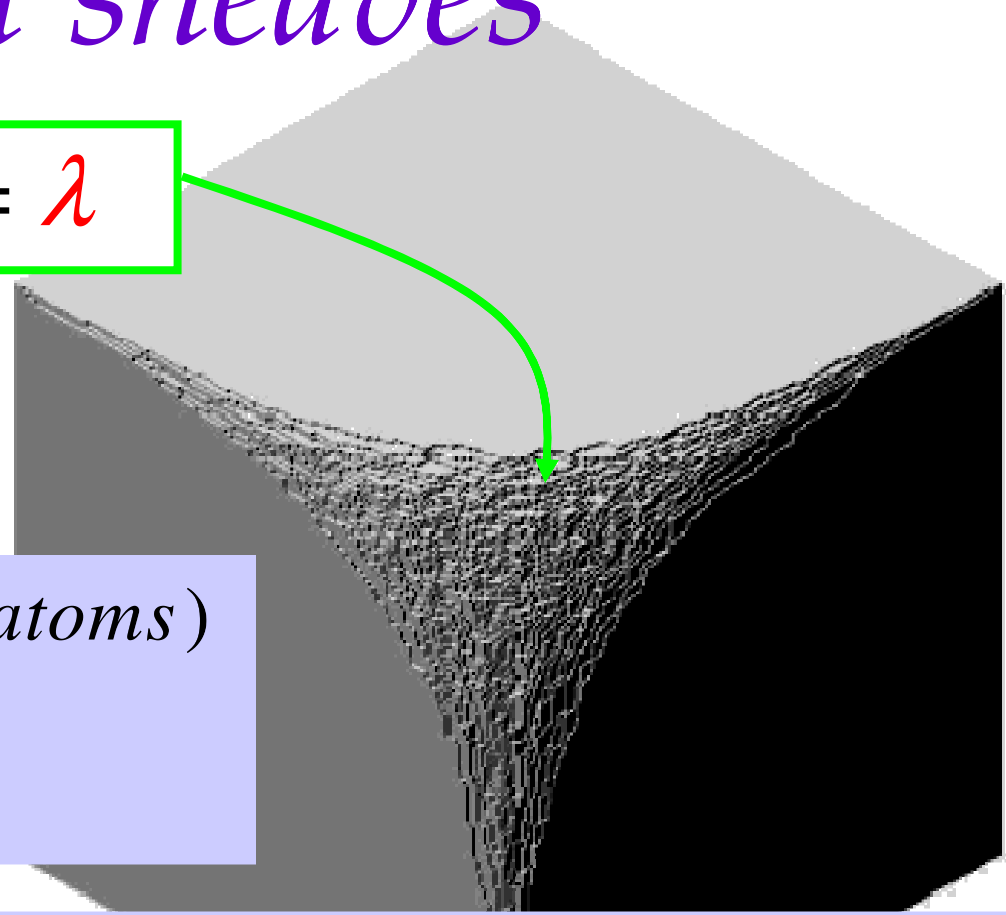
$$\mathbb{C}^3 \to \mathbb{R}^3_+$$



$$\left(|z_1|^2, |z_2|^2, |z_3|^2\right)$$

Ideal sheaves

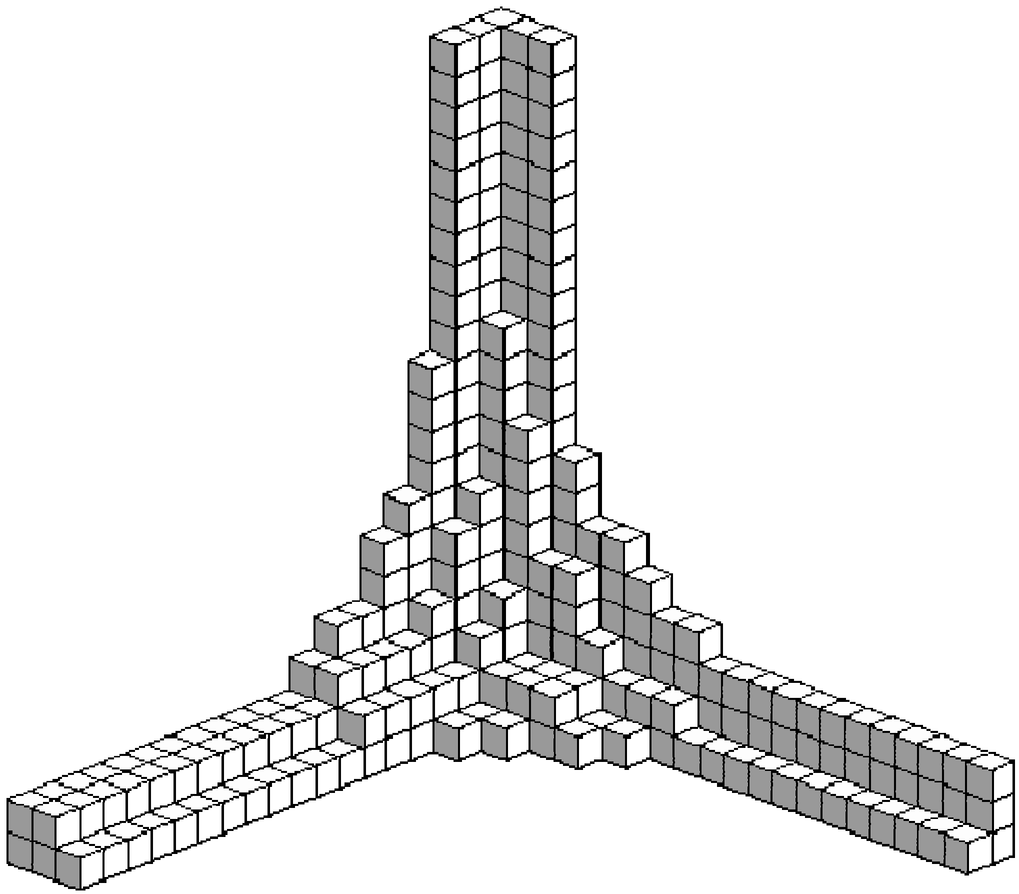
size atom $= \lambda$



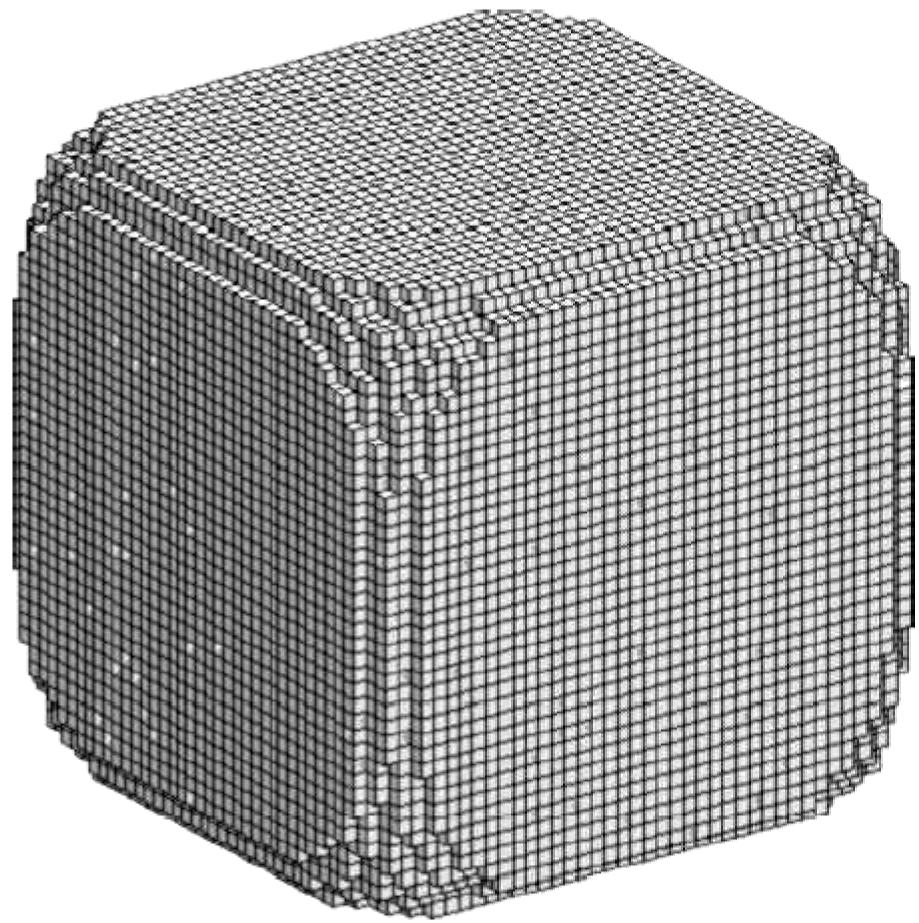
$$Z = \sum_{crystals} e^{-\lambda \#(atoms)}$$

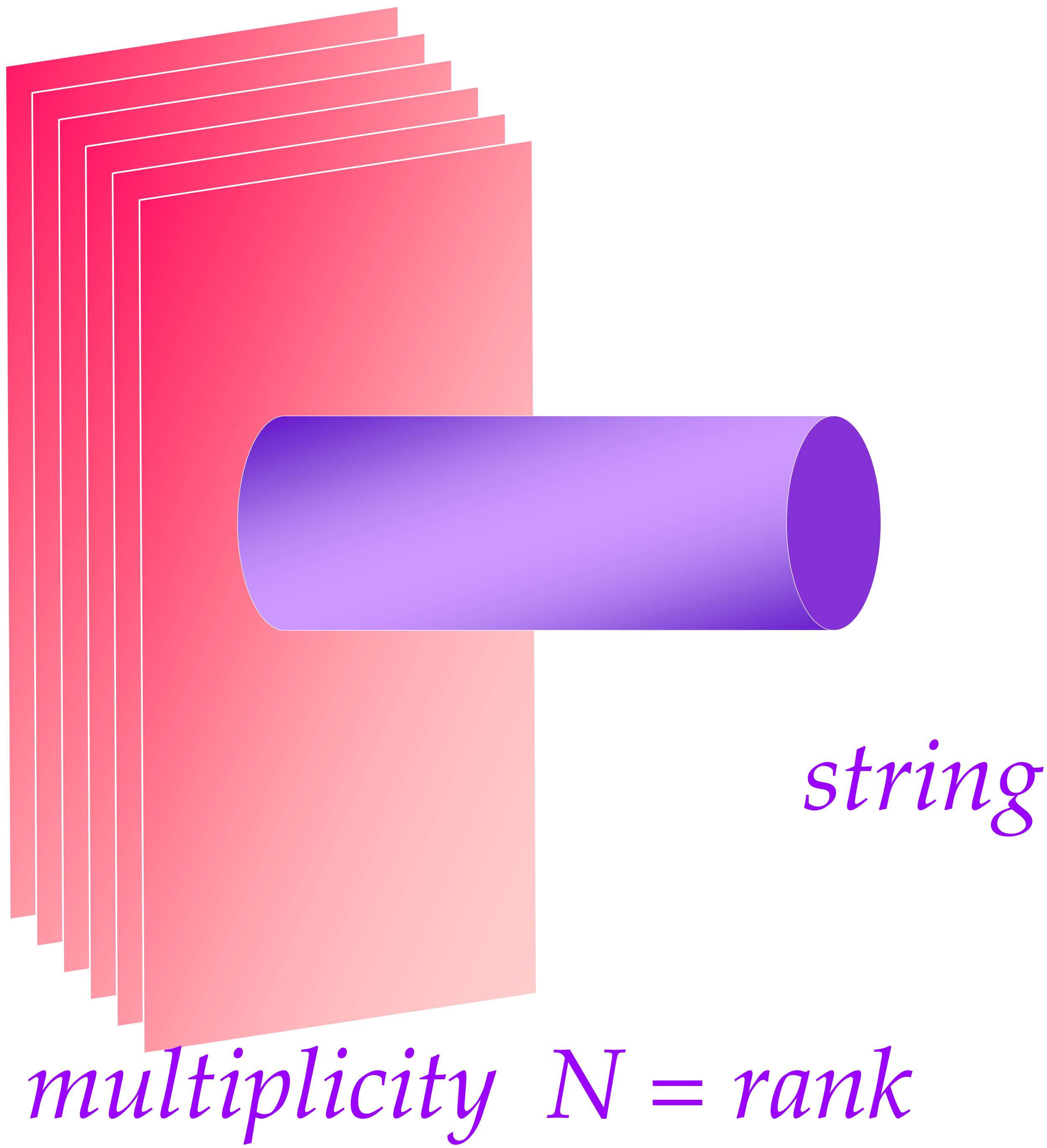
$$= \prod_{n>0} \left(1 - e^{-n\lambda}\right)^{-n} \square \exp \sum_{g} N_{g,0} \lambda^{2g-2}$$

