A bound on the chromatic number of line graphs

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(joint work with B. Reed and A. Vetta)

†Research supported by McGill University and NSERC

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= size of smallest proper colouring.

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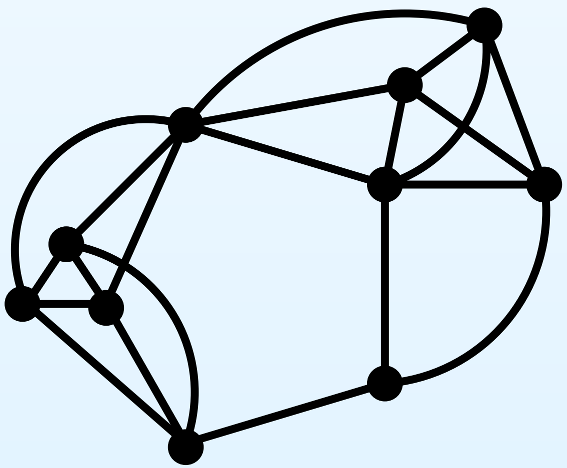

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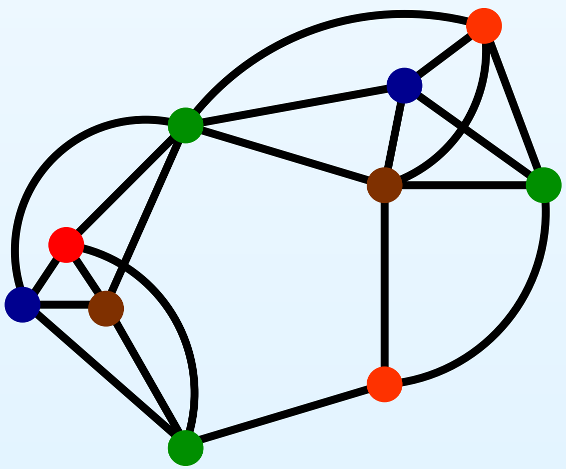

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$$\chi(G) \leq (1-\beta)(\Delta(G)+1) + \beta\omega(G).$$

Conjecture. *For any graph G,*

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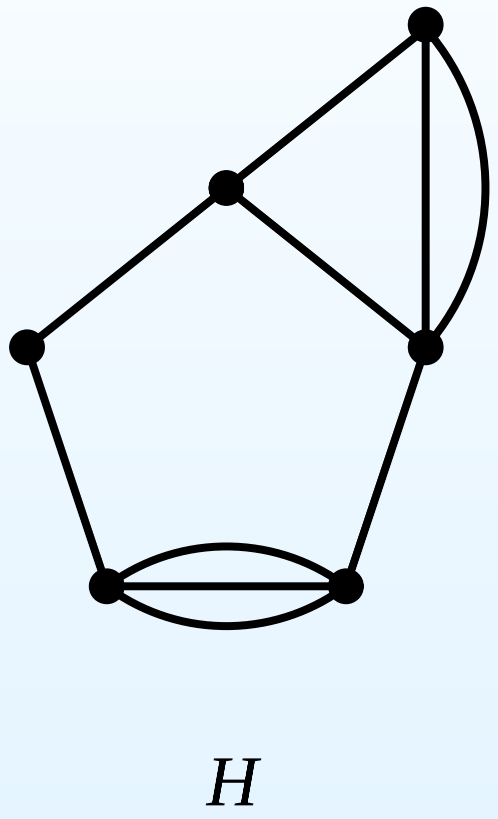
- $\forall\, G, \quad \chi^*(G) \leq \frac{\Delta(G)+1+\omega(G)}{2}$.
- If $\alpha(G) \leq 2$, then $\chi(G) \leq \left\lceil \frac{\Delta(G)+1+\omega(G)}{2} \right\rceil$.

Line graphs

- The *line graph* $L(H)$ of a multigraph $H = (V, E)$ has vertex set E, and two vertices are adjacent if the corresponding edges share an endpoint in H.



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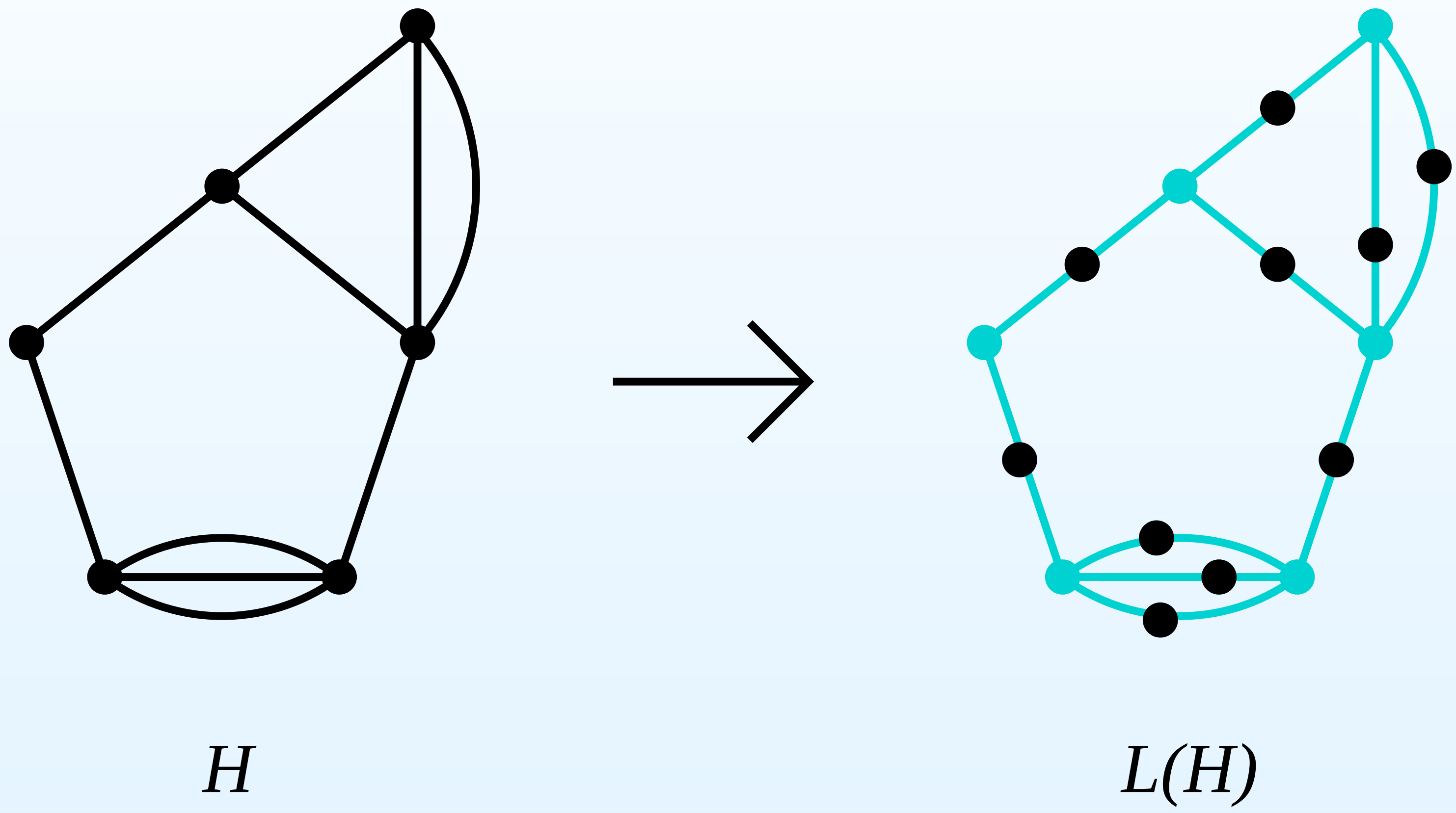

