Packing Steiner Trees and Forests

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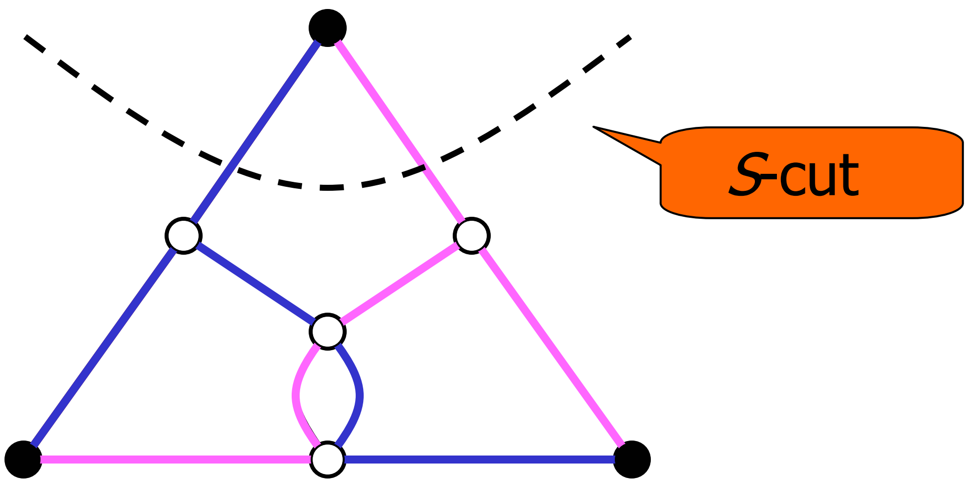
Steiner Tree Packing

Given an undirected multigraph G, $S \subseteq V(G)$.

S – black vertices

$V(G)$-S – white vertices

S-Steiner tree (S-tree)



Steiner Tree Packing
Find a largest collection of edge-disjoint S-trees

Motivations

Theoretical reasons:

- Edge-disjoint s,t-paths problem ($S = \{s,t\}$)

  [Menger] Max edge-disjoint s,t-paths = Min s,t cut

- Edge-disjoint spanning trees problem ($S = V(G)$)

  [Tutte, Nash-Williams] A partition-type min-max relation

  (Corollary) $2k$-edge-connected $\Rightarrow$ k edge-disjoint spanning trees

Practical reasons:

- VLSI circuit design
- Network broadcasting

Approximate Min-Max Relation for Steiner Tree Packing?

Generalization of Tutte's partition-type min-max relation?

Probably not. Steiner Tree Packing is NP-complete.

Kriesell's conjecture:

$2k$-S-connected $\Rightarrow$ k edge-disjoint S-trees

Previous Results

- [Kriesell] If every white vertex is of even degree,
  then $2k$-S-connected $\Rightarrow$ k edge-disjoint S-trees
- [Frank,Kiràly,Kriesell] If no white edge,
  then $3k$-S-connected $\Rightarrow$ k edge-disjoint S-trees
- [Kriesell] If no white bridge of size $> l$
  then $(l+2)k$-S-connected $\Rightarrow$ k edge-disjoint S-trees
  - $l=0$ spanning tree packing
  - $l=1$ no white edge

Previous Results

- [Jain,Mahdian,Salavatipour]
  ≈ *(|S|/4)k*-*S*-connected ⇒ *k* edge-disjoint *S*-trees
  - Improve an exponential bound by [Kriesell]
- [Kriesell]
  (2k+2|V(G)-S|)-*S*-connected ⇒ *k* edge-disjoint *S*-tree
  - Improve an exponential bound by [Pentigi,Rodriguez]
- [Jain,Mahdian,Salavatipour]
  Fractional Steiner Tree Packing ó Minimum Steiner Tree
- [Cheriyan, Salavatipour]
  - Directed Steiner tree packing
  - White-node disjoint Steiner tree packing

Summary of Previous Results

- Integer programming approaches (VLSI applications)
- Practical heuristic methods (networking applications)

- No polynomial time *o(n)*-approximation algorithm

  (i.e. not asymptotically better than *one* spanning tree)

- Kriesell's conjecture was open even when *2k* is replaced

  by *o(n)k* (even for *k=2*)

New Results

- [Main Theorem] If G is $26k$-S-connected,

  then G has k edge-disjoint S-trees

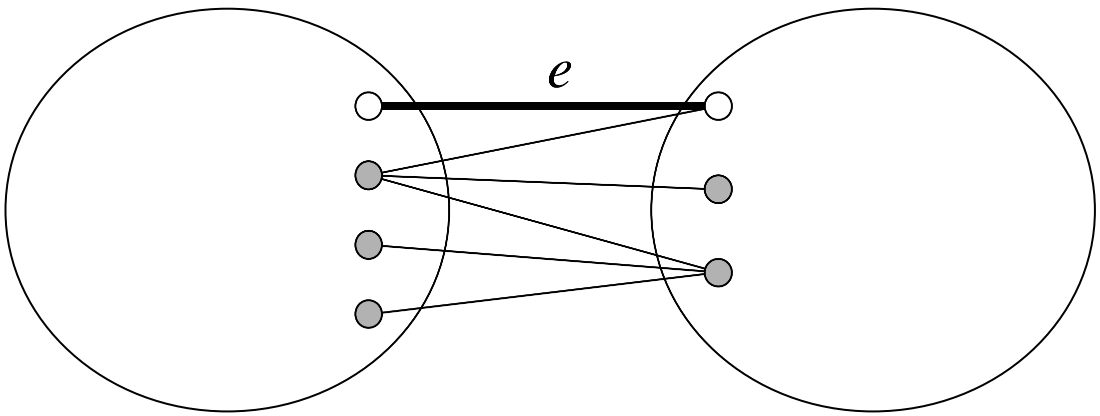
- [Corollary] A polynomial time algorithm to construct

  at least $\lfloor cut_S(G)/26 \rfloor$ S-trees

  ($\Rightarrow$ 51- approximation)

First Try

Lets try to remove the white edges!