brane =3-dim partitions



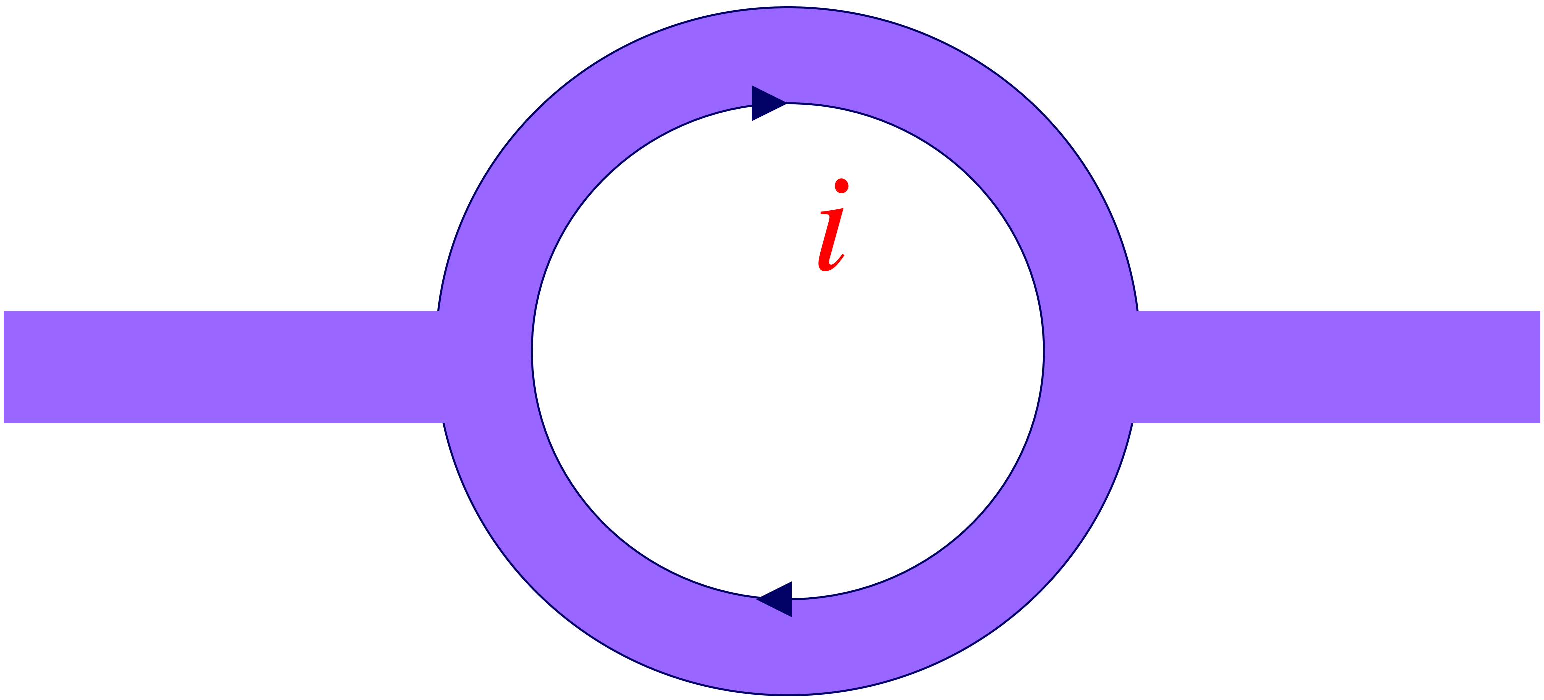
quantum 3-fold, melting crystal





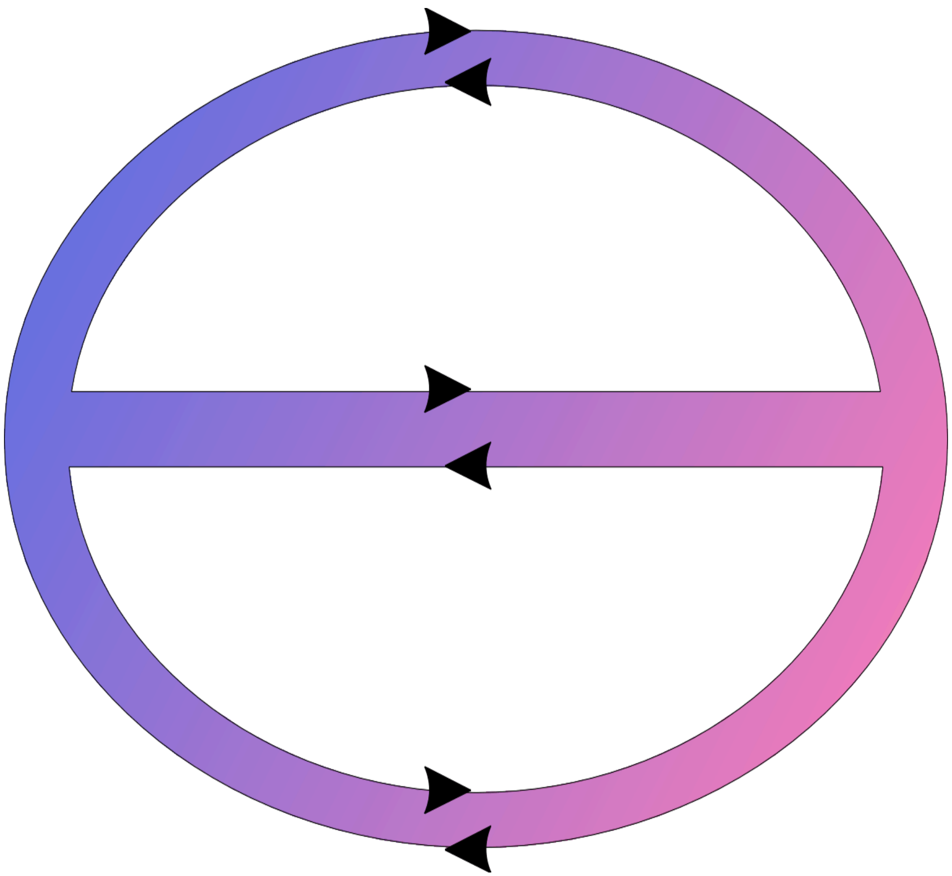
string
multiplicity N = rank

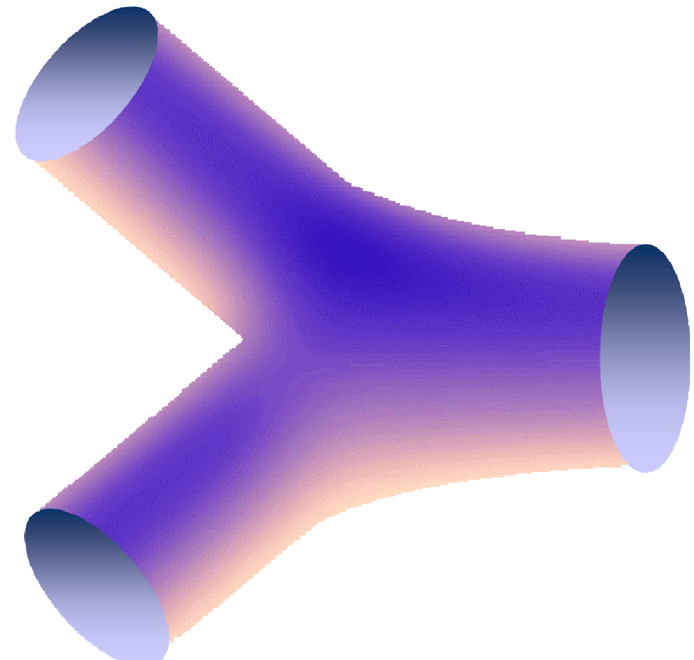
Simple N dependence



$$\sum_i 1 = N$$

Planar diagrams

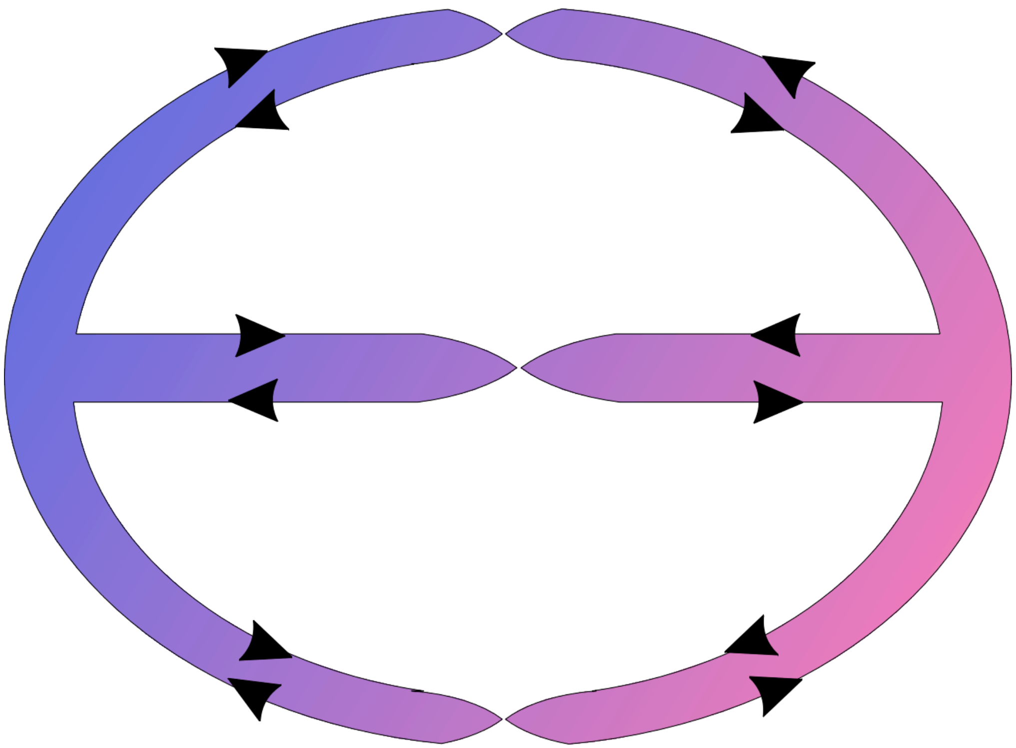


