Mapping Flux to Geometry:

Calabi-Yau Duals of Torus Orientifolds

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Based on

hep-th/020128 S. Kachru, MS, S. Trivedi

hep-th/0211182 S. Kachru, MS, P. Tripathy, S. Trivedi

hep-th/0406001 MS

hep-th/0412270 MS

Motivation

D3/D7 warped compactifications with flux are an exciting arena in which one can readily address

- moduli stabilization
- realizations of de Sitter space
- inflation
- statistics of the landscape
- MSSM-like model building
- SUSY-breaking soft terms

in many cases with a holographic interpretation (due to warping) that lets us view field theory phenomena in an intuitive geometrical way.

Q: To what extent are these just alternative descriptions of more conventional fluxless compactifications?

As usual, $\mathcal{N}=4$ easy $\cong \left.{}^{\text{type I}}_{\text{or Het}}\right/T^6 \cong \left.{}^{\text{IIA}}\right/K3\times T^2$

$\mathcal{N}=2$ nontrivial but tractable

$\mathcal{N}=1$ hard.

We'll focus on the $\mathcal{N}=2$ case. For $\mathcal{N}=2$, $\exists$ large webs of connected CY_3 vacua. Are $\mathcal{N}=2$ flux compactifications part of a CY_3 web?

- Assumption of $\mathcal{N}=2$

  $\Rightarrow$ orientifold of T^6 or $K3 \times T^2$.

- Focus on $T^6/\mathbb{Z}_2$.

  T^6: $x^m \cong x^m + 1$

  $\mathbb{Z}_2$: $\Omega\,(-1)^{F_L}\,I_6$

  WS parity $\quad$ $x^m \to -x^m$ Inversion of T^6

  for $m=0$ states, acts as -1 on L Ramond sector

- For $F_{(3)} = H_{(3)} = 0$,

  $T^6/\mathbb{Z}_2 \cong$ type I on T^6

  via 6 T-dualities.

- 16 D3 + 2^6 O3 $\leftrightarrow$ 16 D9 + 2^0 O9

(O3 planes are located at the fixed points of I_6.)

- $T^6/\mathbb{Z}_2$ w.o. flux preserves $\mathcal{N}=4$.
- More generally, can trade D3s for flux:

$$2M + N_{flux} = 32, \qquad \in H^2(T^6, 2\mathbb{Z})$$

$$M = \#D3, \quad N_{flux} = \frac{1}{(2\pi)^2\alpha'^4}\int_{T^6} H_{(3)} \wedge F_{(3)}.$$

- Metric:

$$ds^2 = Z^{-1/2}\eta_{\mu\nu}dx^\mu dx^\nu + Z^{1/2} ds^2_{T^6}$$

$$-\nabla^2_{T^6} Z = (2\pi)^4\alpha'^2 g_s\left(\sum_i Q_i^{D3}\frac{\delta^6(x-x_i)}{\sqrt{g_{T^6}}} + \frac{N_{flux}}{V_{T^6}}\right).$$

- 5-form flux:

$$\tilde{F}_{(5)} = (1+*)\, g_s^{-1}\, d\left(Z^{-1} dx^0\wedge dx^1\wedge dx^2\wedge dx^3\right).$$

(We do not get to choose $\tilde{F}_{(5)}$.)

- Which $F_{(3)}, H_{(3)}$ give $\mathcal{N}=2$?

$\mathcal{N}=2$ Flux

- $T^6/\mathbb{Z}_2$ has only one known choice of $\mathcal{N}=2$ flux (up to $SL(6,\mathbb{Z}) \times SL(2,\mathbb{Z})$):

$$F_{(3)} = 2m\,(dx^4 \wedge dx^6 + dx^5 \wedge dx^7) \wedge dx^9$$

$$H_{(3)} = 2n\,(dx^4 \wedge dx^6 + dx^5 \wedge dx^7) \wedge dx^8.$$

(Note, S-duality + 90° 89-rotation: $m \leftrightarrow n$.)

Tadpole canc.: $4mn + M = 16$, $M = \#D3$.

- SUSY: $G_{(3)} = F_{(3)} - \tau_{dil} H_{(3)}$

  must be (2,1) and primitive ($J \wedge G_{(3)} = 0$).

- This $\Rightarrow$ $T^4{}_{(4567)} \times T^2{}_{(89)}$

  w.r.t. cpx and Kähler str,

  together with further constraints:

  (*) T^4: $dx^4 \wedge dx^6 + dx^5 \wedge dx^7$ (1,1) and primitive,

  T^2: $dz \equiv dx^8 + \tau dx^9 \Rightarrow -1/\tau = \frac{m}{n} \tau_{dil}$.