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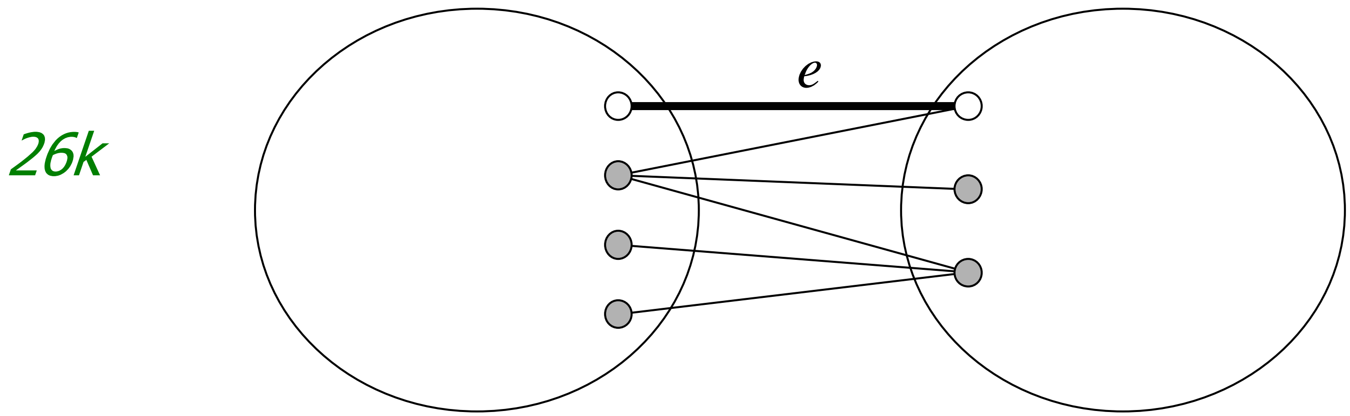