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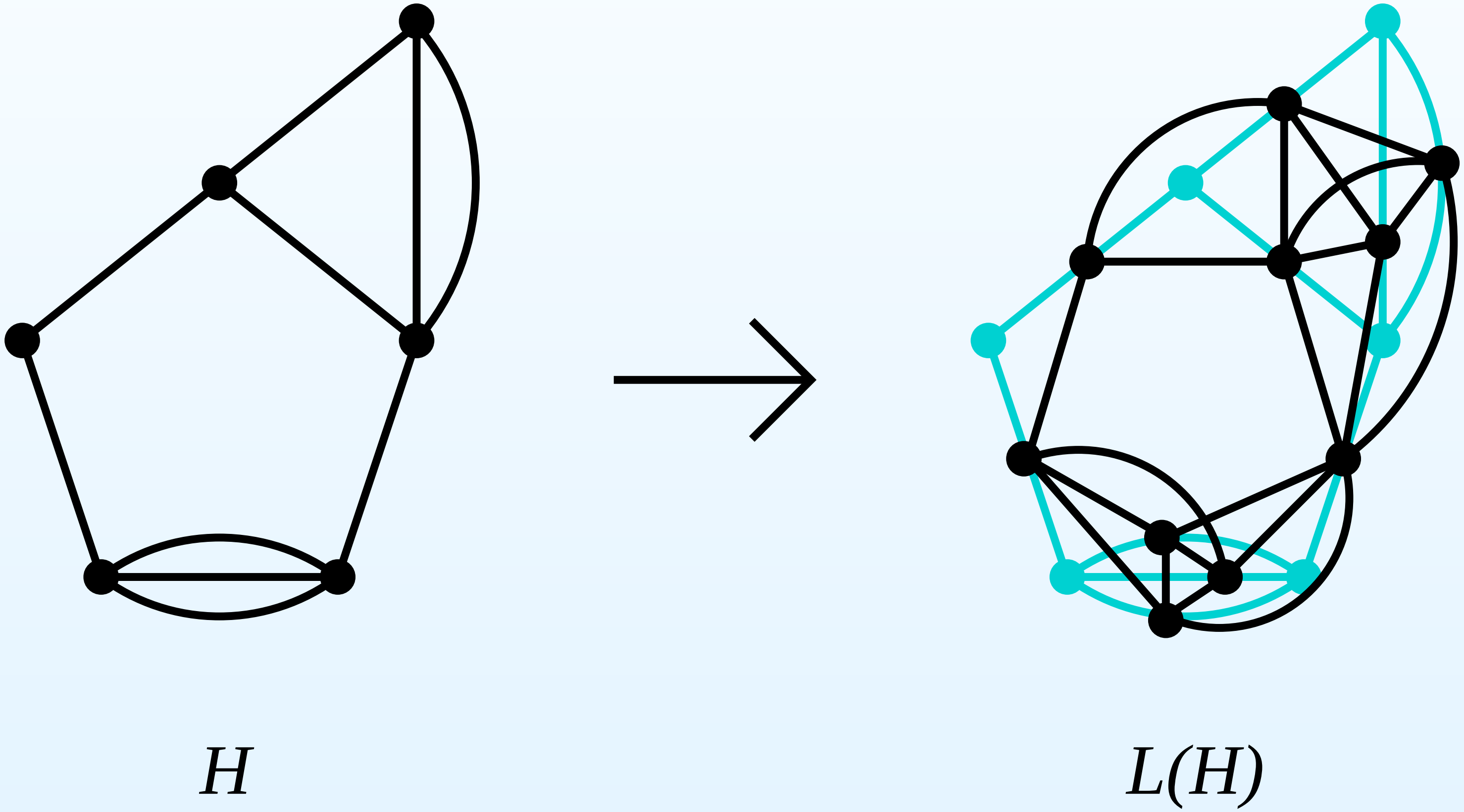

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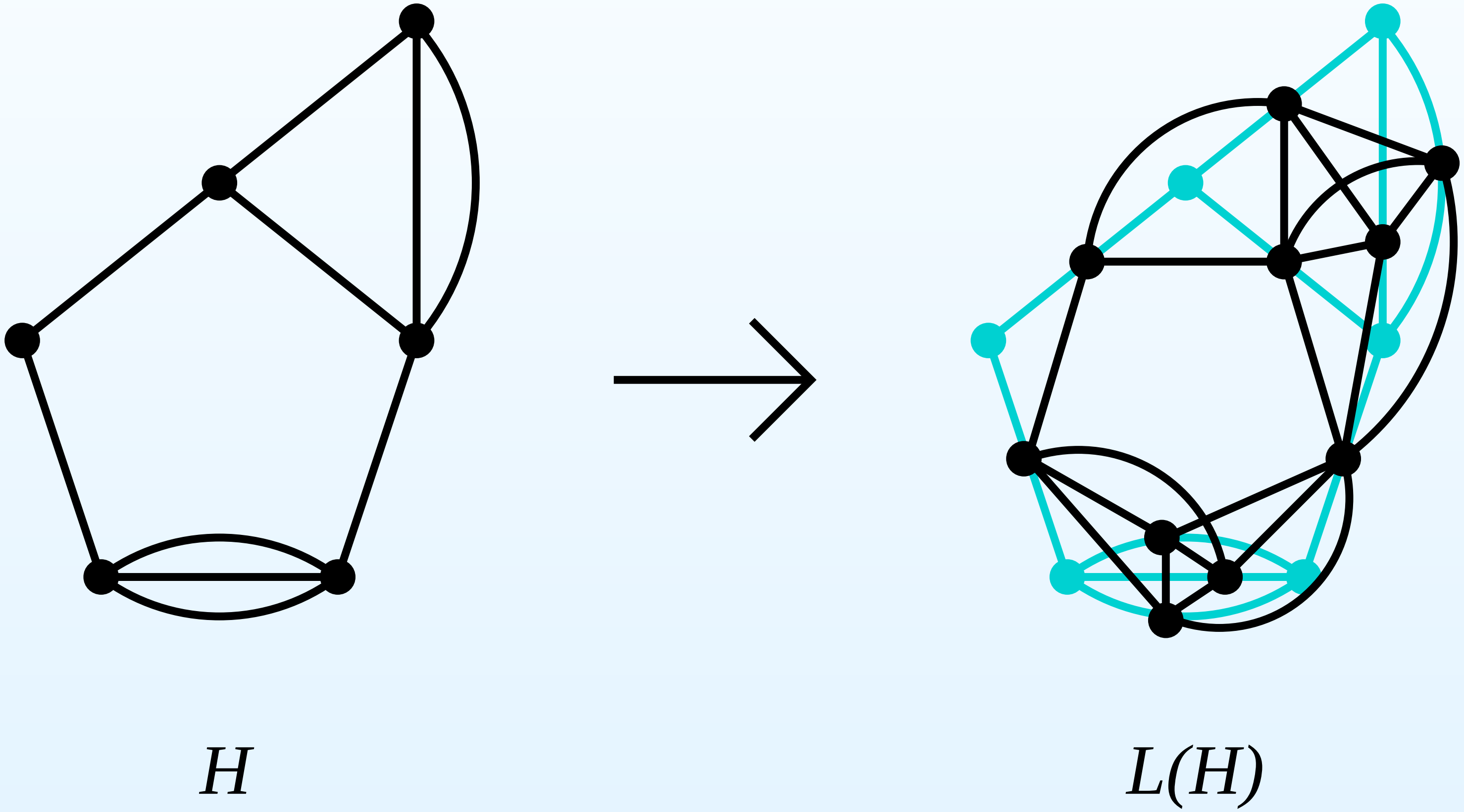