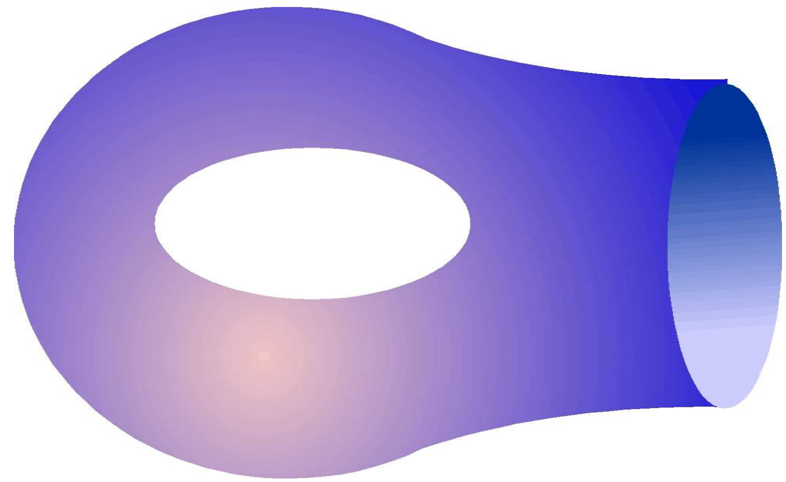


□ N^3

2 loops
3 holes

suppressed by $1/N^2$

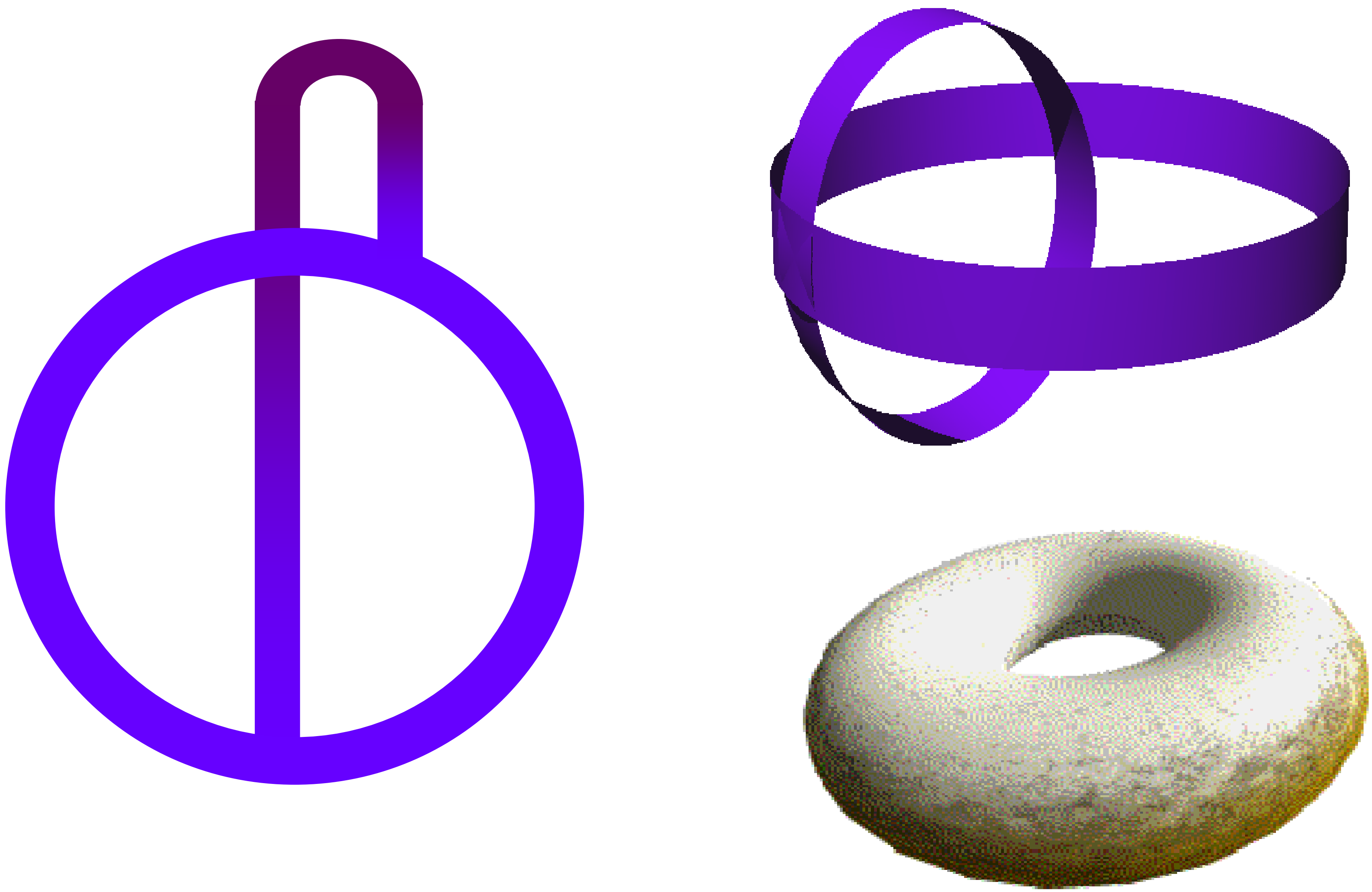




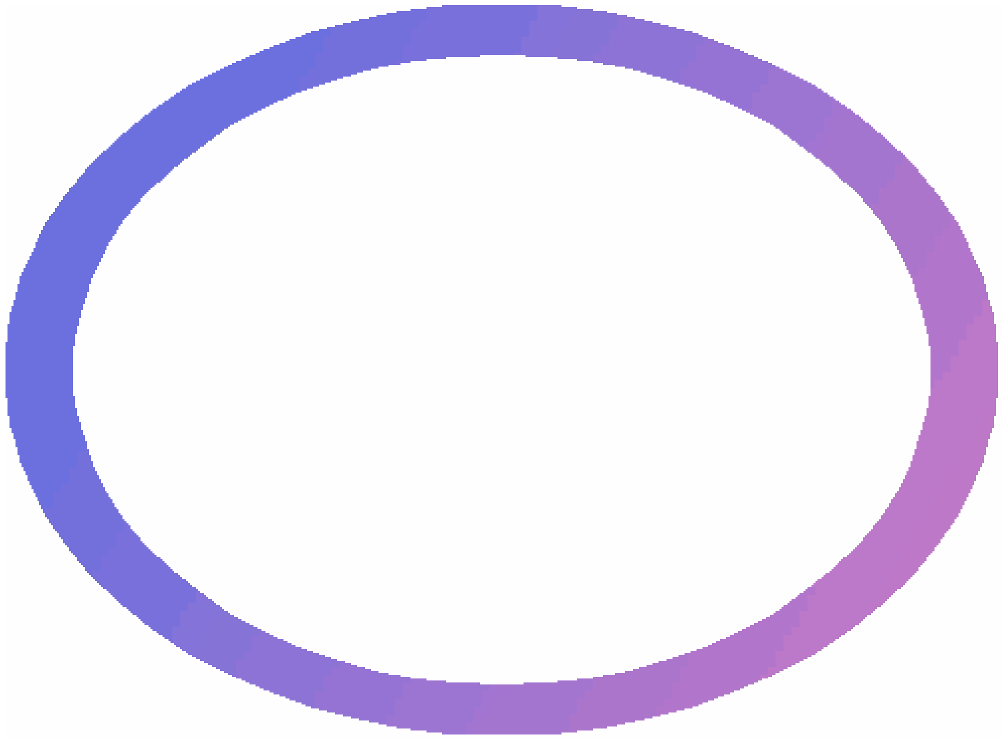
□ N

2 loops
1 hole

Genus one



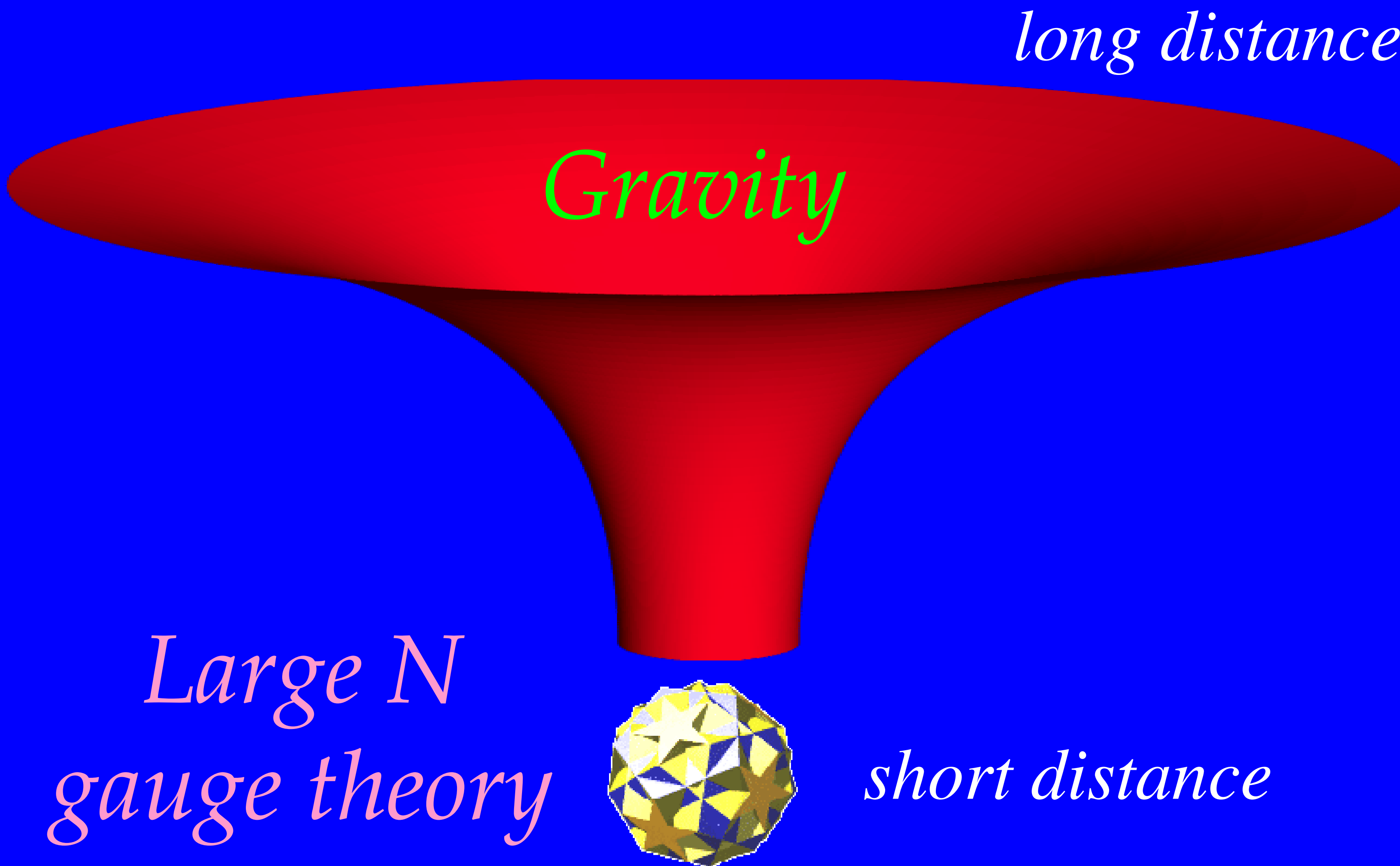
Interactions: Fishnet diagrams



Closed string worldsheet

Quantum Geometry?