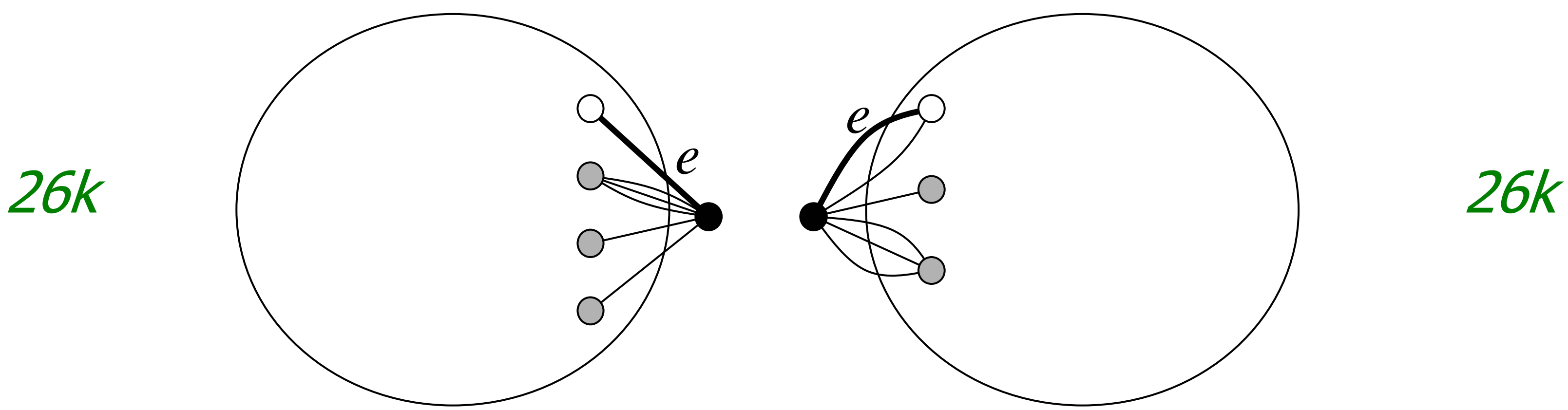

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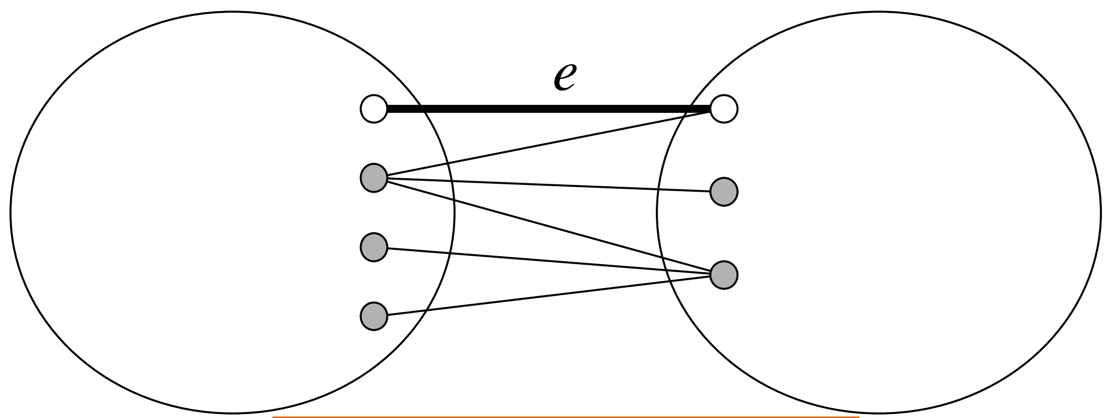
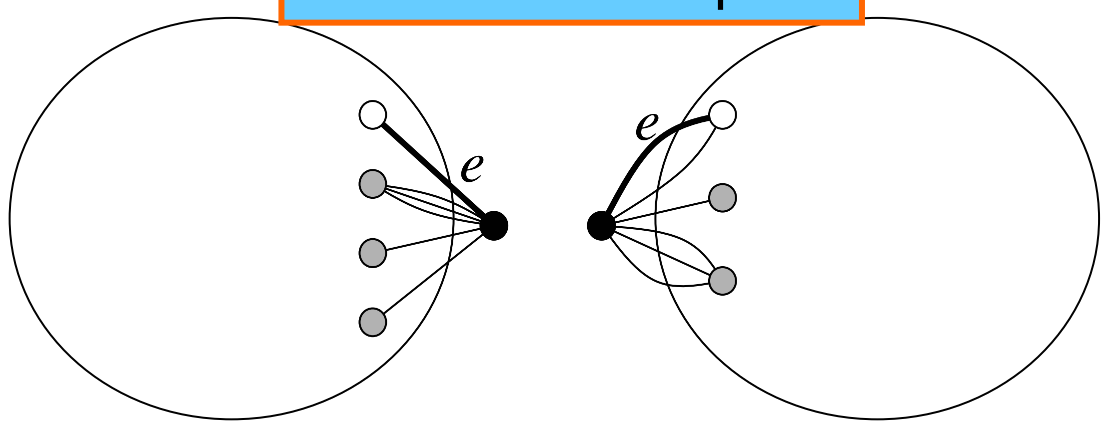
26k



Divide and Conquer?

26k



26k

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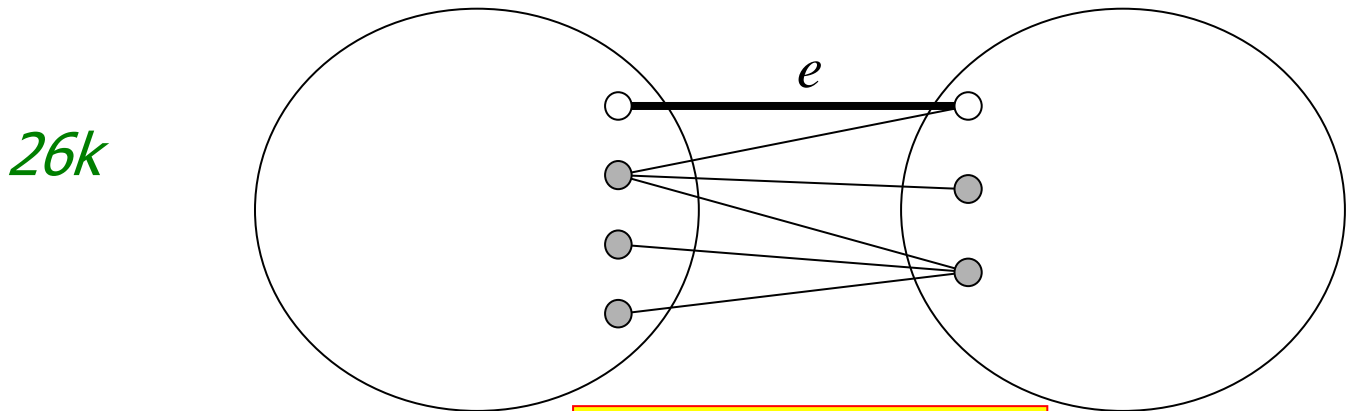
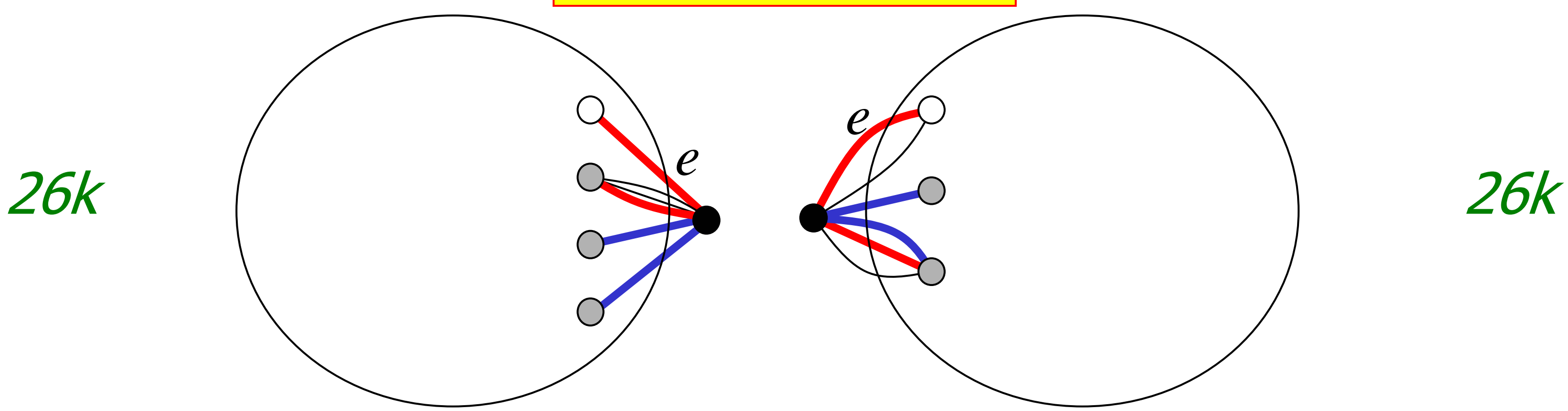