- G is a line graph if it is $L(H)$ for some multigraph H.

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Corollary (MR00). *For any H,*

$$\chi'(H) \leq \max\left\{\lfloor 1.1\Delta(H) + 0.7 \rfloor, \left\lceil \frac{\Delta(G) + 1 + \omega(G)}{2} \right\rceil\right\}.$$

Our approach

Let $G = L(H)$.

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1. $\Delta(G) \geq \frac{3}{2}\Delta(H) - 1$.
   Use Theorem CR98.

2. $\Delta(G) < \frac{3}{2}\Delta(H) - 1$.
   Construct a matching.

The easy case: $\Delta(G) \geq \frac{3}{2}\Delta(H) - 1$

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$\forall \;\; \Delta(H)$,

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so we are done.

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Suppose $\forall\ \emptyset \neq S \subset V$,

$$\chi(G_S) \leq \left\lceil \frac{\Delta(G_S) + 1 + \omega(G_S)}{2} \right\rceil.$$

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If S is a maximal stable set and $\omega(G_S) < \omega(G)$, then

$$\chi(G) \leq \chi(G_S) + 1 \leq \left\lceil \frac{\Delta(G_S) + 3 + \omega(G_S)}{2} \right\rceil \leq \left\lceil \frac{\Delta(G) + 1 + \omega(G)}{2} \right\rceil.$$

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We will show that such an S exists.

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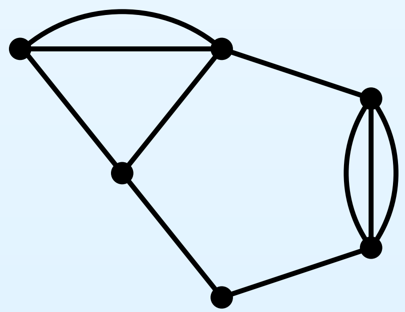

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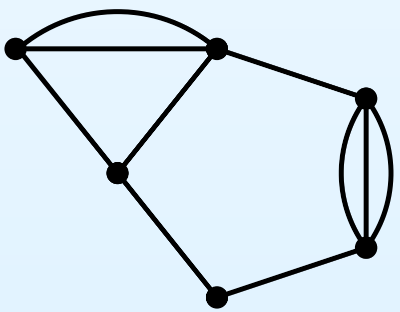

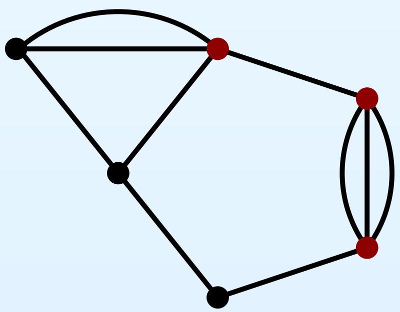

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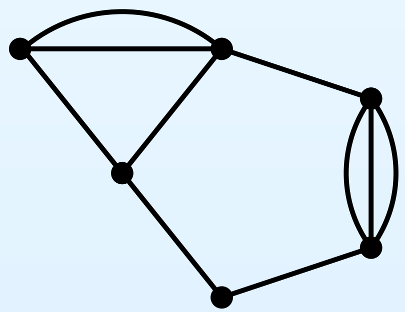

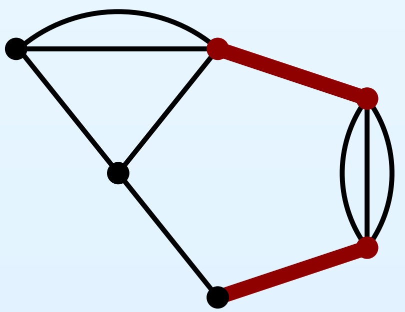

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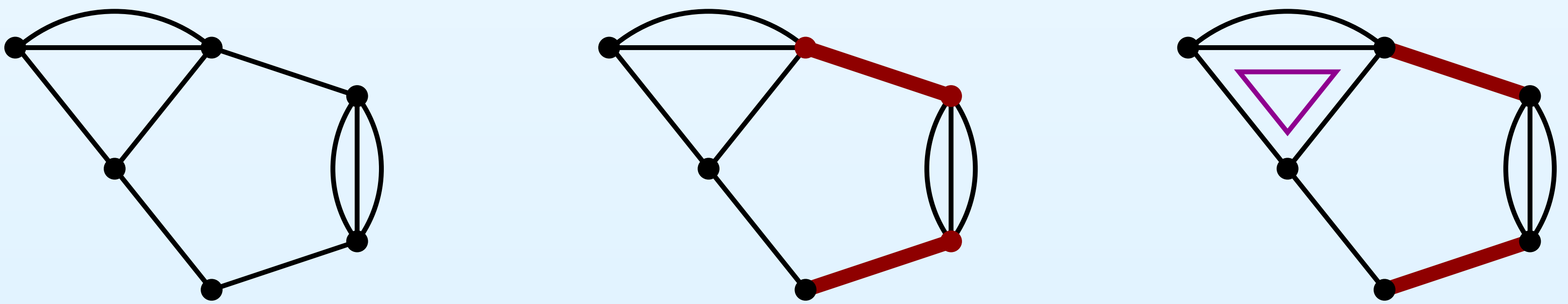