- $\mathcal{N}=2$: $J_a{}^b = i\chi^\dagger \gamma_a{}^b \chi$, so count cpx strs.

  $\exists$ an S^2 of cpx strs on T^4 satisfying (*). ✓

We will ultimately relate this background to IIA on CY_3 via a duality chain:

3 T-dualities + 9-10 circle swap.

To leading order (in g_s, α') this reduces to mapping of massless dof through classical supergravity dualities.

So, let us begin by describing the massless dof of $\mathcal{N}=2$ $T^6/\mathbb{Z}_2$.

First focus on metric.

- On T^4, natural to write metric in terms of

$$z^i = x^i + \underbrace{\tau^i{}_j}_{\text{4 cpx = 8 real dof}} y^j \quad \text{and} \quad \underbrace{g_{i\bar{j}}}_{\text{4 real dof}}$$

but this is redundant since $g^{T^4}_{mn}$ has 10 real dof.

- Fortunately there is another convenient parameterization that avoids this problem.

- Instead, write the T^4 metric as a flat $T^2_{(4,5)}$ fibration over $T^2_{(6,7)}$:

$$ds^2_{T^4} = \frac{V_1}{\mathrm{Im}\,\tau_1}\left| dx^4 + \tau_1 dx^5 + a^4 + \tau_1 a^5 \right|^2$$
$$+ \frac{V_2}{\mathrm{Im}\,\tau_2}\left| dx^6 + \tau_2 dx^7 \right|^2 .$$

a^4, a^5 are flat connections

$\left(= \text{const 1-forms on } T^2_{(6,7)} \right).$

- For T^2, write

$$ds^2_{T^2} = \frac{V_3}{\mathrm{Im}\,\tau_3}\left| dx^8 + \tau_3 dx^9 \right|^2 .$$

$\mathrm{Im}\,\tau$'s are so that $J_{T^2} = V_3\, dx^8 \wedge dx^9$, etc.

- SUSY conditions :

$\tau_1 \tau_2 = -1$, $\quad (m/n)\,\tau_{dil}\,\tau_3 = -1$,

$(a^4)_7 = (a^5)_6$, $\quad V_1, V_2, V_3$ arbitrary.

$\mathcal{N}=2$ $T^6/\mathbb{Z}_2$ massless spectrum

Flux breaks $\mathcal{N}=4$ to $\mathcal{N}=2$.

- Ignoring flux, massless fields of orientifold are

Bulk $\begin{cases} 1 \;\; \text{4D graviton} & g_{\mu\nu} \\ 12 \;\; \text{4D vectors} & B_{m\mu},\; C_{m\mu} \\ 38 \;\; \text{4D scalars} & \underset{15}{C_{mnpq}},\; \underset{21}{g_{mn}},\; \underset{2}{\tau_{dil} = C_0 + ie^{-\phi}} \end{cases}$

M D3 Branes $\begin{cases} 1 \;\; \text{4D vector} & A^I{}_\mu \\ 6 \;\; \text{4D scalars} & \Phi^{Im} \end{cases}$

$\Rightarrow$ 1 $\mathcal{N}=4$ gravity mult
6+M $\mathcal{N}=4$ vector mults.

- Including flux :

Have 10 unlifted metric moduli/dilaton-axion,
V_1, V_2, V_3, 2 indep τ, 3 indep $(a^m)_n$.

From $\left| dC_{(4)} - \frac{1}{2} B \wedge F_{(3)} + \frac{1}{2} C_{(2)} \wedge H_{(3)} \right|^2 = \left| \widetilde{F}_{(5)} \right|^2$,
9 vectors eat 9 $C_{(4)}$ axions
leaving 3 vectors, 6 $C_{(4)}$ axions.

D3 WV fields unlifted.