Geometry is "Effective"



Wigner: Random Matrix

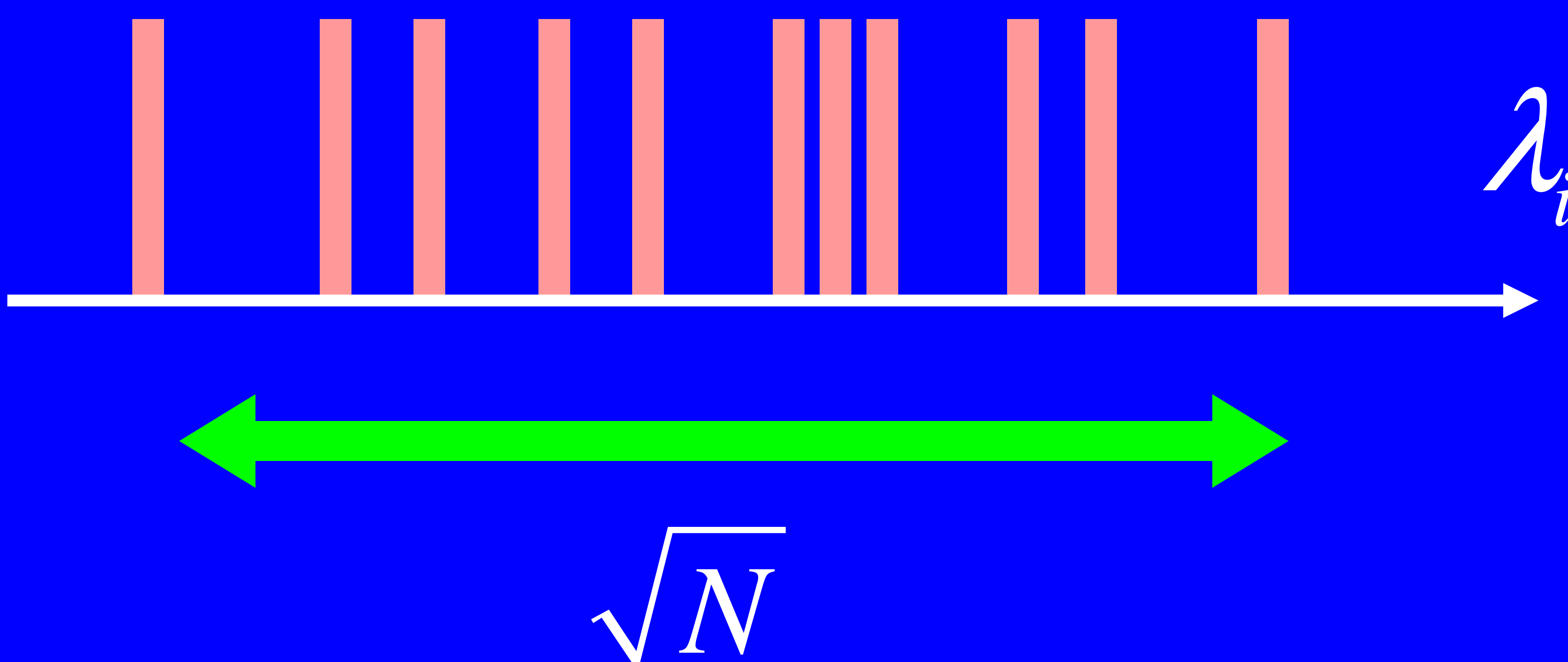
$$\boxed{\lim_{N\to\infty} \int_{N\times N} d\Phi \; e^{-N \,\mathrm{Tr}\,\Phi^2}}$$

Gaussian Ensemble

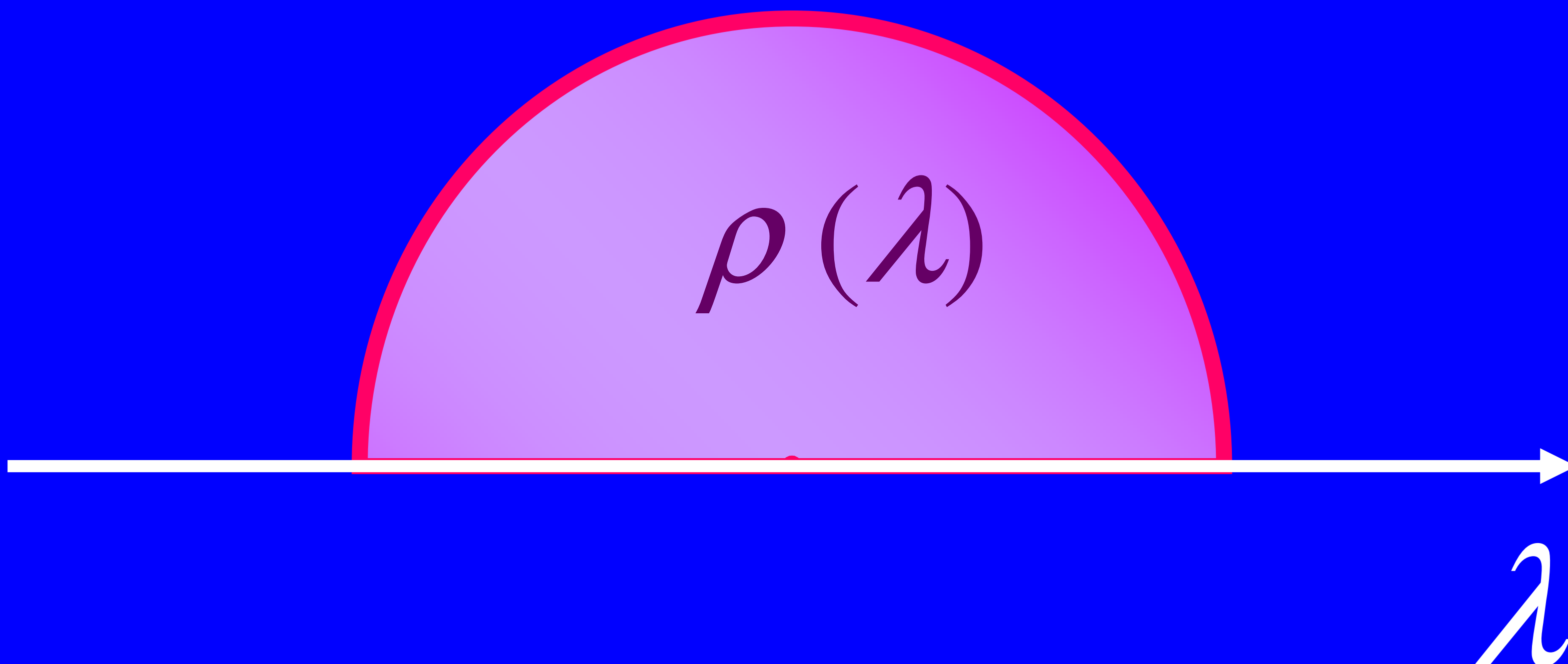
Diagonalize $N \times N$ *matrix* Φ_{ij}

$$\Phi = U \cdot \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_N \end{pmatrix} \cdot U^{-1}$$