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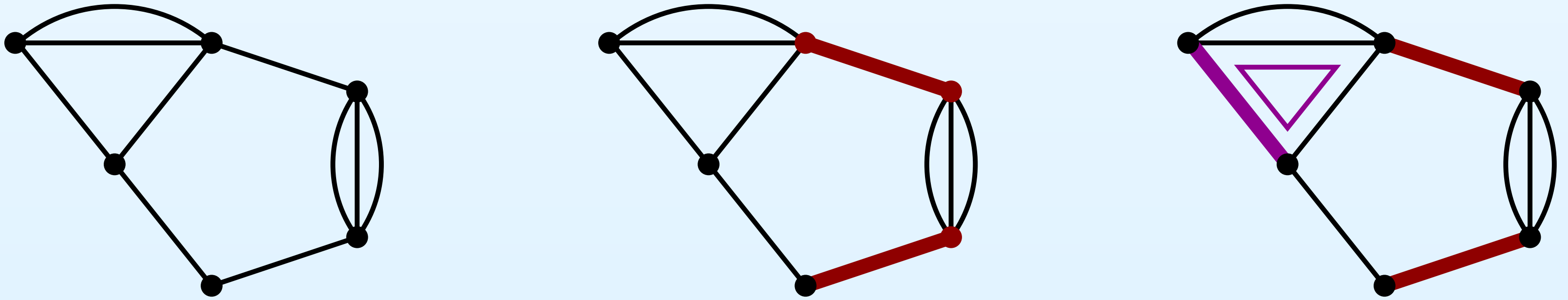

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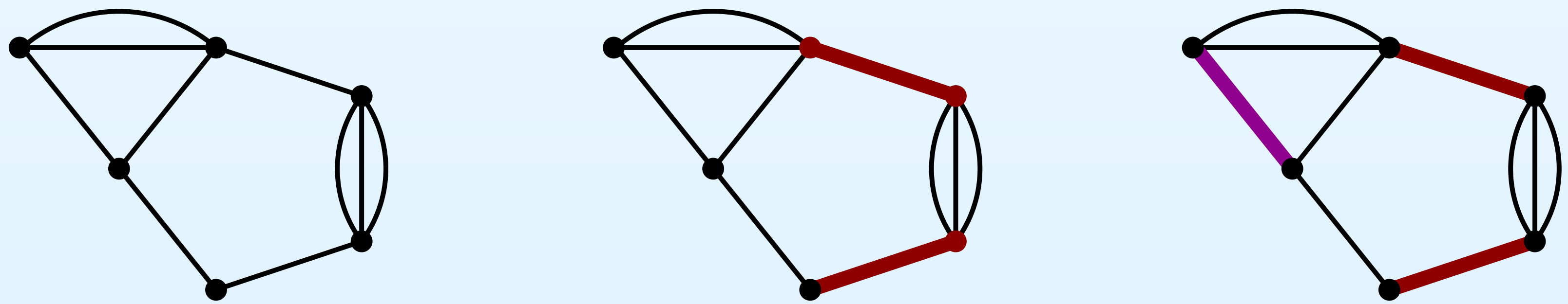

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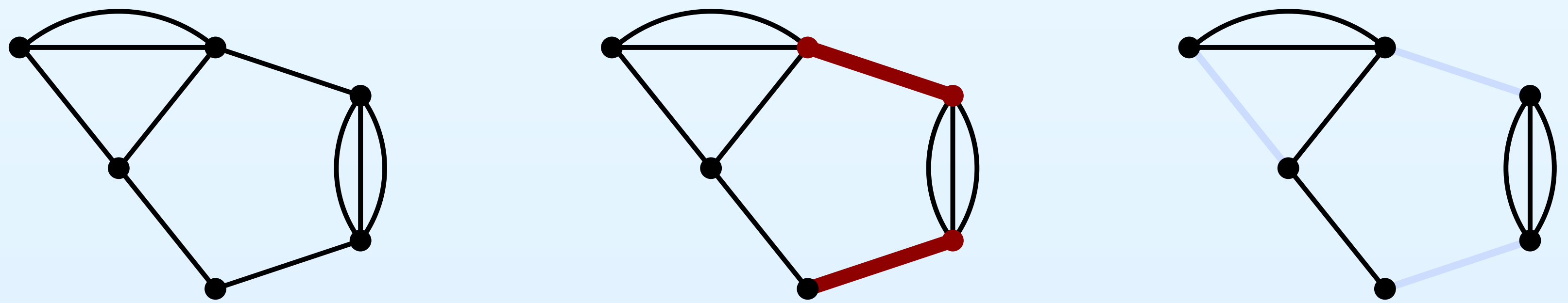

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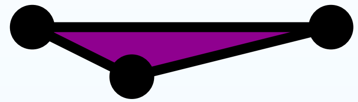
deg $= \Delta(H)$	mult. $\geq \Delta(H)/2$	mult. $< \Delta(H)/2$	weight $= \omega(G)$
			

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	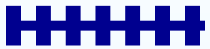		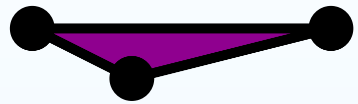

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	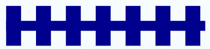		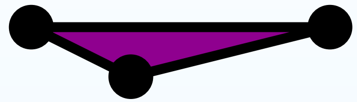

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	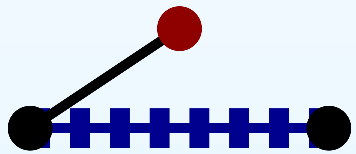	

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			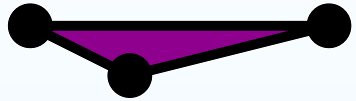

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	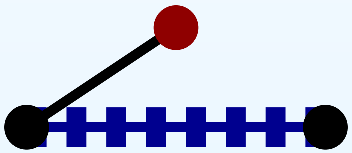	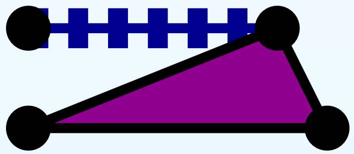

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			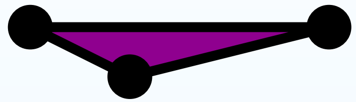

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	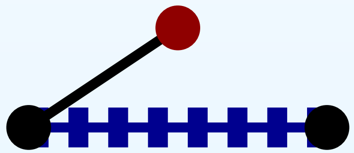	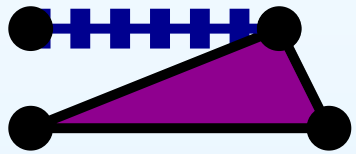
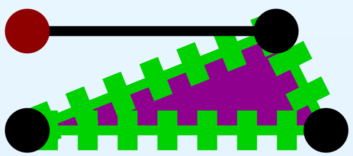		

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			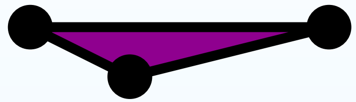