$\Rightarrow$ 1 $\mathcal{N}=2$ gravity mult
2+M $\mathcal{N}=2$ vector mults
3+M $\mathcal{N}=2$ hyper mults

(In IIA on CY_3 :
$= h^{1,1}$
$= h^{2,1} + 1$)

Focus on a T^3 in $T^6/\mathbb{Z}_2$ w. $H_{(3)} \neq 0$.

$$ds^2 = dx^2 + dy^2 + dz^2, \qquad \begin{aligned} H_{(3)} &= N\, dx \wedge dy \wedge dz, \\ B_{(2)} &= N x\, dy \wedge dz. \end{aligned}$$

T-dualize in the z-direction.
Buscher rules $\Rightarrow$

$$ds^2 = dx^2 + dy^2 + (dz + Nx\, dy)^2, \quad H = B = 0.$$

I.e., $\underset{\substack{\uparrow \\ \text{U(1) connection}}}{\mathcal{A}} = Nx\, dy, \qquad \underset{\substack{\uparrow \\ [\mathcal{F}] \text{ chern class}}}{\mathcal{F}} = N dx \wedge dy.$

T-duality has interchanged two S^1-fibrations:

1. geometrical S^1
2. formal $\tilde{S}^1$ of

$$\text{connection} \quad \tilde{\mathcal{A}} = -\int_{S^1} B,$$
$$\text{curvature} \quad \tilde{\mathcal{F}} = \int_{S^1} H.$$

This illustrates a general rule.
(See, e.g., Bouwknegt - Evslin - Mathai,
Fidanza - Minasian - Tomasiello.)

O6/D6 T-dual Orientifold

Starting from $T^6/\mathbb{Z}_2$ w. $\mathcal{N}=2$ flux, T-dualize in 4,5,9 directions.

- Obtain T^3 fibration over T^3 in internal dirs:

$$ds^2 = Z^{-1/2}\left(\eta_{\mu\nu}dx^\mu dx^\nu + ds^2_{T^3_{fib}}\right) + Z^{1/2} ds^2_{T^3_{base}}.$$

Here,

$$ds^2_{T^3_{base}} = \frac{V_2}{\operatorname{Im}\tau}\left|dx^6 + \tau dx^7\right|^2 + R_8{}^2 dx^{8\,2}$$

$$ds^2_{T^3_{fib}} = \frac{V_1'}{\operatorname{Im}\tau}\left|\theta^4 + \tau\theta^5\right|^2 + R_9'{}^2 dx^{9\,2}$$

with $\theta^m = dx^m + \mathcal{A}^m$, $m = 4,5$.

$\left(\mathcal{F}^4 = 2n\, dx^6 \wedge dx^8,\ \mathcal{F}^5 = 2n\, dx^7 \wedge dx^8.\right)$

- The x^9 circle is trivially fibered.
- M D6 + 2^3 O6 wrap T^3 fiber.
- Have $F_{(2)}$ flux only.

$\Rightarrow$ Geometrical M theory lift w. no flux!

- In M theory lift,

  $F_{(2)} \longrightarrow$ fibration of x^{10} over IIA geom.

  D6 $\xrightarrow[\text{locally}]{}$ smooth Taub-NUT space $\times \mathbb{R}^{6,1}$

  O6 $\xrightarrow[\text{locally}]{}$ smooth Atiyah-Hitchin space $\times \mathbb{R}^{6,1}$.

  Also, $S^1_{(9)}$ factorizes

  (since: x^9 not fibered in IIA
  $F_{(2)}$ indep of x^9
  warp factors cancel for $(dx^9)^2$).

- Thus,

  $$T^6/\mathbb{Z}_2 \longleftrightarrow \text{M theory on } X_6 \times S^1_{(9)}$$

  with $\mathcal{N}=2$ and X_6 smooth.

  $\Rightarrow$ X_6 is a Calabi-Yau.

- Compactifying on $S^1_{(9)}$ gives

  $$T^6/\mathbb{Z}_2 \longleftrightarrow \text{IIA on } CY_3 \ X_6.$$

  (Compactifying on $S^1_{(10)} \subset X_6$ gives O6/D6 dual.)

Which CY_3 ?

- Hodge numbers :

  $M+2$ vector mults, $M+3$ hyper mults

  $\Rightarrow h^{1,1} = h^{2,1} = M+2.$

  Here, $M = 16 - 4mn = 0, 4, 8, 12,$
  with different possible (m,n) for each M.

- Intersection numbers ?
- Fibration structure ?
- $\pi_1(X_6)$?
- Discrete isometries ?

- $m \leftrightarrow n$ duality ?

- To answer these questions, we must be more specific about the second part of the duality

$$T^6/\mathbb{Z}_2 \leftrightarrow O6/D6 \text{ dual } \underset{=}{\leftrightarrow} \text{IIA on } CY_3$$

(i.e., the circle swap: up on x^{10}, down on x^9).

- It will help if we first gain intuition on leading order vs. exact M theory lifts [Sen].