Spectrum of Φ_{ij}

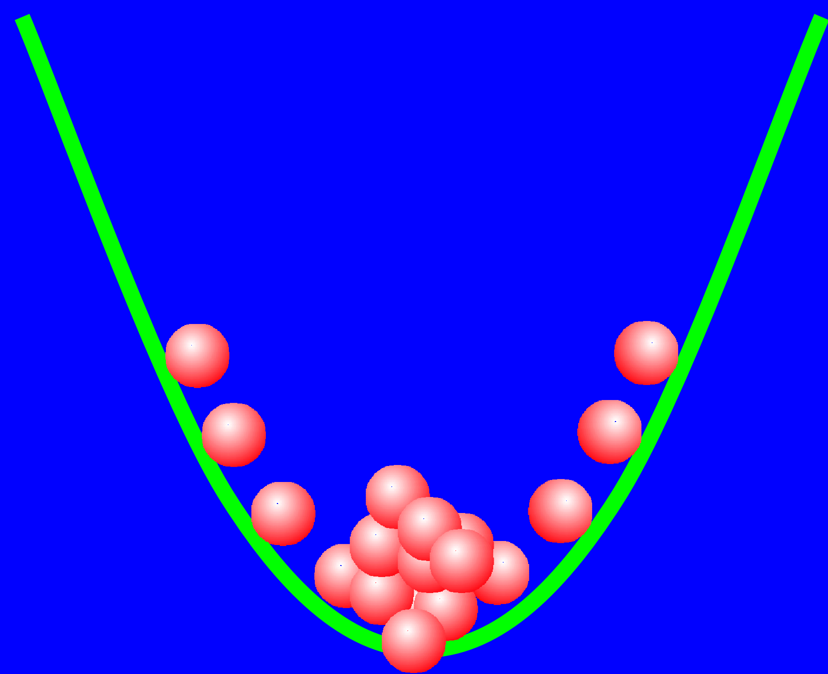


Spectral Density $N \rightarrow \infty$

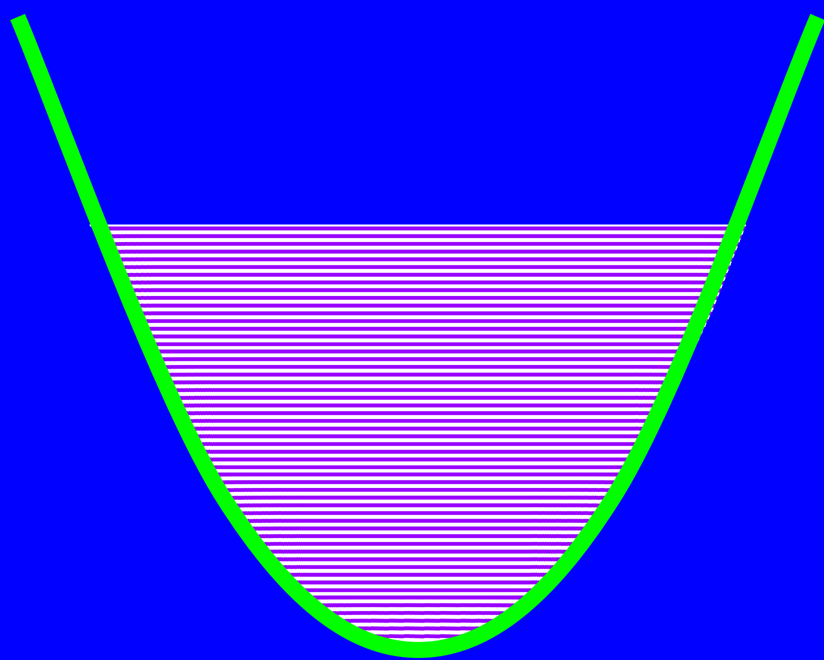


Wigner's semi-circle law

Eigenvalue repulsion

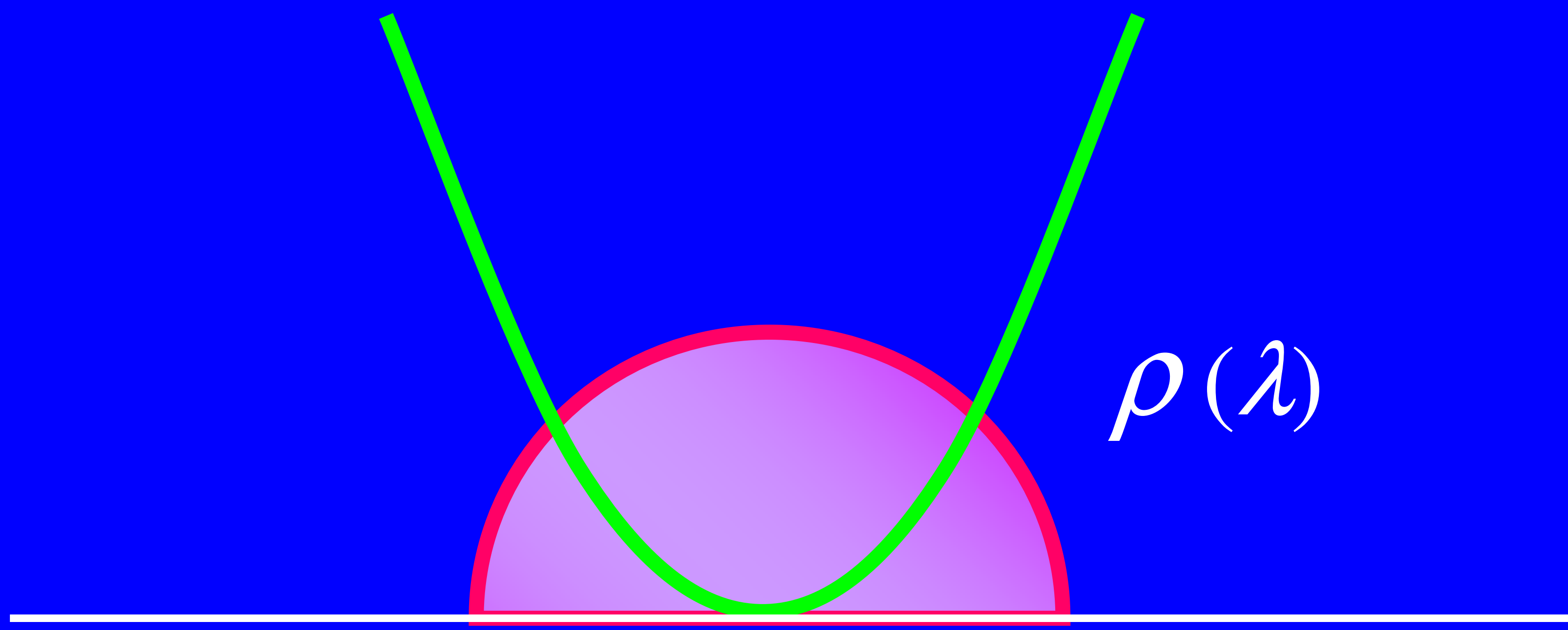


Eigenvalue repulsion



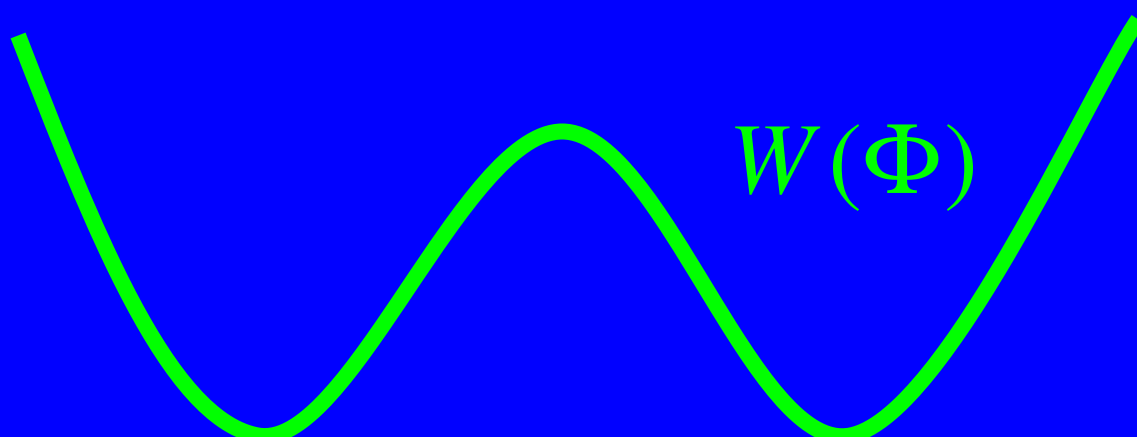
Filling the “Dyson sea”