May not match…



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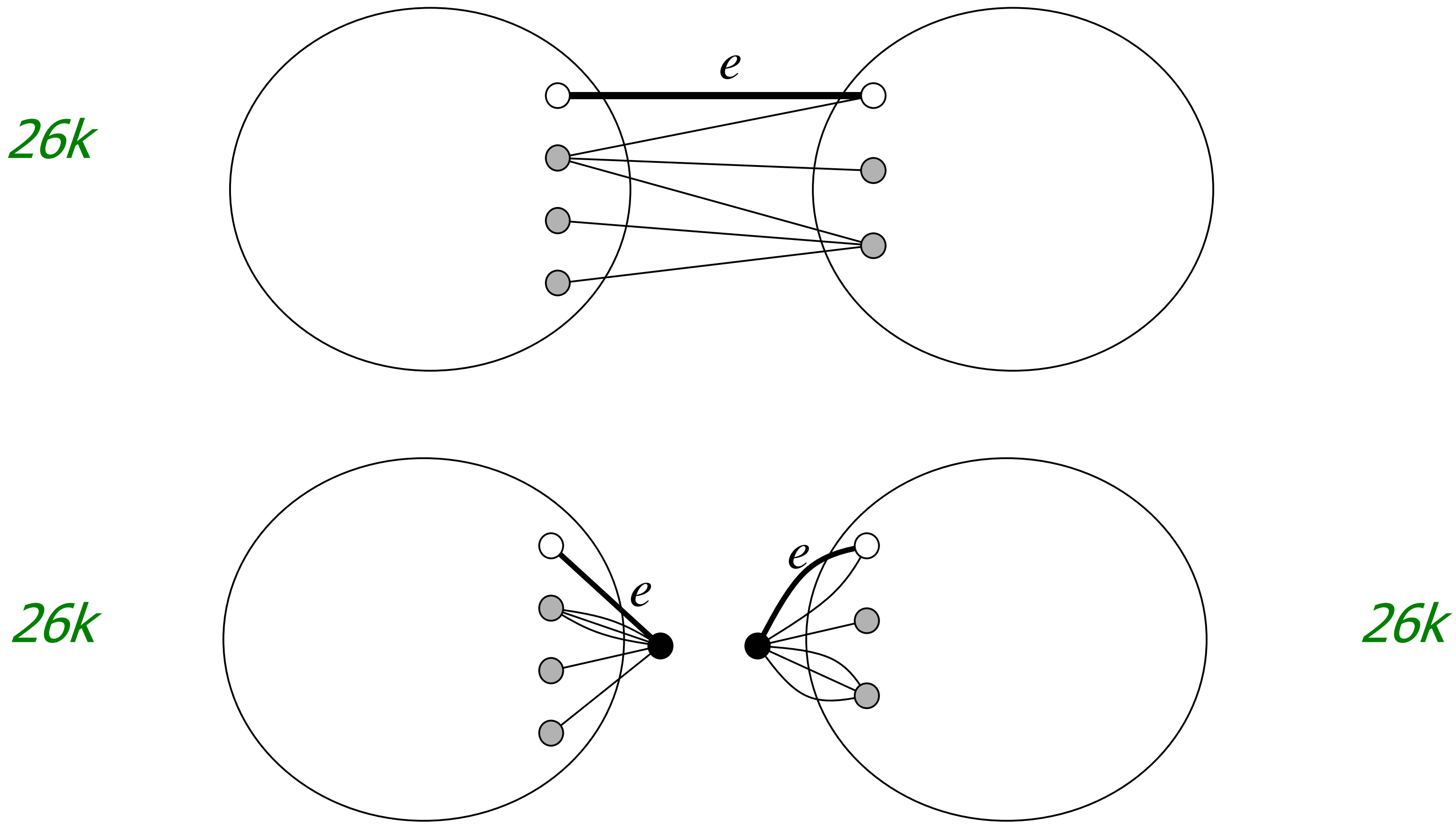