- From the usual IIA $\leftrightarrow$ M theory identifications, lift of (leading sugra) D6 or O6 background is $\mathbb{R}^{6,1} \times X_4$:

$$ds_4^2 = Z^{-1} R_{10}^{\,2} (dx^{10} + A)^2 + Z\, ds^2_{\mathbb{R}^3},$$

where $Z = 1 + \frac{R_{10}}{8\pi^2} \frac{Q}{|\vec{x}|}$ $\quad \vec{x} \in \mathbb{R}^3$

$$dA = R_{10}^{-1} *_3 dZ \qquad \left(= \frac{1}{2\pi\sqrt{\alpha'}} F_{(2)} \text{ IIA} \right).$$

- For D6, $Q = 1$ and this defines a smooth Taub-NUT space.

- For O6, $Q = -4$ and this defines the large radius approximation to an Atiyah-Hitchin space.

It is singular at small enough $\vec{x}$, where $Z = 0$.

- However, this metric is just the truncation of the A-H metric obtained by discarding all but the lowest Fourier mode along x^{10}.

  The complete A-H metric is smooth (also need $x^{10}, \vec{x} \cong -x^{10}, -\vec{x}$).

- The approximation is worst at small $\vec{x}$ where $Z^{-1} \to \infty$ and the M theory circle decompactifies.

- In IIA language, D0 branes become light near O6 planes, where $e^{\Phi} \to \infty$.

  So, need to include sugra fields from tower of D0 bound states.

(• D2 probing O6 : 1-loop moduli space metric, corrected to full A-H metric by 3D instantons;

[Seiberg, Seiberg-Witten].)

Lift of M D6s : Homology

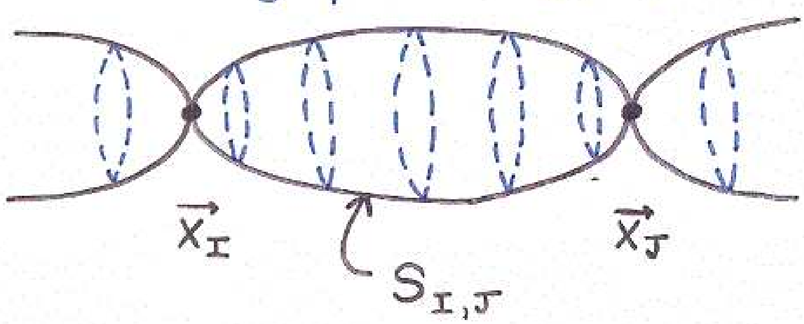
M D6s

- Same form of metric

$$ds^2 = Z^{-1} R_{10}^{\,2} (dx^{10} + A)^2 + Z\, ds^2_{\mathbb{R}^3} \,,$$

but now $Z = 1 + \sum_{J=1}^{M} Z^J$, $Z^J = \frac{R_{10}}{8\pi^2} \frac{Q}{|\vec{x} - \vec{x}_J|}$.

- S^1 fiber shrinks over each $\vec{x}_J$ on $\mathbb{R}^3$ base, so obtain S^2 from fibration over curves connecting pairs $\vec{x}_I, \vec{x}_J$:



- Basis of $H_2(Z)$ is $\{S_{12}, S_{23}, \ldots, S_{M-1,M}\}$.
- Intersections : $S_{I,I+1} \cdot S_{I+1,I+2} = 1$,

$$S_{I,I+1}^{\;\;2} = -2,$$

$$\Rightarrow I = \begin{pmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & 1 & \ddots & & \\ & & & \ddots & 1 \\ & & & 1 & -2 \end{pmatrix} = -\mathrm{Cartan}(A_{M-1}).$$

- Similarly, obtain one L_2 harmonic form for each center:

$$F^J = (Z^J/Z)_{,m}\left(-dx^m \wedge (dx^{10}+A) + \tfrac{1}{2} R_{10}^{-1} \mathrm{Vol}_{\mathbb{R}^3}{}^m{}_{np}\, dx^n \wedge dx^p\right)$$

$$\left(\text{locally } d\left[A^J - (Z^J/Z)(dx^{10}+A)\right]\right).$$

- The F^J are anti-selfdual, and satisfy

$$\int_{X_4} F^I \wedge F^J = -\delta^{IJ}.$$

- The 2-form $\omega^{I,J} = F^I - F^J$ is Poincaré dual to the sphere $S_{I,J}$:

$$\int_{X_4} \omega^{I,J} \wedge \omega^{K,L} = S_{I,J} \cdot S_{K,L}.$$

- It will be more convenient to work with cohomology when we discuss the CY_3 duals of T^6/Z_2.