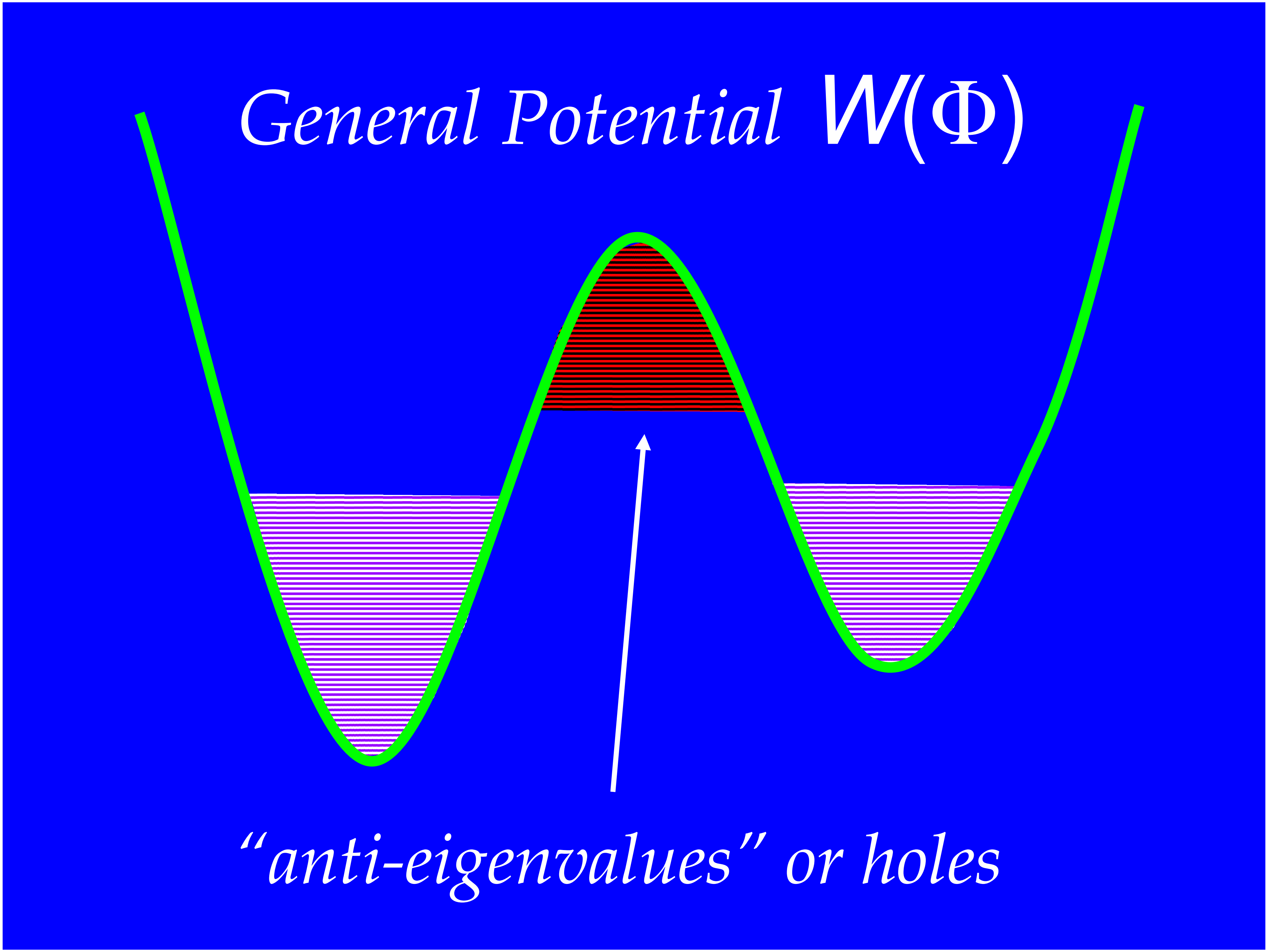
Density Profile



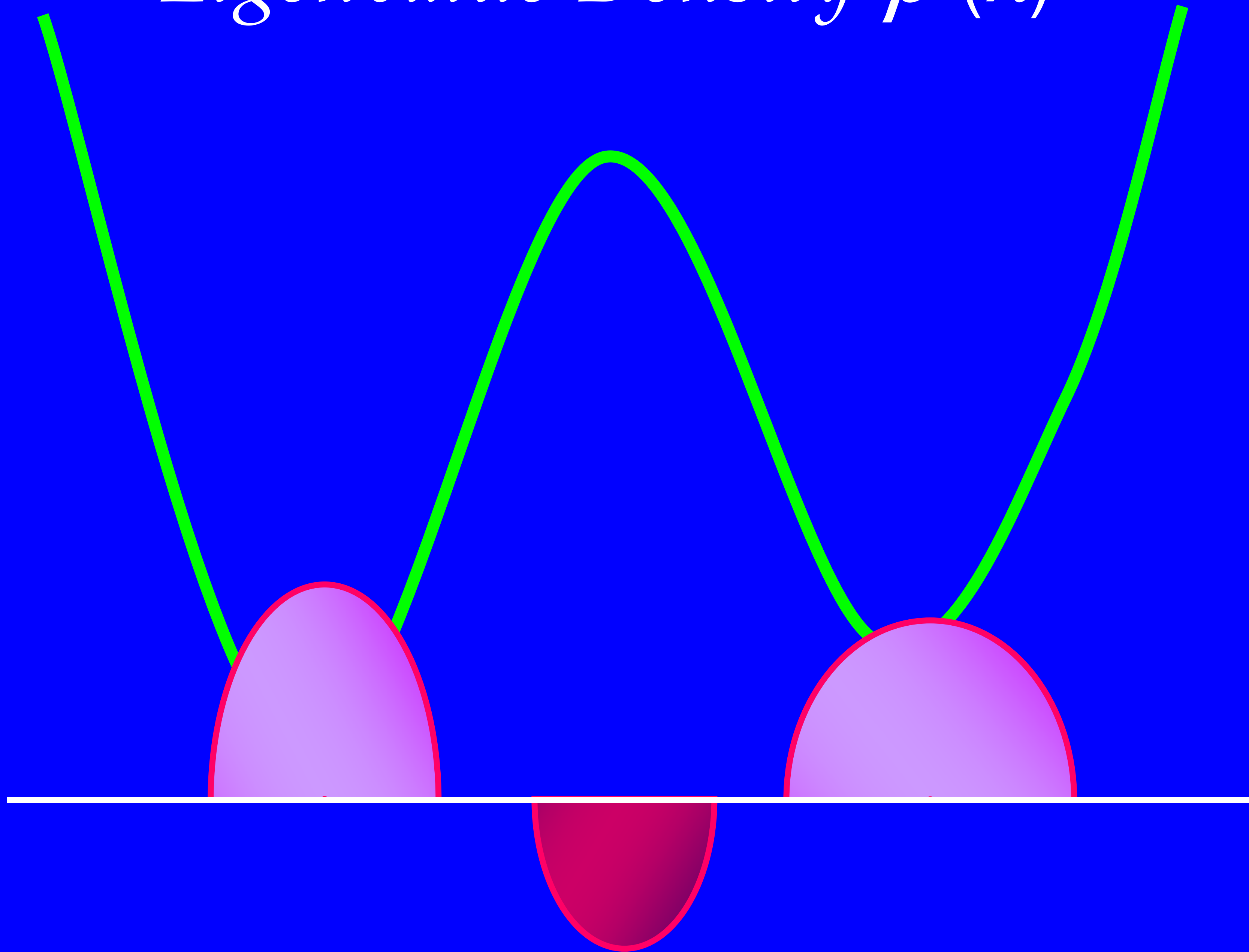
Random Matrix Model

$$\lim_{N\to\infty}\int_{N\times N} d\Phi\; e^{-\frac{1}{\hbar}\mathrm{Tr}\,W(\Phi)}$$