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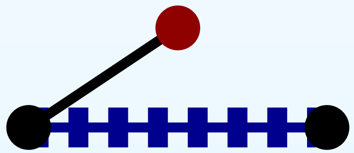
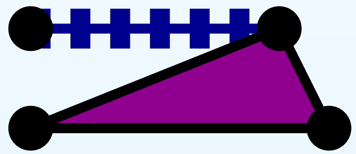
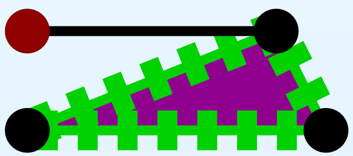
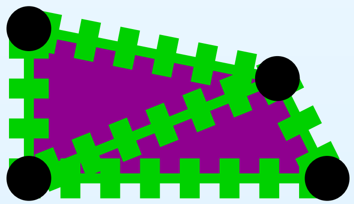
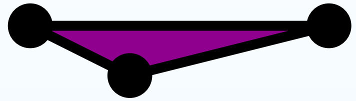
		
		

Structural properties (forbidden configurations)

deg $= \Delta(H)$	mult. $\geq \Delta(H)/2$	mult. $< \Delta(H)/2$	weight $= \omega(G)$
			

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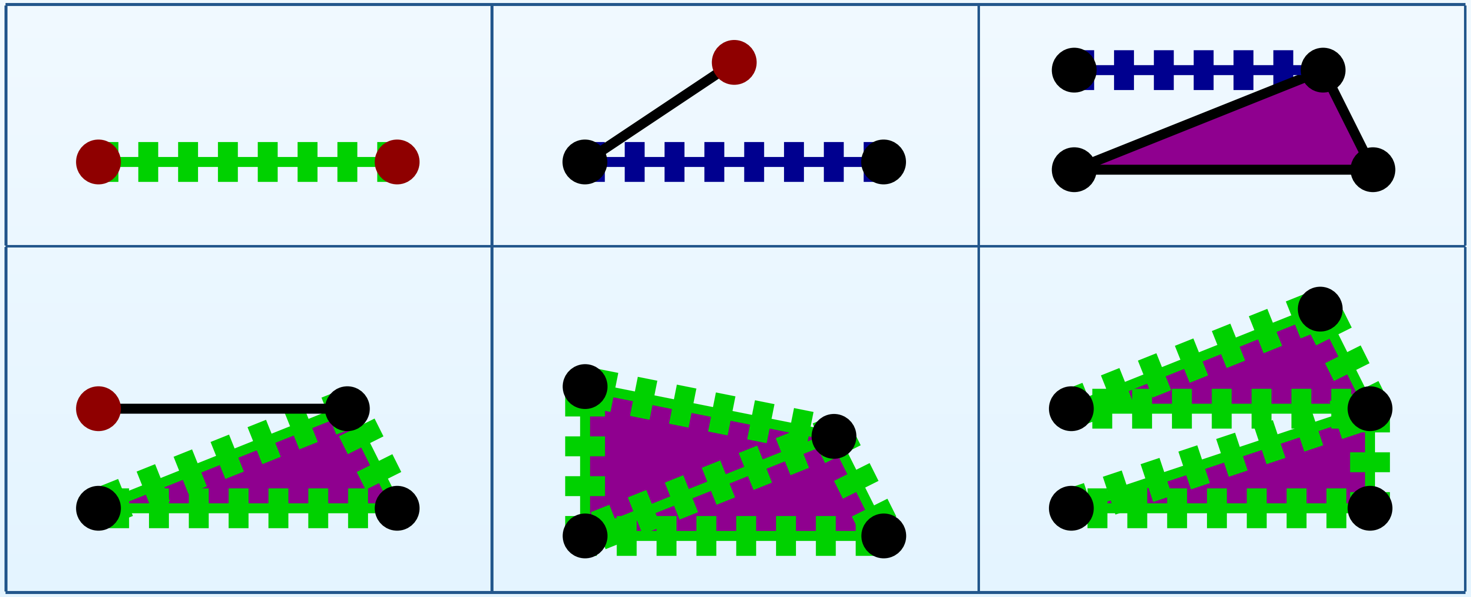

The construction method

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We construct the matching M incrementally:

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Use Hall's Theorem.

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The construction method

We construct the matching M incrementally:

- Take high-multiplicity edges $(\geq \frac{1}{2}\Delta(H))$.

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  Use Hall's Theorem.

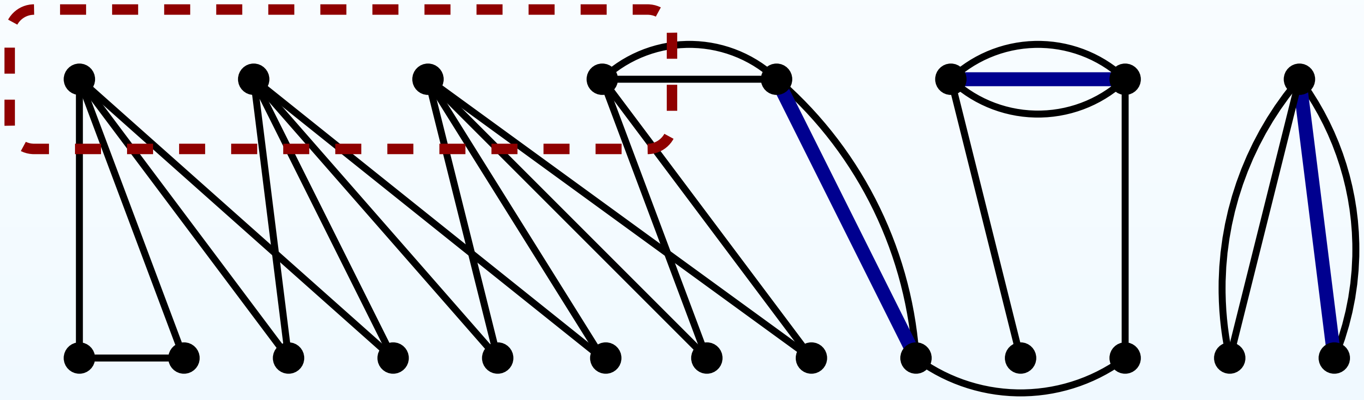
- Cover maximum weight triangles.

  Use structure... greedily!