Lift of O6 + M D6s (end of review)

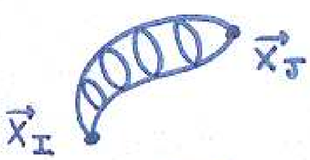
- Now have

$$ds_4^2 = Z^{-1} R_{10}^2 (dx^{10} + A)^2 + Z\, ds^2_{\mathbb{R}^3}$$

with
$$Z = 1 + \frac{R_{10}}{8\pi^2}\Bigg[\underset{\substack{\uparrow \\ \text{O6}}}{-\frac{4}{|\vec{x}|}} + \sum_{J=1}^{M}\Bigg(\underset{\substack{\uparrow \\ \text{M D6}}}{\frac{1}{|\vec{x}-\vec{x}_J|}} + \underset{\substack{\uparrow \\ \text{M image D6}'}}{\frac{1}{|\vec{x}+\vec{x}_J|}}\Bigg)\Bigg]$$

modulo $x^{10}, \vec{x} \cong -x^{10}, -\vec{x}$.

- Obtain S^2 from S^1-fibration btw centers:



$+ \; \mathbb{Z}_2$ image $\quad (S_{I,J})$

$+ \; \mathbb{Z}_2$ image $\quad (S_{I,J'})$.

- The spheres $S_{I,I+1}$ $(I = 1, \dots, M)$ and $S_{M-1,M'}$ form a simple basis w. D_M intersection matrix.

- Metric is singular at small $\vec{x}$ but can choose representative cycles that avoid this region.

- So, obtain correct topological data (intersection matrix) from singular leading order lift. (Likewise for cohomology.)

- Now return to O6/D6 dual of $T^6/\mathbb{Z}_2$ and perform leading order M theory lift + $S^1(x^9)$ compactification. Obtain IIA on

$$ds^2_{CY_3} \simeq \frac{V_1'}{\mathrm{Im}\,\tau}\left|\theta^4+\tau\theta^5\right|^2 + Z^{-1}R_{10}^2(dx^{10}+A)^2 \quad \text{fib}$$

$$+\frac{V_2}{\mathrm{Im}\,\tau}\left|dx^6+\tau dx^7\right|^2 + Z R_8^2 \, {dx^8}^2, \quad \text{base}$$

modulo $x^{10}, \vec{x} \cong -x^{10}, -\vec{x} \quad (\vec{x} = x^6, x^7, x^8)$.

- Here : $V_1' = (m/n) R_8 R_{10}$,

  4,5 fibrations are as before,

$$dA = \underset{\substack{\uparrow \\ \text{like Taub-NUT/A-H}}}{R_{10}^{-1} *_3 dZ} - 2m(\theta^4 \wedge dx^7 \underset{\substack{\uparrow \\ \text{new}}}{-} \theta^5 \wedge dx^6).$$

- There is a natural cpx pairing, w. (1,0) forms

$$\theta^4+\tau\theta^5, \quad dx^6+\tau dx^7, \quad Z dx^8 + it(dx^{10} + \underset{\substack{\uparrow \\ \text{modulus } \delta A = a dx^8}}{A}), \; t = \frac{R_{10}}{R_8}.$$

- Can compute corresponding J, Ω; find that $dJ = d\Omega = 0$ away from 8 pts on T^3_{base} (from O6s).

$\Rightarrow SU(3)$ holonomy away from $Z \leq 0$ region.

Validity of leading order description

Can show that $Z = 1 + \frac{n}{m}\frac{V_1'}{V_2}\hat{Z}$,
with $\hat{Z} = O(1)$ as $V_1'/V_2 \to 0$.

- By tuning V_1'/V_2, can make bad $Z \leq 0$ regions smaller and smaller.
- Leading order metric becomes an arbitrarily good approximation at most points.
- Nothing special happens at bad loci ($\sim$ A-H $\times$ $\mathbb{R}^2$ in full lift).

So, expect homology to be reliably computable by leading order description of duality.

- Define Kähler moduli s, h by

$$v_1' = \bar{m}h, \quad v_2 = 2s, \quad R_8 R_{10} = \bar{n}h,$$

with $(\bar{m}, \bar{n}) = (m,n)/\gcd(m,n)$.