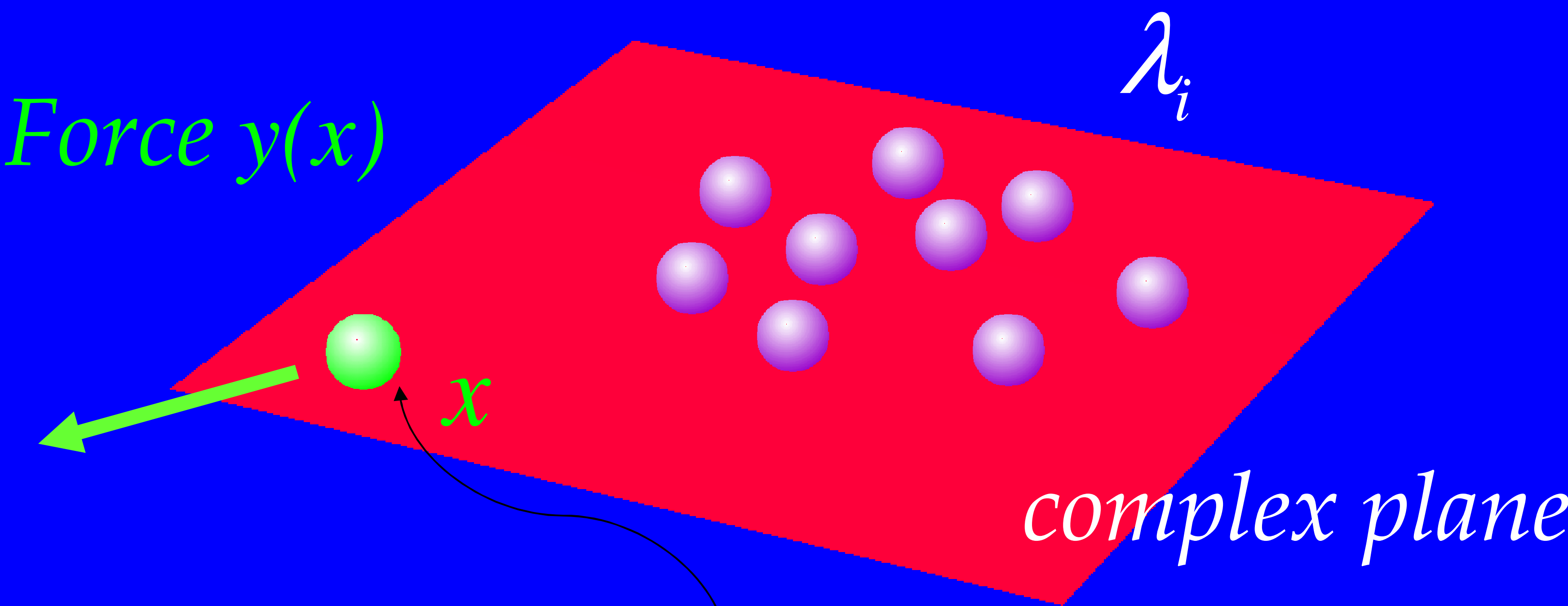



General Potential $W(\Phi)$
"anti-eigenvalues" or holes

Eigenvalue Density $\rho(\lambda)$

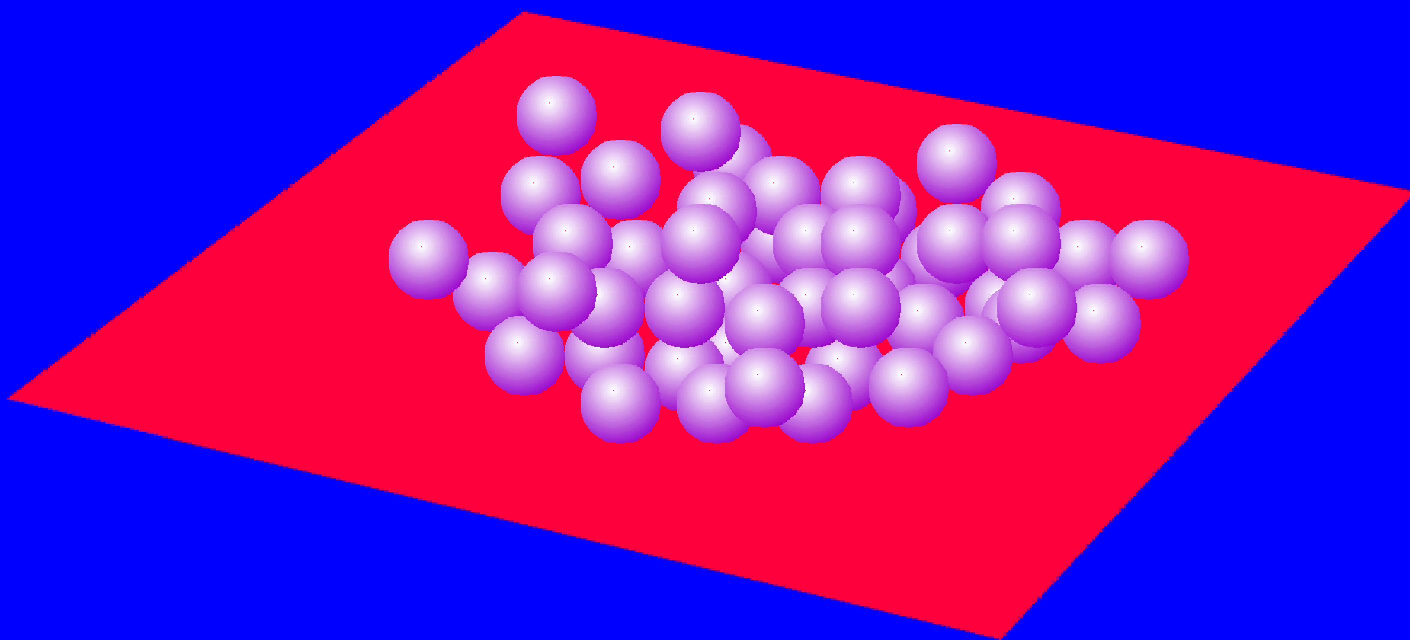


Probe eigenvalue

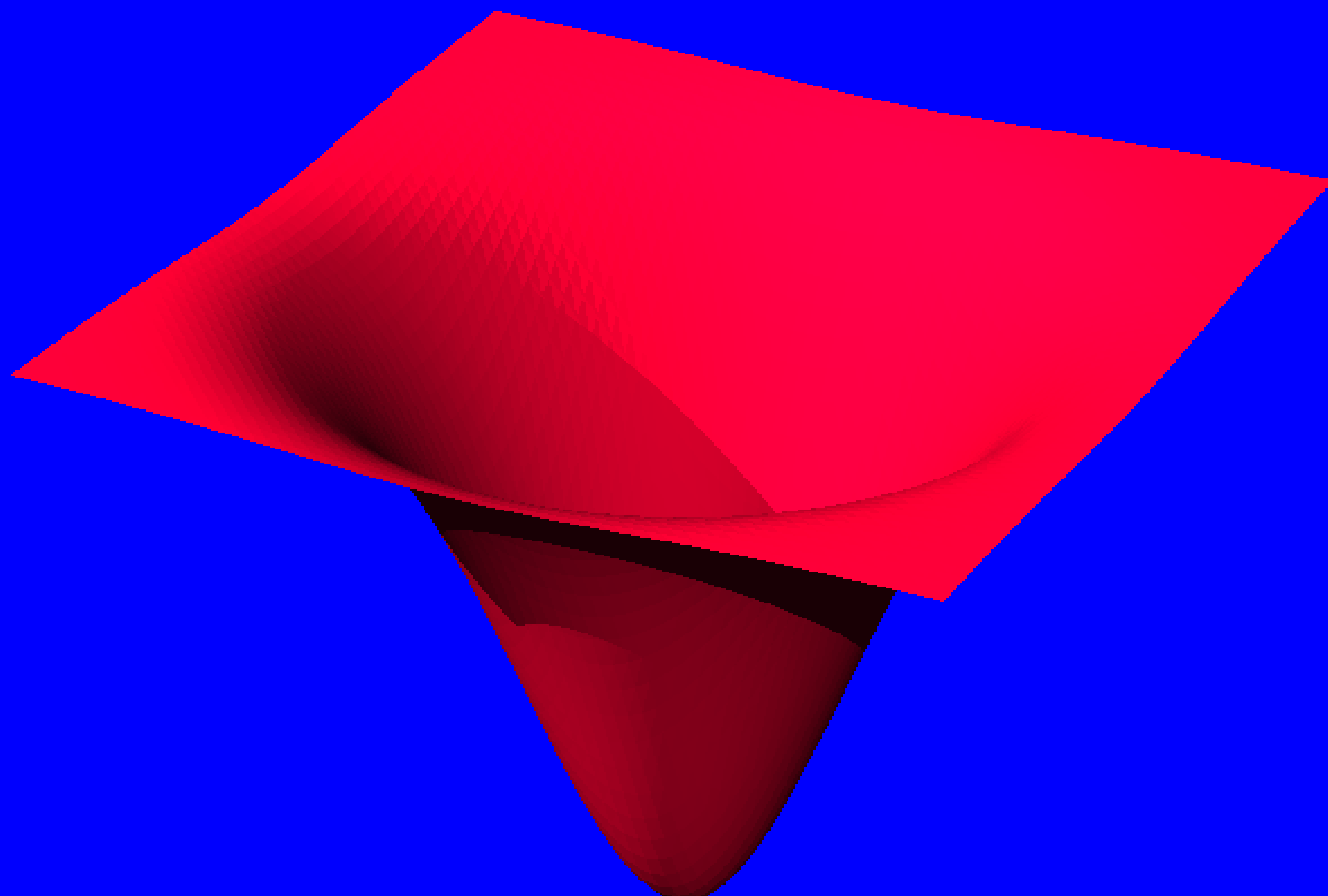


$$y(x) = W'(x) - 2\hbar \sum_i \frac{1}{x - \lambda_i}$$

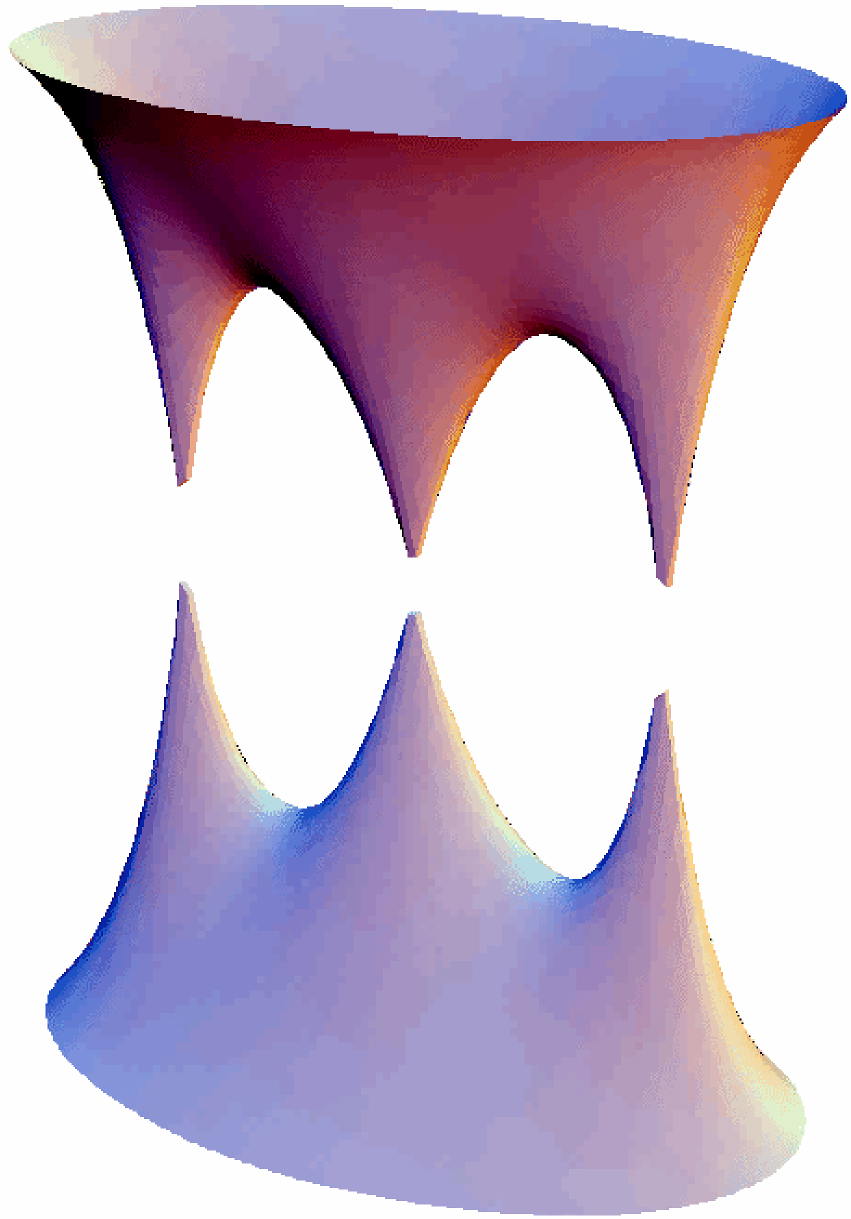
$N \to \infty$ *Effective Geometry*



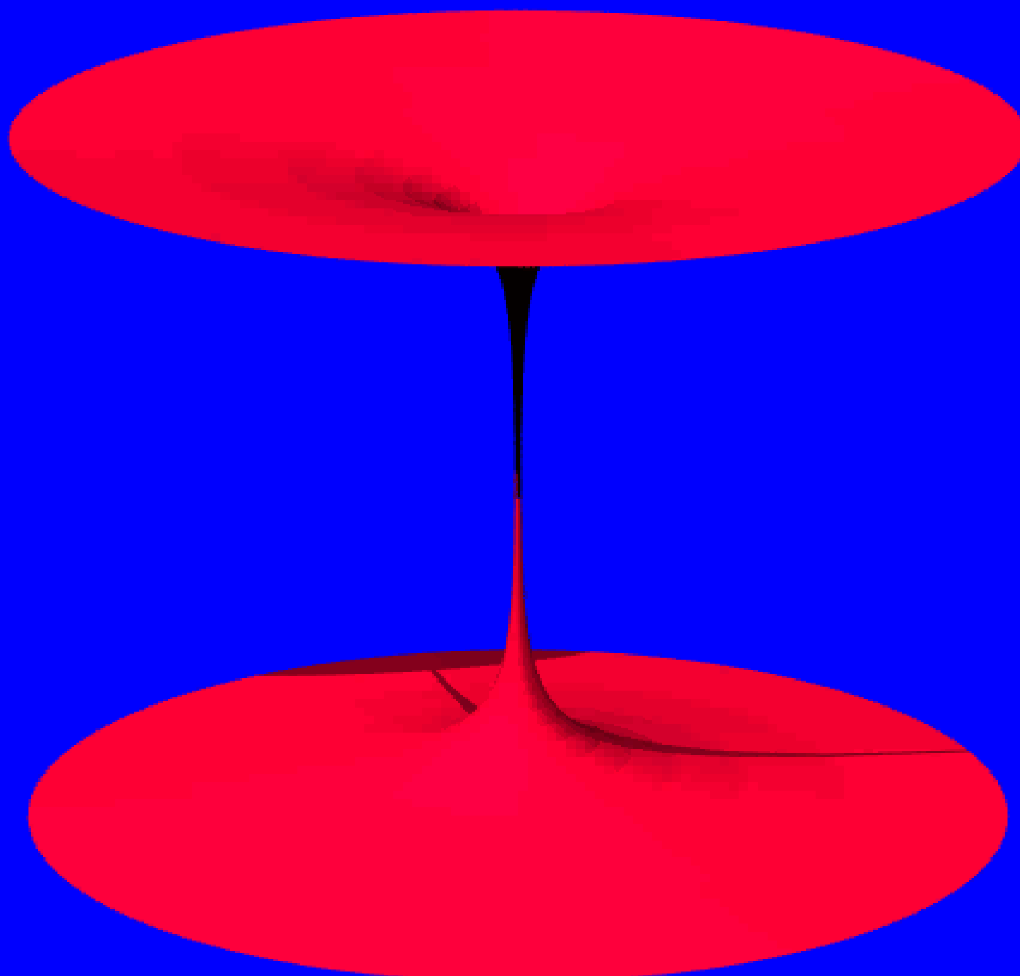
$N \to \infty$ *Effective Geometry*



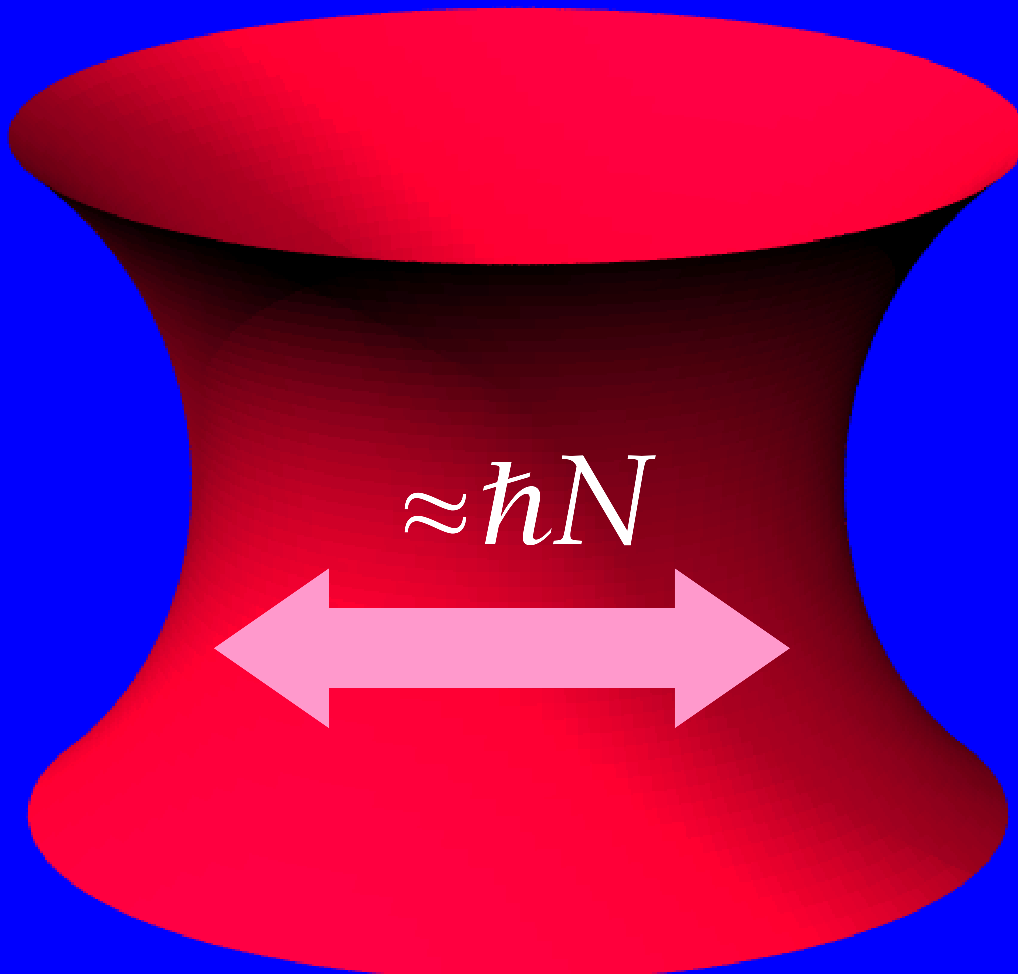
Smooth curve