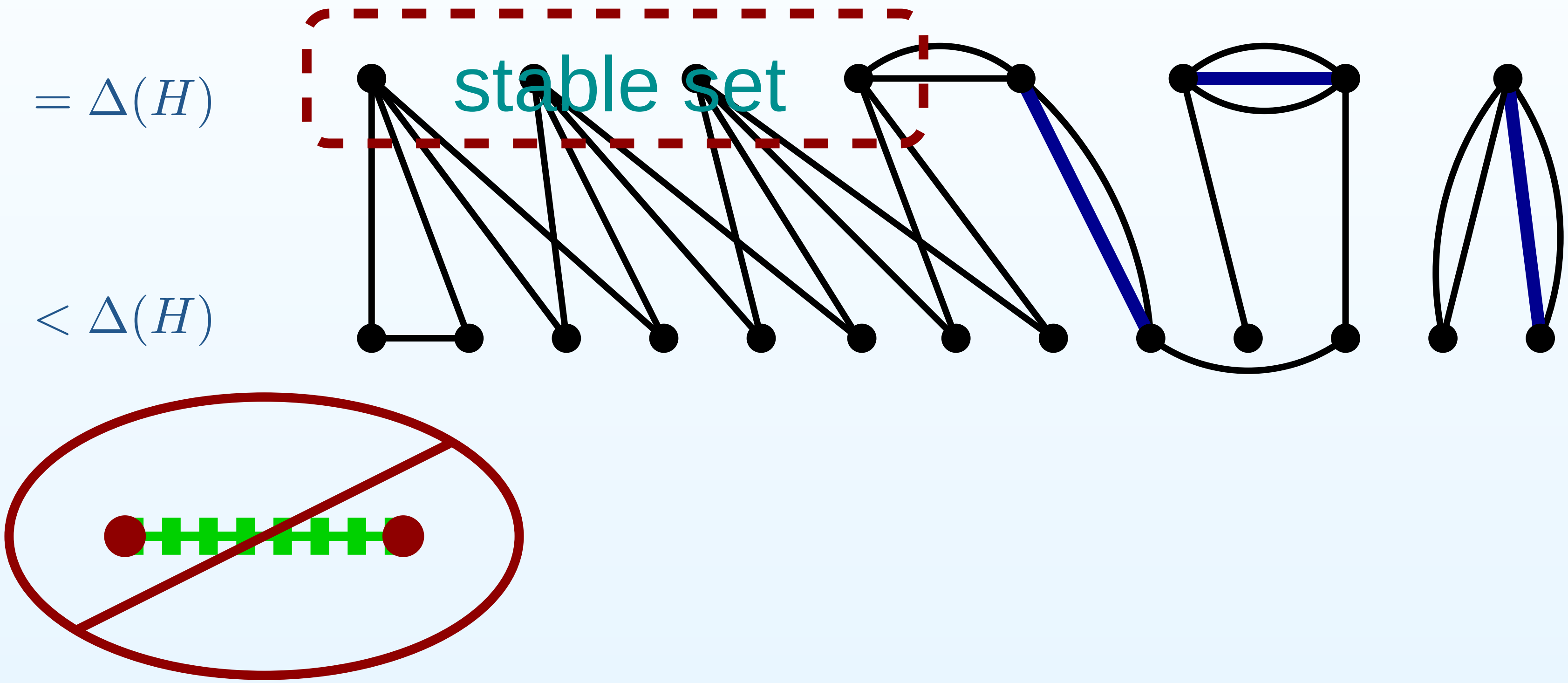
Hitting maximum degree vertices

$= \Delta(H)$

$< \Delta(H)$



Hitting maximum degree vertices



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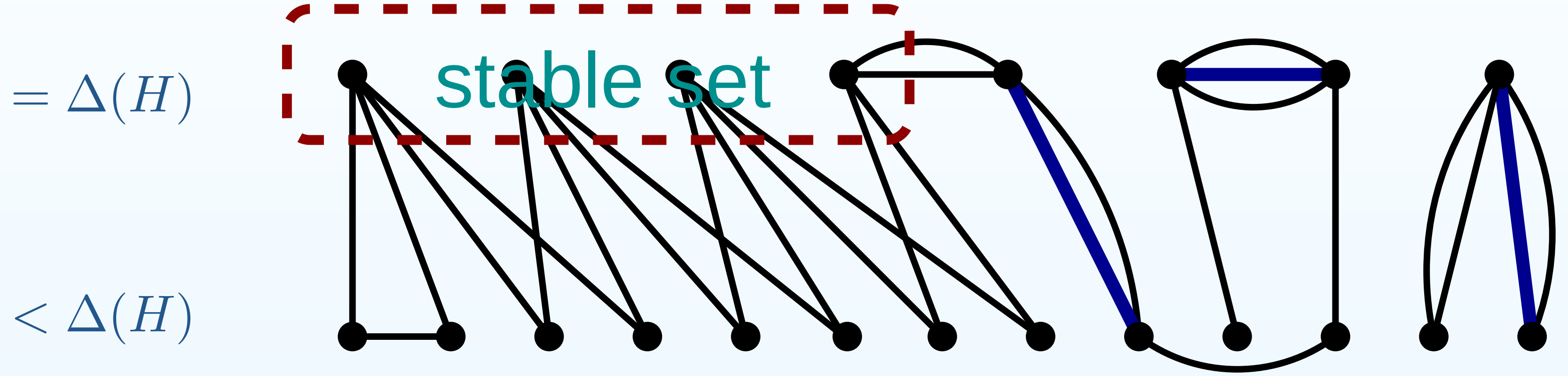
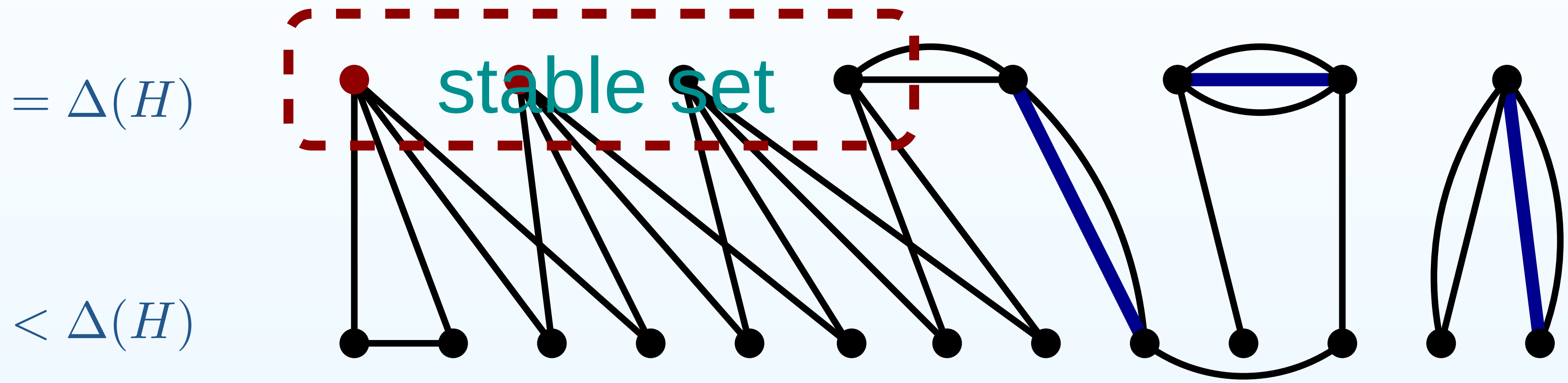