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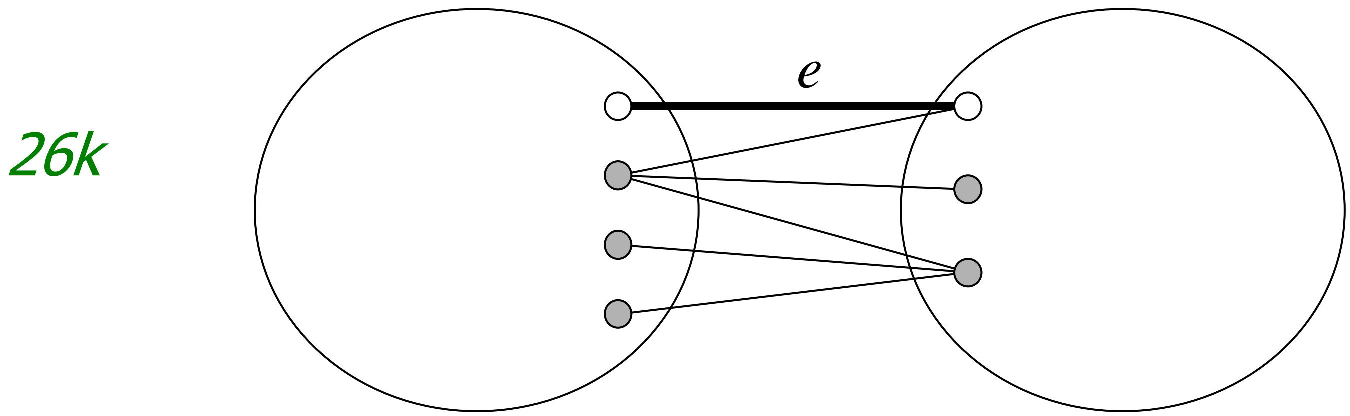