- The Kähler form is $J = s\omega_S + h\omega_H$,

$$\omega_S = 2\,dx^6 \wedge dx^7,$$

$$\omega_H = \bar{m}\,\theta^4 \wedge \theta^5 + 2\hat{Z}\,dx^6 \wedge dx^7 + \bar{n}\,dx^8 \wedge (dx^{10} + A)$$

Also have M harmonic forms

$$\omega_J = F^J - F^{J'},$$

$$F^J = (Z^J/Z)_{,m}\left[-dx^m \wedge (dx^{10} + A) + \tfrac{1}{2} Z R_{10}^{-1} \mathrm{Vol}_3{}^m{}_{np}\, dx^n \wedge dx^p\right] + (Z^J/Z)\, 2m\,(\theta^4 \wedge dx^7 - \theta^5 \wedge dx^6).$$

- Can check that the ω span $H^2(X_6, \mathbb{Z})$ using $B_{(2)} \in H^2(\mathbb{R})/H^2(\mathbb{Z})$ and periodicities like $C_{(0)} \cong C_{(0)} +$ in $T^6/\mathbb{Z}_2$ dual.

- Intersection numbers are $A \cdot B \cdot C = \underset{\substack{\uparrow \\ \mathbb{Z}_2 \text{ coordinate identification}}}{\tfrac{1}{2}} \int \omega_A \wedge \omega_B \wedge \omega_C$.

$$\Rightarrow \boxed{H^2 \cdot S = 2\bar{m}\bar{n}, \quad H \cdot I \cdot J = -\bar{m}\,\delta_{IJ}, \quad \text{others} = 0.}$$

- CY_3 metric came out as a fibration over $T^3/\mathbb{Z}_2$, but $T^3/\mathbb{Z}_2$ nonorientable, so not a useful fibration.

- Instead, view as fibration over $T^2_{(6,7)} \cong \mathbb{P}^1$.

  X_6 was $T^2_{(4,5)}$ fibered over $T^3_{(6,7,8)}$,
  $S^1_{(10)}$ fibered over that,
  mod $\mathbb{Z}_2(6,7,8,10)$.

  All fibration curvatures restrict trivially to $x^{6,7}=$ cons

  $\Rightarrow$ generic fiber over $\mathbb{P}^1_{(6,7)}$ is just T^4 !

- Fiber is an abelian surface :

- The Kähler form is $J = h\omega$,

$$\omega = \bar{m}\, dy^4 \wedge dy^5 + \bar{n}\, dy^8 \wedge dy^{10},$$

where $(\bar{m}, \bar{n}) \in \{(1,1), (1,2), (1,3), (1,4), + \text{interchanges}\}$.

- In general, when T^{2d} has $J \propto \omega$,

$$\omega = \sum_{i=1}^{d} a_i\, dx^{2i-1} \wedge dx^{2i}, \qquad a_i \mid a_{i+1},$$

$[\omega]$ is called a polarization and determines an embedding of T^{2d} in a projective space.

Then T^{2d} = abelian variety $(d=2$, abelian surface$)$

Can now give a simple description of the divisors dual to our cohomology basis.

- ω_S

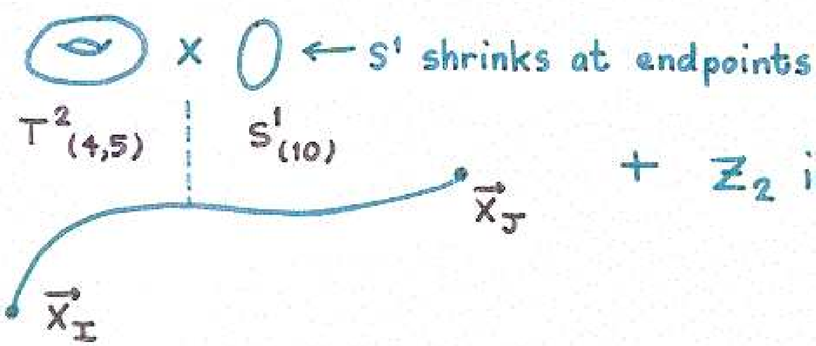
  Dual to $S = T^4$ (abelian surface fiber).

- ω_H

  Dual to $H =$ hyperplane divisor of T^4 fibered over $\mathbb{P}^1$ base.

- $\omega_{I,J} = \omega_I - \omega_J$

  Dual to 4-cycle from $T^2_{(4,5)} \times S^1_{(10)}$ fibration over $T^3/\mathbb{Z}_2$ base:



+ $\mathbb{Z}_2$ image .

(Again, can choose cycles that avoid singular loci.)