For any set S of remaining max-degree vertices,

$$N(S) \geq S.$$

Hitting maximum degree vertices



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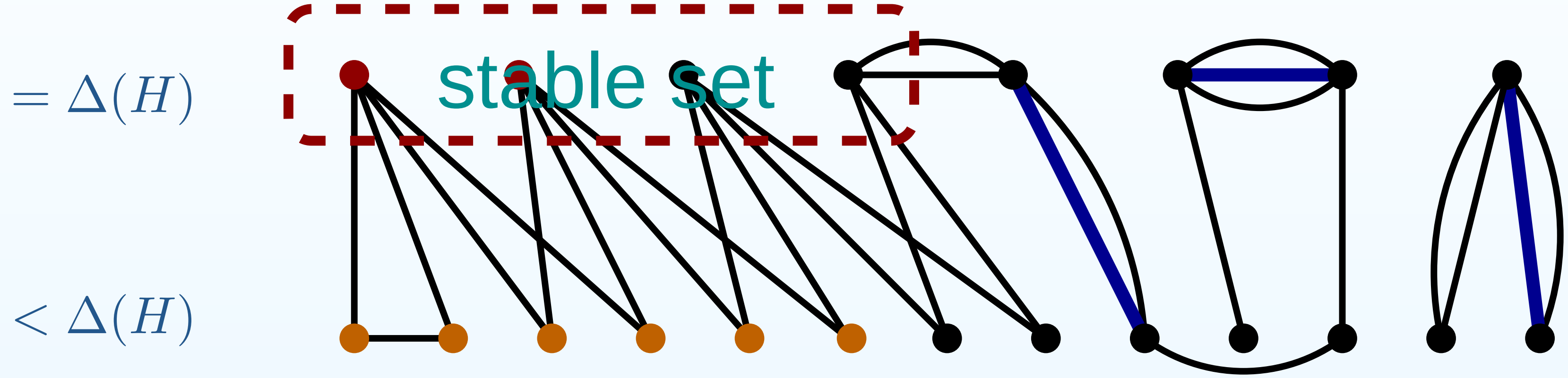

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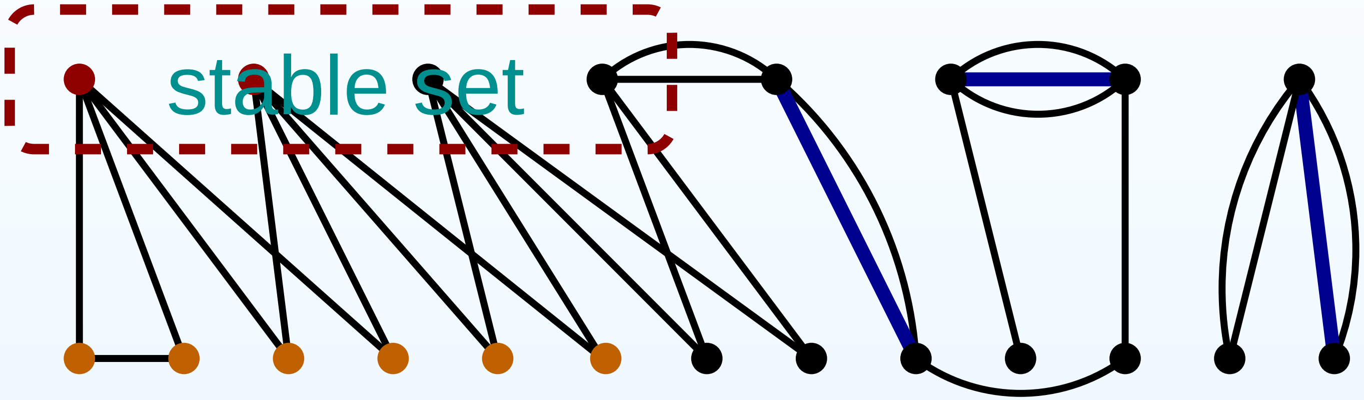

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So by Hall's Theorem we can hit them with a matching.

Hitting maximum degree vertices

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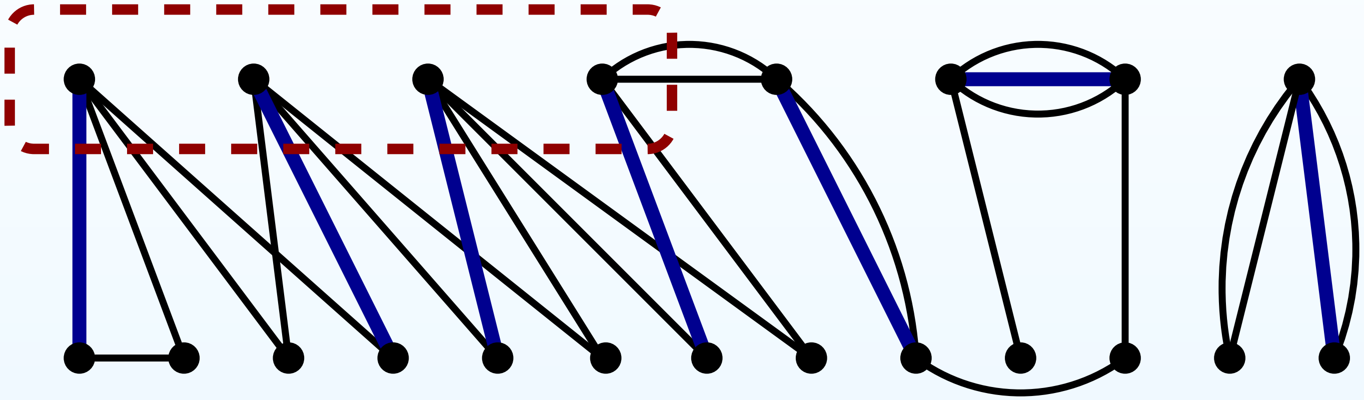
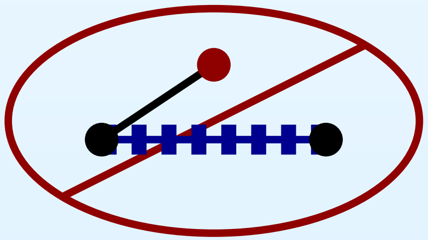


For any set S of remaining max-degree vertices,

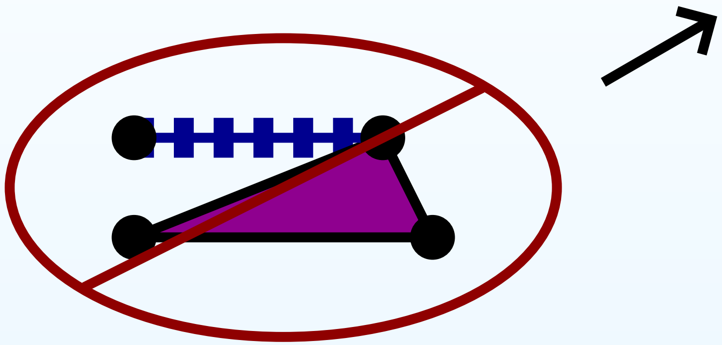
$$N(S) \geq S.$$

So by Hall's Theorem we can hit them with a matching.