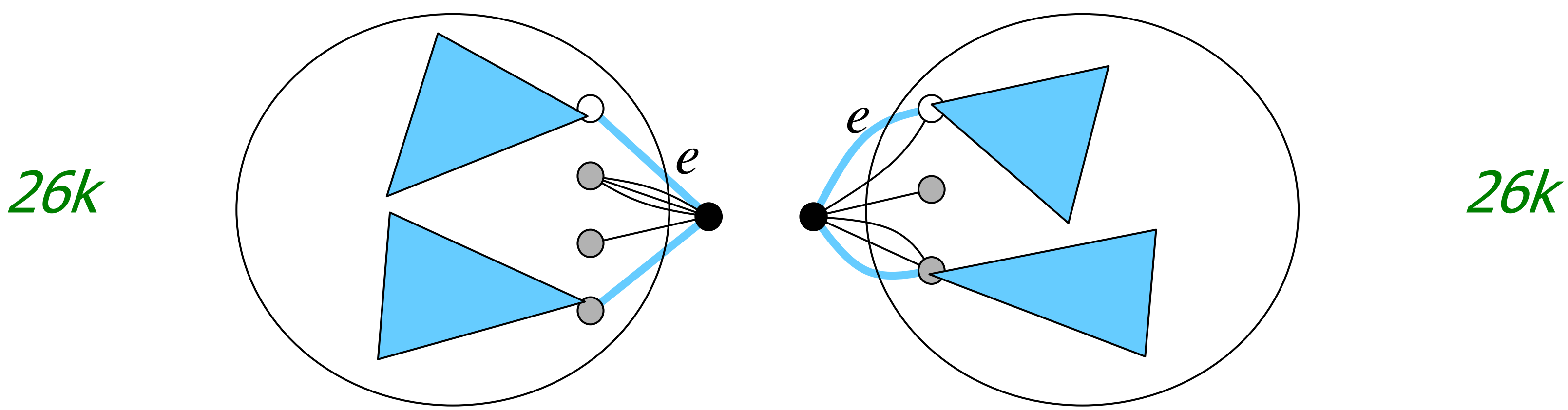

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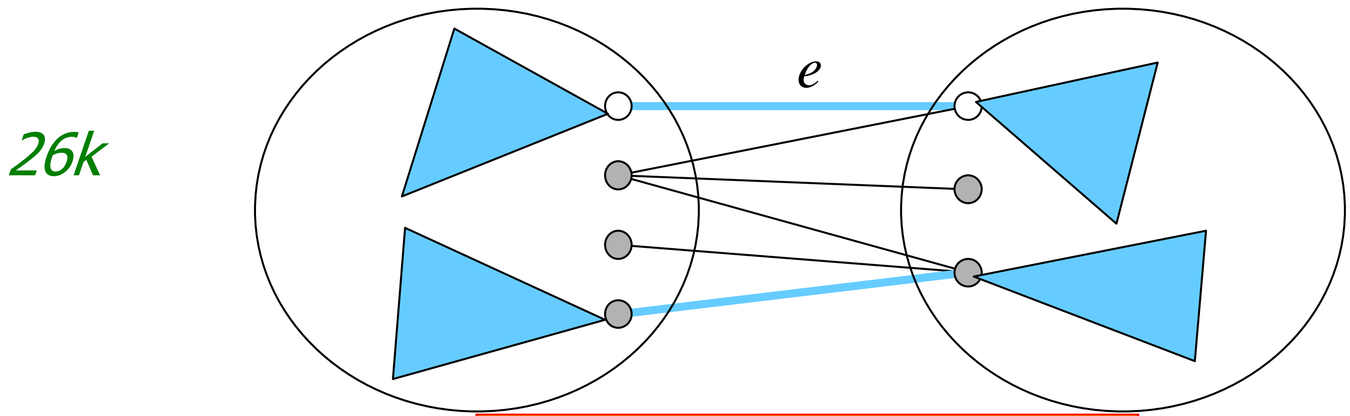
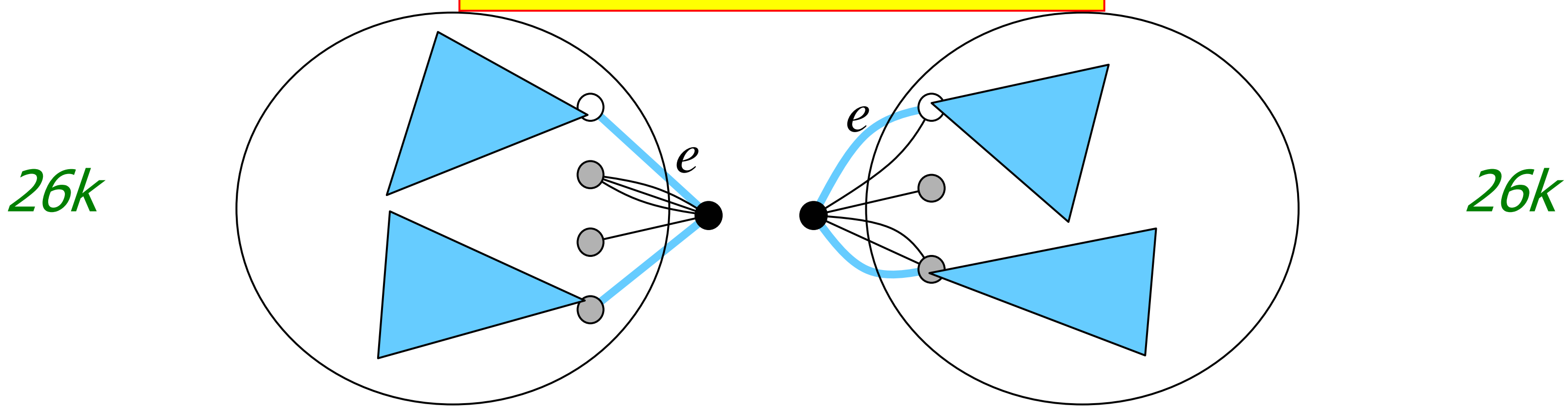


May not be connected…



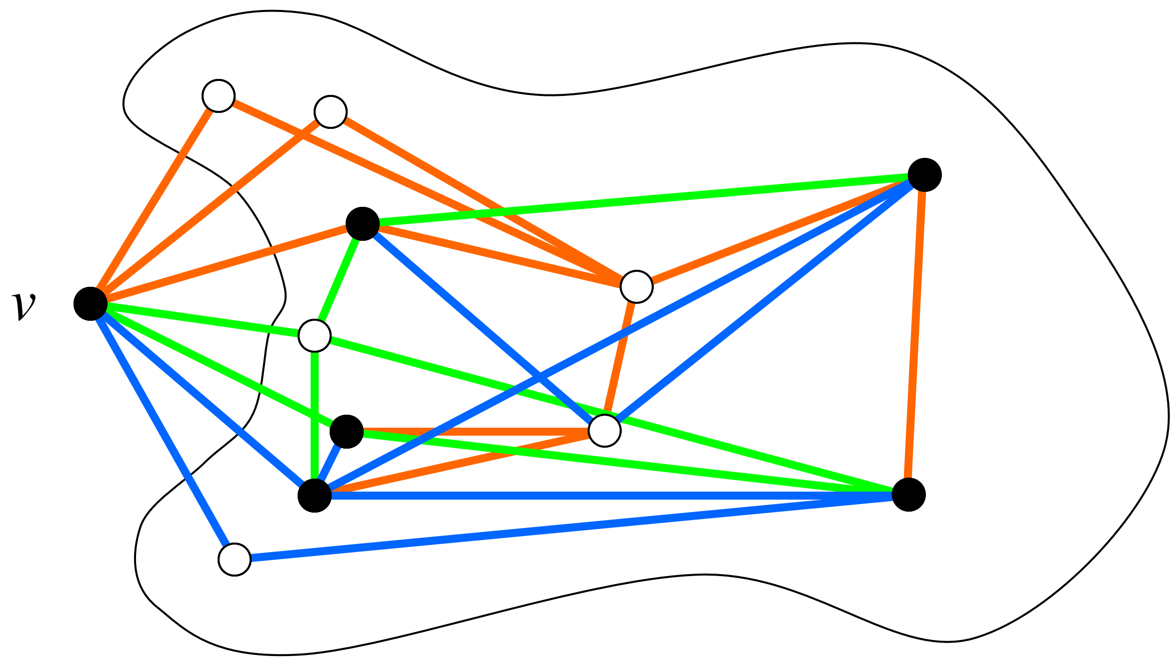
Our Method

The Main Theorem holds with a rich combinatorial property, which we call **"the extension property"**

For any vertex v of degree ***26k***

For any "***k*-edge-partition**" $\mathcal{P}$ of the edges incident to v

there are k disjoint Steiner subgraphs in $G - v$ that "**extend**" $\mathcal{P}$