- As a check that the CY_3 duals of $\mathcal{N}=2$ $T^6/\mathbb{Z}_2$ are abelian surface fibrations over $\mathbb{P}^1$, note

Thm (Oguiso): Given CY_3 with a divisor D s.t. $D^2=0$ and D is nef (i.e., $D\cdot C \geq 0$ for all algebraic curves C),

 - CY_3 is fibration over $\mathbb{P}^1$
 - fibers are K3 or abelian surface.

We have $S^2=0$.
S: $z^2 = \text{const.}$ is effective $\Rightarrow$ nef.
$\quad$ $\hookleftarrow x^6+\tau x^7$

- K3 or T^4?

Have $S\cdot c_2(X_6) = \chi(S) = \begin{cases} 0 & T^4 \\ 24 & K3. \end{cases}$

The LHS appears in

$$F_1 = -\frac{4\pi i}{12}\sum_\alpha \left(D_\alpha \cdot c_2(X_6)\right) t^\alpha + \text{WS instantons}.$$

F_1 enters in the R^2 coupling $\propto \mathrm{Re}\int F_1 \,\mathrm{tr}(R-iR^*)^2$.

In $T^6/\mathbb{Z}_2$, as in $\begin{smallmatrix}\text{Het}\\ \text{or I}\end{smallmatrix}/T^6$, $F_1 = \tau_{dil} + \cdots$.
We will see that $1/g_s$ of $T^6/\mathbb{Z}_2 \leftrightarrow h$ of CY_3.

$\Rightarrow \quad \chi(S) = S\cdot c_2(X_6) = 0$, abelian surface.

- From kinetic terms in $T^6/\mathbb{Z}_2$,

$$\left| dC_{(4)} - \tfrac{1}{2} B_{(2)} \wedge F_{(3)} + \tfrac{1}{2} C_{(2)} \wedge H_{(3)} \right|^2 ,$$

$C_{(4)}$ has axionic couplings to $B_{(2)a\mu}$ and $C_{(2)b\mu}$.

- When $C_{(4)}$ couples to $NB_{(2)a\mu}$ or $NC_{(2)b\mu}$, the corresponding gauge symmetry is broken to $\mathbb{Z}_N$.

  This tells us about torsion cycles and discrete isometries:

Charge	Gauge sym (CY_3)	$T^6/\mathbb{Z}_2$ field	IIA CY_3 field
m	$\mathbb{Z}_m$ (winding)	$C_{(2)4\mu}$	$B_{(2)5\mu}$
m	$\mathbb{Z}_m$ (winding)	$C_{(2)5\mu}$	$B_{(2)4\mu}$
n	$\mathbb{Z}_n$ (isometry)	$B_{(2)4\mu}$	$\mathcal{A}^4{}_\mu$ (KK vector)
n	$\mathbb{Z}_n$ (isometry)	$B_{(2)5\mu}$	$\mathcal{A}^5{}_\mu$ (KK vector)

(+ other discrete gauge symmetries that correspond to higher dimensional torsion cycles.)

$\Rightarrow$ $\mathbb{Z}_m \times \mathbb{Z}_m$ winding, $\mathbb{Z}_n \times \mathbb{Z}_n$ isometry.

$H_1(X_6, \mathbb{Z}) = \mathbb{Z}_m \times \mathbb{Z}_m$ $\left(= \pi_1/\text{commutators} \right)$.

- T-duality interchanges winding and isometries.

  $m \leftrightarrow n$ duality a T-duality ?

- In the $T^6/\mathbb{Z}_2$ orientifold with $\mathcal{N}=2$ flux

$$F_{(3)}/(2\pi)^2\alpha' = 2m\,(dx^4\wedge dx^6 + dx^5\wedge dx^7)\wedge dx^9\,,$$
$$H_{(3)}/(2\pi)^2\alpha' = 2n\,(dx^4\wedge dx^6 + dx^5\wedge dx^7)\wedge dx^8\,,$$

we can interpret $m\leftrightarrow n$ interchange as S-duality $\otimes$ $90°$ 89-rotation. Here,

$$S:\begin{cases} g_s \longrightarrow \tilde{g}_s = 1/g_s\,, \\ R_m/\alpha' \longrightarrow \widetilde{(R_m/\alpha')} = \frac{1}{g_s}(R_m/\alpha')\,. \end{cases}$$

- We can map this duality to the IIA CY_3 dual description. It acts on the Kähler modulus h as

$$\frac{\bar{m}h}{(2\pi)^2\alpha'} \longrightarrow \widetilde{\left(\frac{\bar{m}h}{(2\pi)^2\alpha'}\right)} = \frac{(2\pi)^2\alpha'}{\bar{m}h}\,.$$

(I.e., it inverts the T^4 volume.)