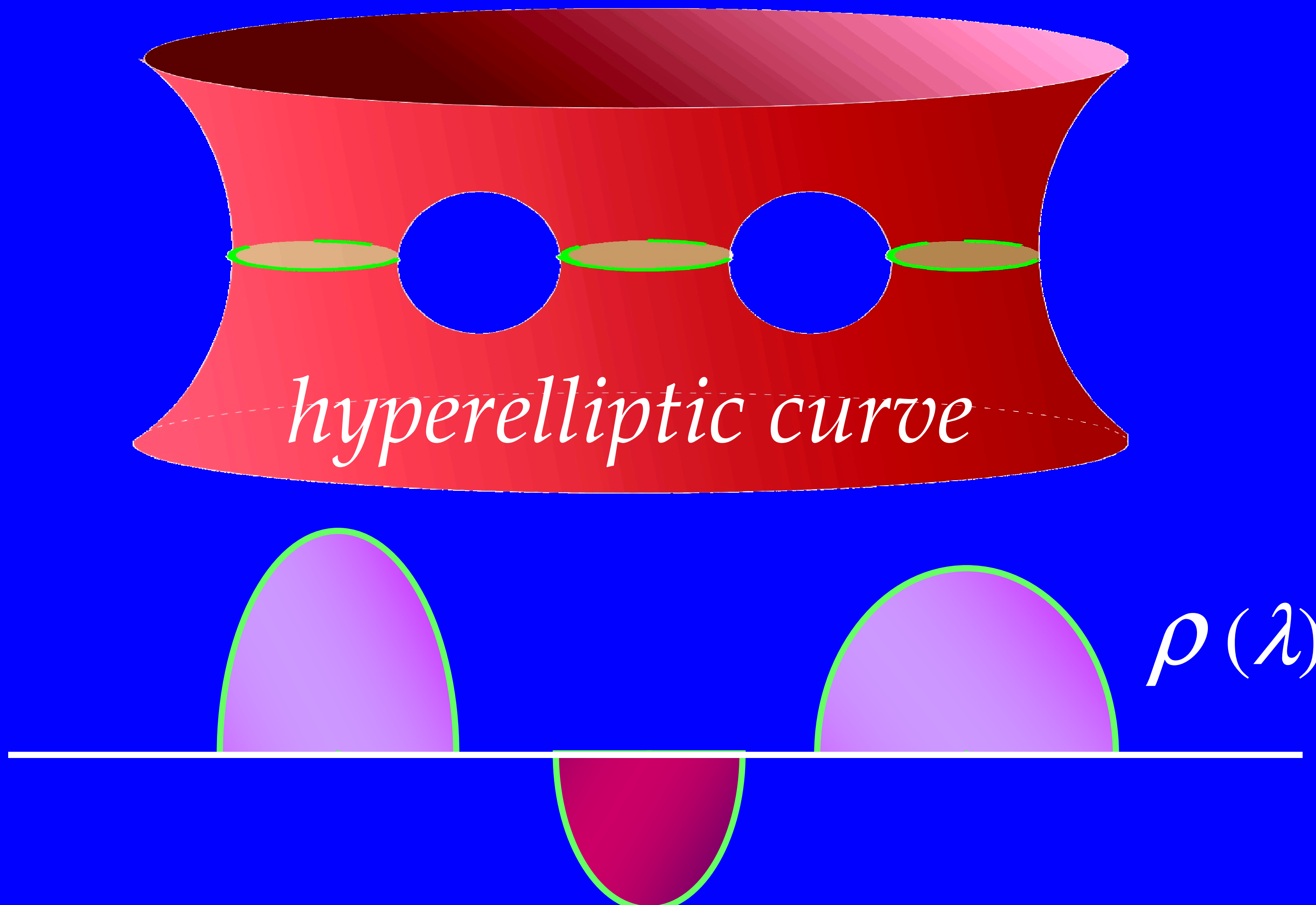
Effective Geometry



Size ~ Degrees of Freedom



Density Profile

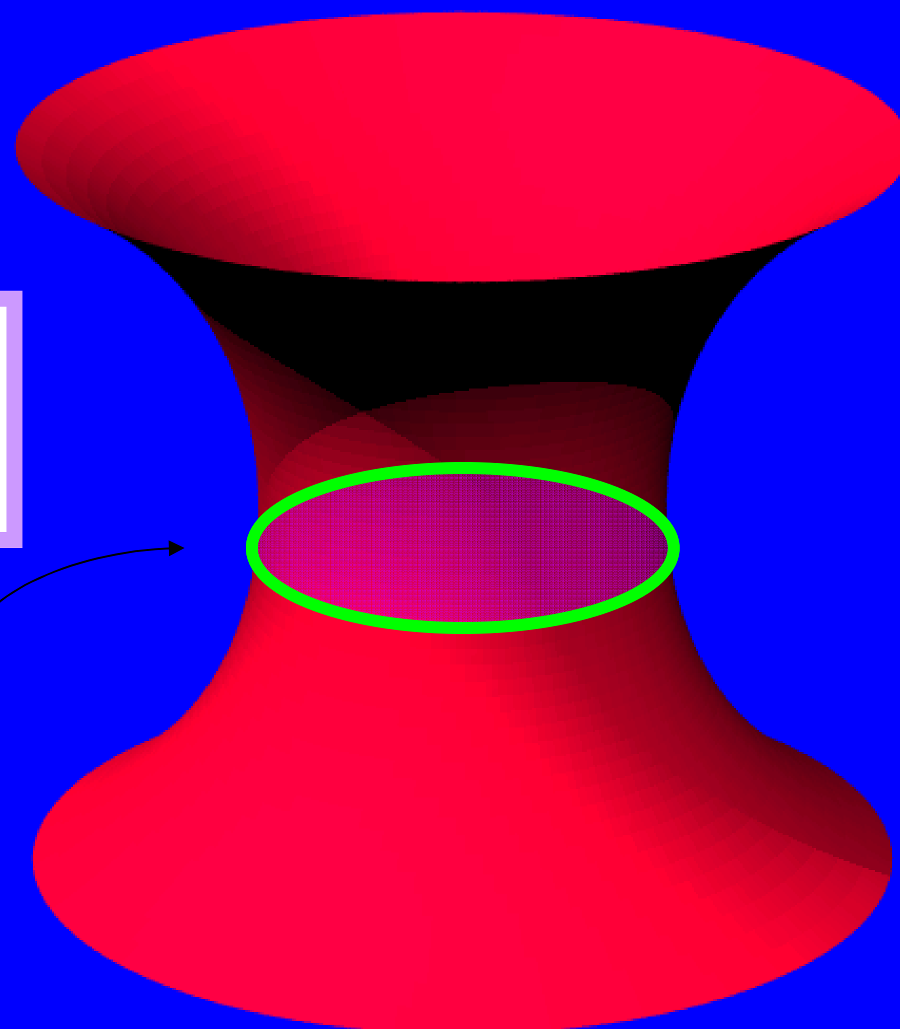


Gaussian Model

$$W = \mathrm{Tr}\left(\Phi^2\right)$$

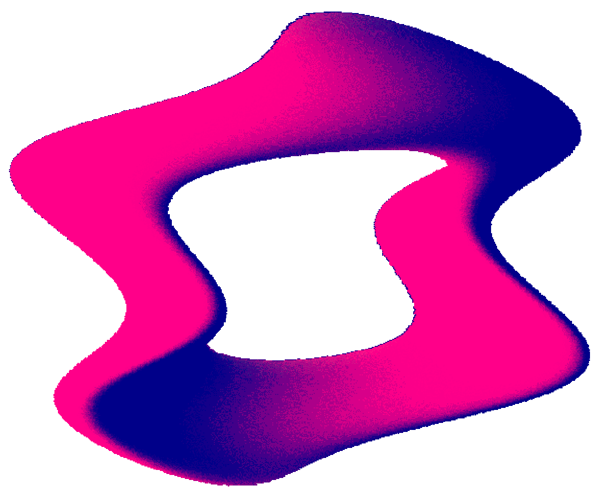
$$\boxed{x^2 + y^2 = \hbar N}$$

Wigner's circle



General Case

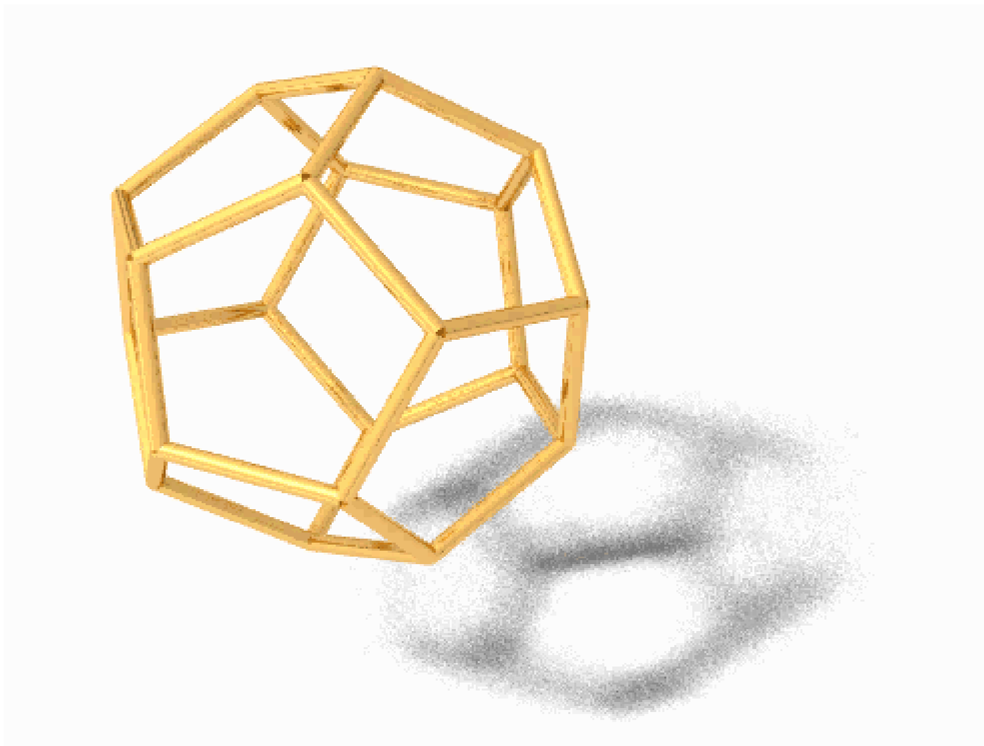
Effective geometry



Calabi-Yau 3-manifold

Plato's Cave

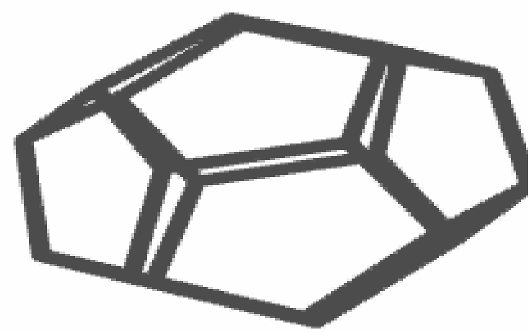
Mathematical Dream



Physical Reality

Quantum Cave

Physical Dream



Mathematical Reality