Fourier Analysis and Large Independent Sets in Powers of Complete Graphs.

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We build too many walls and not enough bridges. (Isaac Newton)

There is a deep unity of our science that gives it its strength and vitality. (László Lovász)

Things have changes in the past two decades. (Bill Gates)

Fourier Analysis

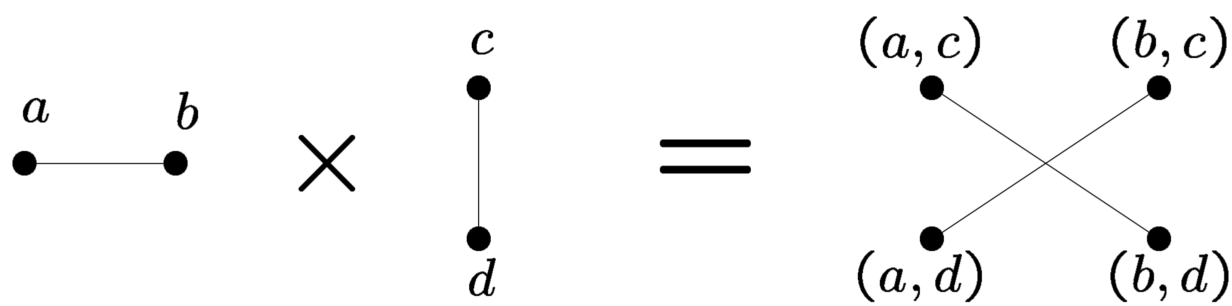
- Combinatorial and Additive Number Theory
- Study of Boolean Functions
- Complexity Theory
- Discrete Geometry
- Graph Theory

The graph $G \times H$ is defined as:

1. $V(G \times H) = V(G) \times V(H)$.

2. $(g_1, h_1) \sim (g_2, h_2)$ iff $g_1 \sim g_2$ and $h_1 \sim h_2$.

Example:



Complete Graphs

- Label the vertices of K_r by $\{0, \ldots, r-1\}$.

$$G = K_r^n = \underbrace{K_r \times K_r \times \ldots \times K_r}_{n}$$

- Edges:

$$(x_1, \ldots, x_n) \sim (y_1, \ldots, y_n) \text{ iff } x_1 \neq y_1, \ldots, x_n \neq y_n.$$

- An Independent Set:

$$I_{ij} = (*, \ldots, *, j, *, \ldots, *) = \{x : x_i = j\}$$

- Size:

$$\frac{|I_{ij}|}{|G|} = \frac{r^{n-1}}{r^n} = \frac{1}{r}.$$

- D. Greenwell and L. Lovász 1978:

I_{ij} are the only maximum independent sets for $r > 2$.

- Not necessarily true for $r = 2$.

- What can we say, if

$$\frac{|I|}{|G|} = \frac{1-\epsilon}{r}$$

- N. Alon, I. Dinur, E. Friedgut, and B. Sudakov 2004:
  There exists a function $M(r)$ s.t.

$$\frac{|I_{ij} \triangle I|}{|G|} \leq \frac{M(r)\epsilon}{r},$$

  for some i and j.

- According to the proof in ADFS, $M(r) > cr^4$.

- To obtain any nontrivial bound $\epsilon < \frac{2}{cr^4}$.

- **Theorem:** For $r \geq 20$ and $\epsilon < 10^{-8}$, $\frac{|I|}{|G|} = \frac{1-\epsilon}{r}$,

$$\frac{|I_{ij} \triangle I|}{|G|} \leq \frac{40\epsilon}{r}.$$

- **Remark:** For $\epsilon \geq 10^{-8}$, trivially

$$\frac{|I_{ij} \triangle I|}{|G|} \leq \frac{2 \times 10^8 \epsilon}{r}.$$

Fourier Analysis

We think of $V(G)$ as $\mathbb{Z}_r^n$.

Let $f, g : V(G) \to \mathbb{C}$.

Define $\langle f, g \rangle = \frac{1}{|G|} \sum_{x \in G} f(x)\overline{g(x)}$.

- $\langle u_y, u_z \rangle = 0$.

- $\langle u_y, u_y \rangle = 1$.

- $u_y(x)$ is a function of $\{x_i : y_i \neq 0\}$.