- So, $m\leftrightarrow n$ duality is

$$\frac{T^6/\mathbb{Z}_2}{\text{S-duality} \otimes R_{89}(90°)} \quad \xleftrightarrow{m\leftrightarrow n} \quad \frac{CY_3}{\text{T-duality of entire } T^4 \text{ fiber.}}$$

Are these CY_3 known?

First place to look:

- Kreuzer and Skarke have tabulated all 473,800,766 reflexive polyhedra in 4D.

  $$\left.\begin{array}{l}\text{So, all hypersurface } CY_3\text{s known, and} \\ \text{all connected through flops + conifolds.}\end{array}\right\} \pi_1 = 0$$

- Look for

  $(h^{1,1}, h^{2,1}) = (2,2), (6,6), (10,10), (\underline{\underline{14,14}})$, w. $m = 1$.

  Find three $(14,14)$ candidates ...

  but do not appear to be abelian surface fibrations.

$\Rightarrow$ CY_3 duals of $T^6/\mathbb{Z}_2$ do not seem to be in the known web, but could still be in other large webs that remain to be explored

(or, could be isolated).

Another guess :

- Appearance of T^2 moduli

  τ in 4,5 and 6,7 directions
  $\tilde{\tau}$ in 8,10 directions

suggests

$$X_6 = \left(E_\tau \times \underset{\substack{\uparrow \\ \text{elliptic curves}}}{E_\tau} \times E_{\tilde{\tau}}\right) / \underset{\substack{\uparrow \\ \text{discrete group}}}{\Gamma} .$$

- For $\Gamma = D_8$, obtain $(h^{1,1}, h^{2,1}) = (2,2)$ ✓
  but wrong $c_2(X_6)$ and $\pi_1(X_6)$. ✗

- Still an open problem to identify explicit constructions of the CY_3 duals.

What could we learn from explicit constructions?

- Could ask about flops and extremal transitions. (Web or isolated?)

- Rational curves $\subset CY_3 \Rightarrow$ WS instantons
  $\Leftrightarrow$ dual $T^6/\mathbb{Z}$ instantons.

- This would tell us about instantons in $T^6/\mathbb{Z}_2$ that are not currently known.

  (E.g., $\mathbb{P}^1$ section of T^4 fibration $\Rightarrow$ D3 instanton on $T^4 \subset T^6/\mathbb{Z}_2$.)

  Potentially useful for model building on chiral orientifolds of T^6/Γ.

- Most (all?) CY_3 other than free quotients of T^6 have rational curves.

  Our CY_3 are not free quotients of T^6 since $c_2 \neq 0$.

- So, either learn about new CY_3 w.o. rational curves or gain prediction for instanton corrections to $T^6/\mathbb{Z}_2$.

Conclusions

We have seen that

CY_3 duals of $T^6/\mathbb{Z}_2$ with $\mathcal{N}=2$ flux have:

- $(h^{1,1}, h^{2,1}) = (2,2)^3, (6,6)^2, (10,10)^2, (14,14)^1$
- Degeneracies distinguished by (m,n)

  $$4mn = 16 - M\,, \quad h^{1,1} = h^{2,1} = M+2$$
- $\mathbb{Z}_m \times \mathbb{Z}_m$ winding, $\mathbb{Z}_n \times \mathbb{Z}_n$ isometry
- Intersection numbers

  $$H^2 \cdot S = \frac{2mn}{(\gcd(m,n))^2}\,, \quad H \cdot I \cdot J = -\frac{m}{\gcd(m,n)}\delta_{IJ}$$
- Abelian surface (T^4) fibrations over $\mathbb{P}^1$.

We identified $m \leftrightarrow n$ interchange with

- S-duality $\otimes$ $R_{89}(90°)$ $\qquad T^6/\mathbb{Z}_2$
- T-duality of entire T^4 fiber $\qquad CY_3$.

Would like to identify these CY_3 in terms of concrete algebro-geometric constructions.

- Could then ask about flops and extremal transitions (i.e., web).
- Rational curves / ws instanton ?

  This would teach us about instantons in $T^6/\mathbb{Z}_2$ or new CY_3 without rational curves.

Finally,

- To what exent are there standard (fluxless) duals of $\mathcal{N}=1$ flux compactifications ?