There will be no conflict:



Covering maximum weight triangles



We already covered every $\omega(G)$-weight triangles with a mult. $\geq \Delta(H)/2$ edge.

We can extend the matching to cover the remaining $\omega(G)$-weight triangles.

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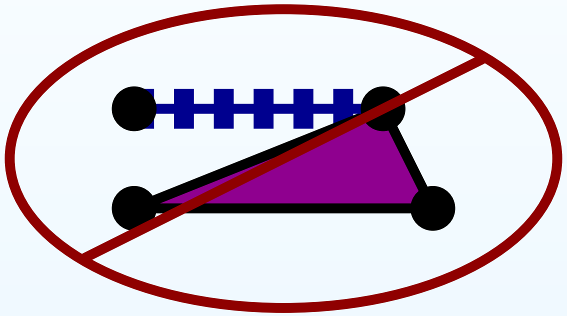

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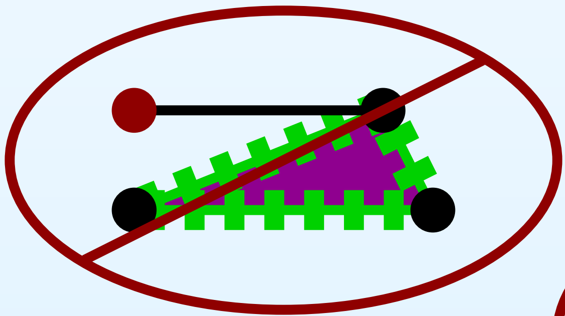

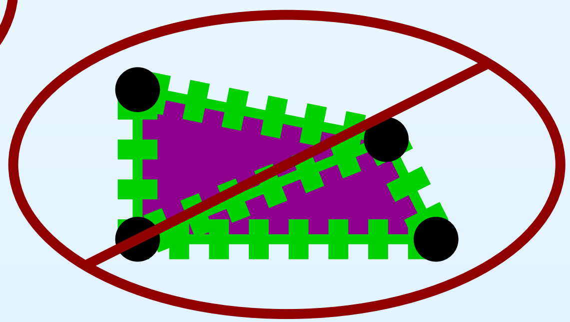

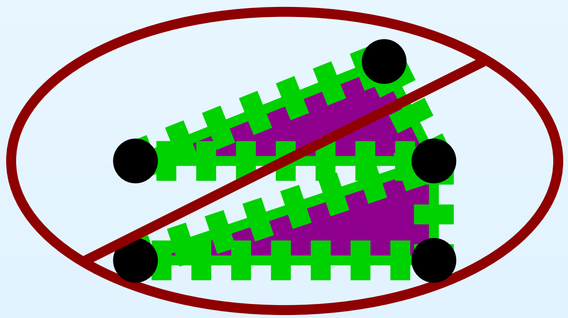

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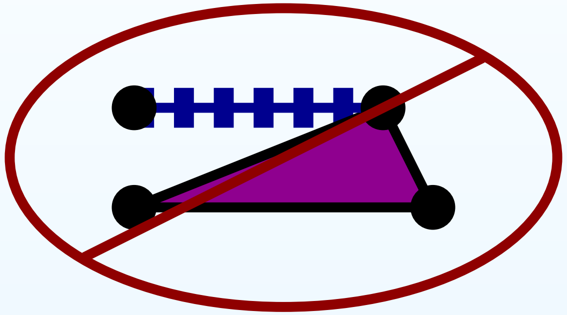
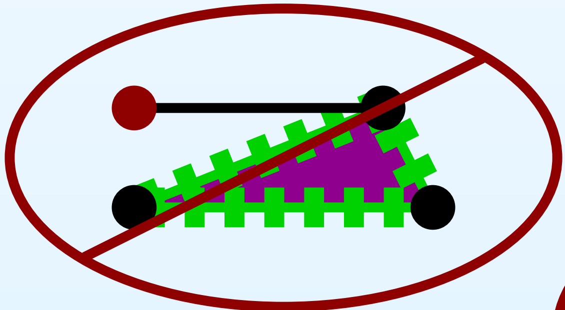


We already covered every $\omega(G)$-weight triangles with a mult. $\geq \Delta(H)/2$ edge.

All bad triangles are hit!



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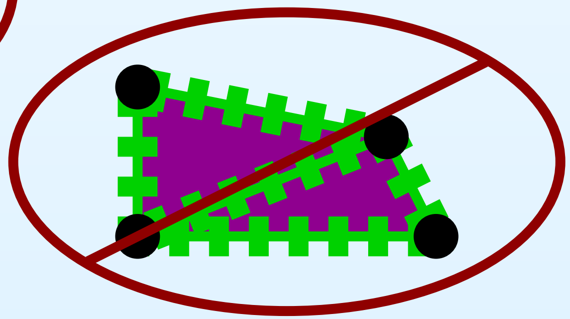

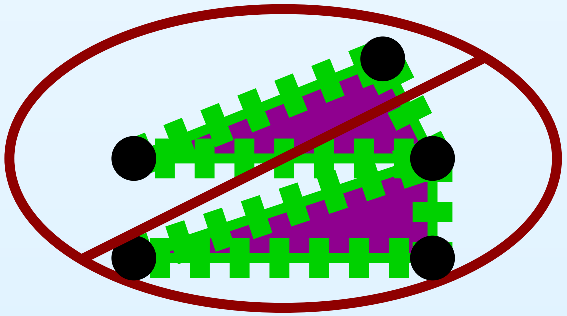

Finishing up

We have shown:

$$\Delta(G) < \frac{3}{2}\Delta(H) - 1 \quad \Rightarrow \quad \begin{array}{c} H \text{ contains a matching } M \\ \text{s.t.} \\ \omega(L(H-M)) < \omega(G). \end{array}$$

This completes the proof of the Main Theorem.

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This completes the proof of the Main Theorem.

Recall:

Main Theorem. *For any line graph G,*

$$\chi(G) \le \left\lceil \frac{\Delta(G) + 1 + \omega(G)}{2} \right\rceil .$$

Future work

The bound

$$\chi(G) \leq \left\lceil \frac{\Delta(G) + 1 + \omega(G)}{2} \right\rceil$$

is conjectured to hold for all graphs.

Promising graph classes:

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Promising graph classes:

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  Every vertex is bisimplicial

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Line graphs $\subset$ Quasi-line graphs $\subset$ Claw-free graphs

References

[1] A. Caprara and R. Rizzi. Improving a family of approximation algorithms to edge color multigraphs. *Information Processing Letters*, 68:11–15, 1998.

[2] M. Molloy and B. Reed. *Graph Colouring and the Probabilistic Method*. Springer-Verlag, Berlin, 2000.

[3] B. Reed. ω, δ, and χ. *Journal of Graph Theory*, 27:177–212, 1998.