Eigenvalues of some of my Favourite Graphs

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Eigenvalues of Graphs

The **adjacency matrix** of a graph G on n vertices (labelled $1, 2, \ldots, n$) is an

- $n \times n$, 01-matrix denoted A(G)
- 1 in the i, j position if vertices i and j are adjacent
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The eigenvalues of G are the eigenvalues of $A(G)$.

Complete Graph

The adjacency matrix for the complete graph K_5 is:

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

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The characteristic polynomial is

$$\phi(K_5, \lambda) = (1 + \lambda)^4 (4 - \lambda).$$

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Ratio Bound Let G be a k-regular graph on n vertices with least eigenvalue τ. Then

$$\alpha(G) \leq \frac{n}{1 - \frac{k}{\tau}}.$$

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An example of a uniform 3-partition of a 9-set is

$$P = 123|456|789.$$

Qualitative Independence

Let A, B be uniform k-partitions of an n-set,

$$A = \{A_1, A_2, \ldots, A_k\} \text{ and } B = \{B_1, B_2, \ldots, B_k\}.$$

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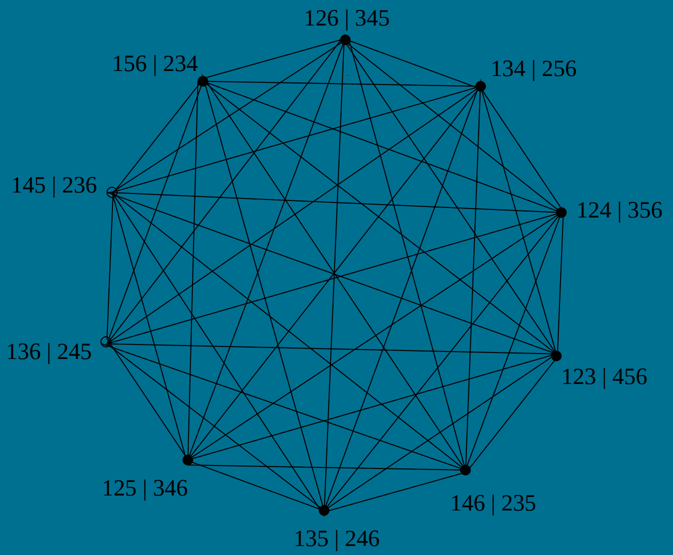

Equitable Partitions

Equitable partition for a graph G:

- $\star$ partition π of $V(G)$ with cells $C_1, C_2, \ldots, C_r$,
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Quotient graph of G over π, G/π is the directed graph with

- ⋆ r cells C_i as its vertices
- ⋆ b_{ij} arcs between the $i^{th}$and j^{th} cells.

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Theorem 1. *If G is a vertex-transitive graph and π is the orbit partitions of some subgroup of $Aut(G)$, then if π has a singleton cell $\{u\}$, every eigenvalue of G is an eigenvalue of G/π.*

A Partition on $UQI(ck, k)$

For a partition $P \in V(UQI(n, k))$ and $s \in Sym_n$
let P^s be the partition with $s(a) \in (P^s)_i$ if and only if $a \in P_i$.

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For any P,

- $\mathrm{fix}(P)$ is a subgroup of $Aut(UQI(ck, k))$,
- the partition P is a singleton cell.

The Same but Different

For $P, Q \in V(QI(n,k))$ define **meet table of P and Q** to be the $k \times k$ array with the i, j entry $|P_i \cap Q_j|$.

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For $P = 123|456|789$ and $Q = 147|258|369$,

$$M_{P,Q} = \begin{array}{c|ccc} & Q_1 & Q_2 & Q_3 \\ \hline P_1 & 1 & 1 & 1 \\ P_2 & 1 & 1 & 1 \\ P_3 & 1 & 1 & 1 \end{array}$$

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- Two meet tables are **isomorphic** if there is some permutation of the rows and columns of one array that produces the other array.

Why this Partition works

Theorem 2. *Let* $P, Q, R \in V(QI(n,k))$. *Then the meet table for* P *and* Q *is isomorphic to the meet table for* P *and* R *if and only if there is a* $g \in \mathrm{fix}(P)$ *so that* $g(Q) = R$.

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- Assume $M_{P,Q}$ is isomorphic to $M_{P,R}$. For permutations $\sigma, \phi \in Sym_k$,

$$[M_{P,Q}]_{i,j} = [M_{P,R}]_{\sigma(i),\phi(j)}, \text{ for } i,j \in \{0,1,\ldots,k-1\}.$$

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- $|P_i \cap Q_j| = |P_{\sigma(i)} \cap R_{\phi(j)}|$. Let $P_i \cap Q_j = \{a_1, \ldots, a_m\}$ and $P_{\sigma(i)} \cap R_{\phi(j)} = \{b_1, \ldots, b_m\}$.

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- Let $g_{i,j}$ be the permutation that maps a_l to b_l for $l = 1, \dots m$.

proof con't

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- Define a permutation on the rows $i \in \{0, \ldots, k-1\}$ of $M_{P,Q}$ by $\sigma(i) = i'$ if and only if $g(P_i) = P_{i'}$.

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- Define a permutation ϕ on the columns $i = 0, \ldots, k-1$ of $M_{P,Q}$ by $\phi(j) = j'$ if and only if $g(Q_j) = R_{j'}$.

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- Thus,

$$[M_{P,Q}]_{\sigma(i),\phi(j)} = [M_{P,R}]_{i,j}, \text{ for } i, j \in \{0, 1, \ldots, k-1\}$$

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- Store the non-isomorphic tables as a single partition that has the meet table with P
- For each non-isomorphic meet table, count the number of partition from each orbit which are qualitatively independent with it.

Spectrums of Small UQIs

Graph	Eigenvalues and corresponding multiplicities
$9, 3$	(-4, 2, 8, -12, 36) (84, 120, 48, 27, 1)
$12, 3$	(0, 8, -12, 18, -27, 48, 108, -252, 1728) (275,2673,462,616,1408,132,154,-54,1)
$15, 3$	(4, 8, -10, -22, 29, 34, -76, 218, -226, 284, 1628, -5060, 62000) (1638, 21450, 910, 25025, 32032, 22113, 11583, 1925, 7007, 2002, 350, 90,1)
$18, 3$	(8, 15, 18, -60, 60, -102, -120, 120, 368, 648, -655, -2115, 2370, -2115, 2370, 2460, -4140, 24900, -89550, 1876500, $954 \pm 18\sqrt{10209}$) (787644, 678912, 136136, 87516, 331500, 259896, 102102, 219912, 99144, 11934, 88128, 22848, 4641, 5508, 2244, 663, 135, 1, 9991)
$16, 4$	(-72,$-56 \pm 8\sqrt{193}$, $-96 \pm 96\sqrt{37}$, $24 \pm 24\sqrt{97}$, -96, 96, -288, 8, -144, 24, 192, 32, 1728, -64, -16, 432, 48, 1296,-48, -576, 128, -3456, 576, 13824, -1152, 144) (266240, 137280, 7280, 76440, 69888, 91520, 24960, 262080, 73920, 24024, 65520,150150,440,51480,753324,20020,420420,1260,23100,10752,60060,104,4070