- For every $f : V(G) \to \mathbb{C}$,

$$f(x) = \sum_{y \in G} \widehat{f_y} u_y(x).$$

$$\sum_{y \in G} \widehat{f_y}^2 = \langle f, f \rangle = \frac{1}{|G|} \sum_{x \in G} |f(x)|^2 = \|f\|_2^2.$$

- Let $f(x) = \chi_{[x \in I]}$.
- $\frac{|I|}{|G|} = \sum_{y \in G} \widehat{f_y}^2$.
- Let $|y| = |\{y_i : y_i \neq 0\}|$.
- ADFS 2004:

$$\sum_{y \in G, |y|>1} \widehat{f_y}^2 < \frac{2\epsilon}{r}.$$

Let

$$f^{\le 1} = \sum_{|y|\le 1} \widehat{f_y} u_y = \widehat{f_0} + g_1 + \ldots + g_n,$$

where

$$g_i = \sum_{|y|=1, y_i \neq 0} \widehat{f_y} u_y.$$

W.L.O.G. assume

$$\|g_1\|_2^2 \ge \ldots \ge \|g_n\|_2^2.$$

Goal: $\sum_{i=2}^{n} \|g_i\|_2^2$ is small ($< \frac{8\epsilon}{r}$).

This would imply: f is close to $\widehat{f_0} + g_1$.

Because

$$\|f\|_2^2 = \widehat{f_0}^2 + \|g_1\|_2^2 + \sum_{i=2}^{n} \|g_i\|_2^2 + \|f^{>1}\|_2^2,$$

Or

$$\|f - \widehat{f_0} - g_1\|_2^2 = \sum_{i=2}^{n} \|g_i\|_2^2 + \|f^{>1}\|_2^2 < \frac{10\epsilon}{r}.$$

So

$$\|f - \text{round}(\widehat{f_0} + g_1)\|_2^2 \le 4\|f - (\widehat{f_0} + g_1)\|_2^2 \le \frac{40\epsilon}{r}.$$

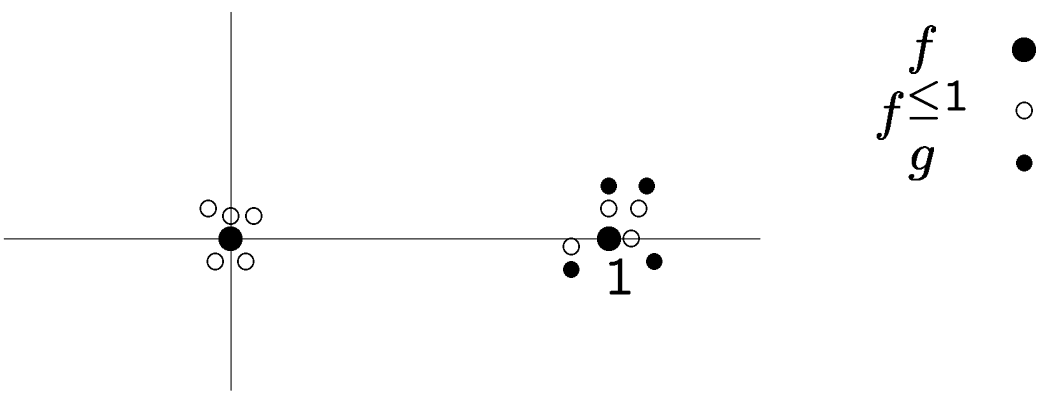
Proof of Goal

$f^{\leq 1}$ is close to f.

There exist δ_i's s.t.

$g(x_2) = \widehat{f_0} + g_1(\delta_1) + g_2(x_2) + g_3(\delta_3) + \ldots + g_n(\delta_n)$

is close to $\{0, 1\}$.



$\|g_2\|_2^2 = \text{Var}(g) \leq \frac{4000\epsilon}{r}$.

So

$$\frac{4000\epsilon}{r} \geq \|g_2\|_2^2 \geq \ldots \geq \|g_n\|_2^2.$$

Let $m \geq 2$ be the minimum

$$\sum_{i=m}^{n} \|g_i\|_2^2 \leq \frac{10^4\epsilon}{r}.$$

Since $\|d(f^{\leq 1}, \{0,1\})\|_2^2 \leq \frac{4\epsilon}{r}$:

There exist $\delta_1, \ldots, \delta_{m-1}$ s.t. if

$$\begin{aligned} g(x_m, \ldots, x_n) &= \overbrace{\widehat{f_0} + \ldots + g_m(\delta_{m-1})}^{b} \\ &\quad + g_m(x_m) + \ldots + g_n(x_n) \end{aligned}$$

then

$$\|d(g(x_m, \ldots, x_n), \{0,1\})\|_2^2 \leq \frac{4\epsilon}{n}.$$

So $\|d(b, \{0,1\})\|_2^2$ is small.

$$\sum_{i=m}^{n} \|g_i\|_2^2 \leq \frac{8\epsilon}{r}.$$

So $m = 2$ which proves the goal.

Open Problems

Corollary: $G = K_n^n$, $r \geq 20$. If $\frac{|I|}{|G|} \leq \frac{1-10^{-9}}{r}$, then

$$I \subseteq I_{ij}.$$

Consider the induced subgraph of K_n^n on the vertices with distinct coordinates.

Conjecture:(Cameron and Ku2003) There is a constant c such that every independent set of size at least $c(n-1)!$ is a subset of an independent set of size $n!$.