Refined Statement

[Main Theorem]

If G is $26k$-S-connected,

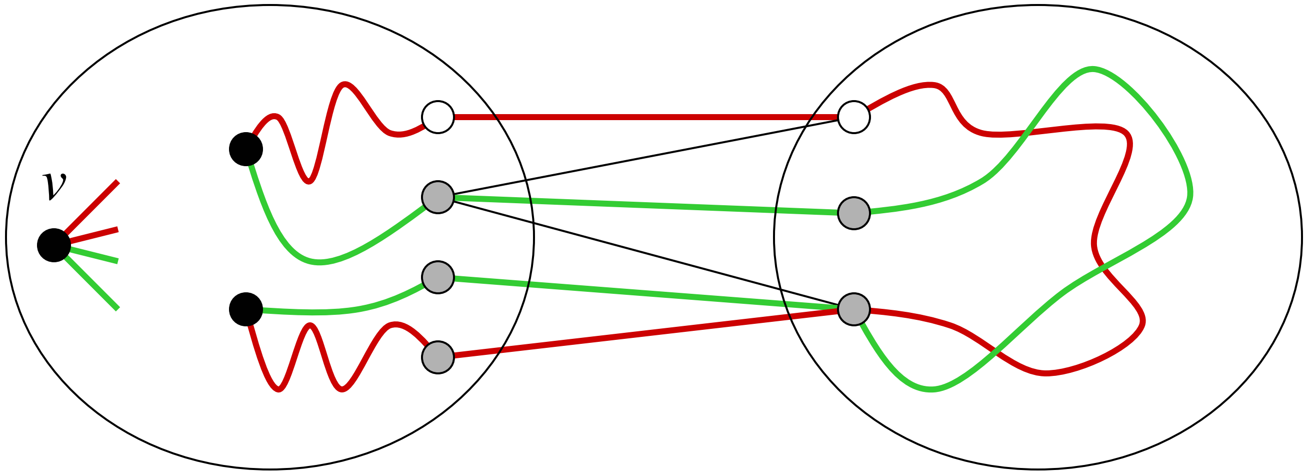
then G has k edge-disjoint S-trees.

Furthermore, the "extension property" holds.

Consider a counterexample $\mathcal{G}$

with the minimum number of edges

Cut Decomposition



1. New graphs have fewer edges than $\mathcal{G}$
2. New graphs have the same Steiner connectivity
3. New vertices are of degree *26k*

Therefore, $\mathcal{G}$ has no white edge!

About the Extension Property

- ü No white edge (“cut decomposition”)

- v [Frank,Kiràly,Kriesell] If no white edge,
  then $3k$-S-connected $\Rightarrow$ k edge-disjoint S-trees

The harder part is to show “the extension property”

Main tool: Mader’s splitting lemma

$\Rightarrow$ Every white vertex is of degree 3

And also: No white edge and $26k$-S-connected

Packing Steiner Forests

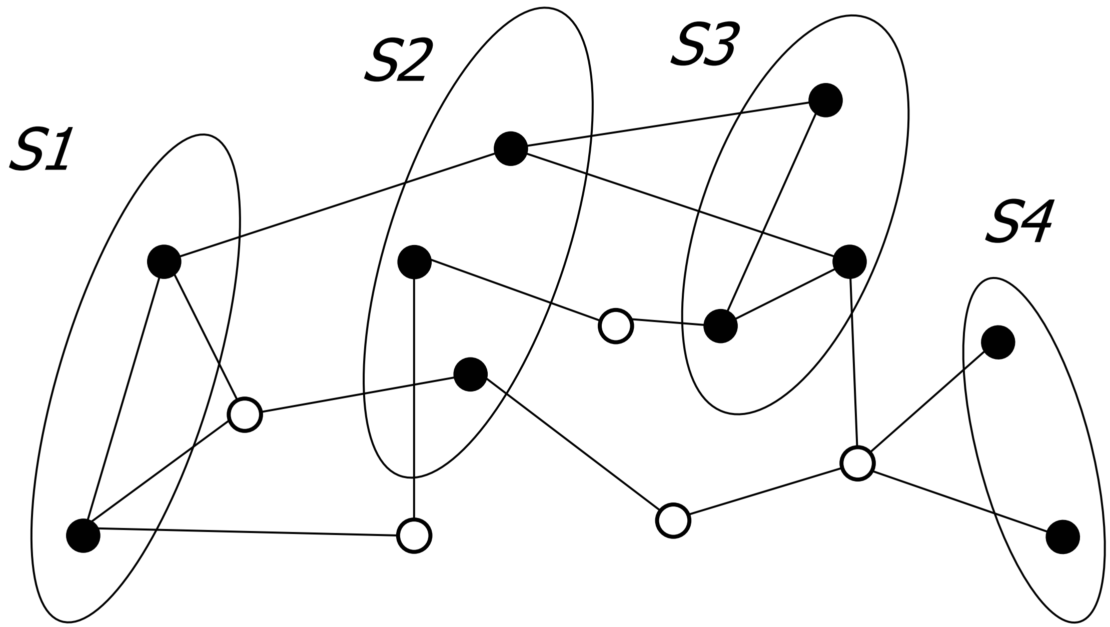
Acknowledgement:

The Egerváry Research Group on Combinatorial Optimization

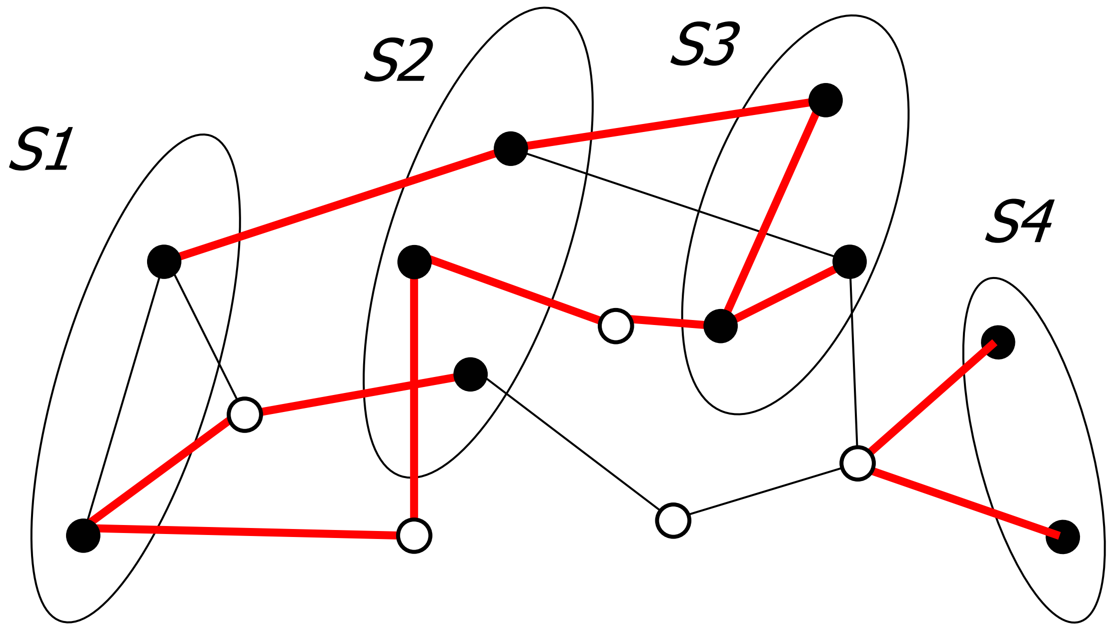
(EGRES)

Operations Research Department, Eötvös University, Budapest.

Steiner Forest Packing



Steiner Forest Packing



A Steiner forest (*S* -forest) is a subgraph *H* such that each *Si* is connected in *H*

Steiner Forest Packing
Find a largest collection of edge-disjoint *S*-forests

Results

- [Chekuri,Shepherd]

  If each S_i is *2k* edge-connected in G

  and G is Eulerian,

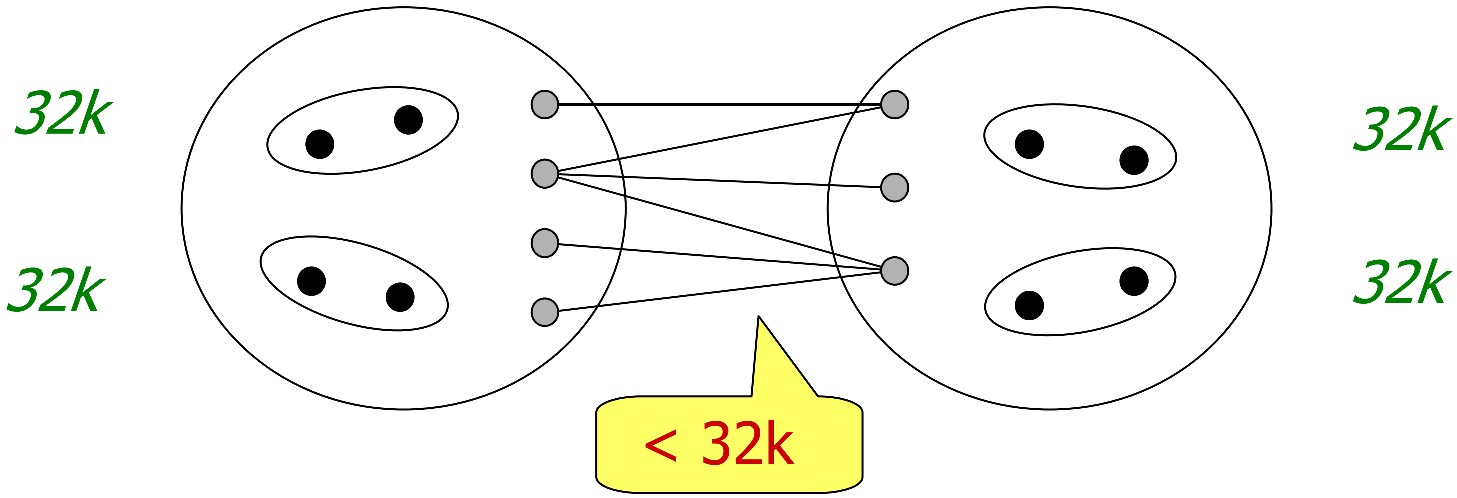
  then G has k edge-disjoint S-forests

- [L] If each S_i is *32k* edge-connected in G,

  then G has k edge-disjoint S-forests

Proof Idea

Reduce to Steiner Tree Packing?



Proof Idea

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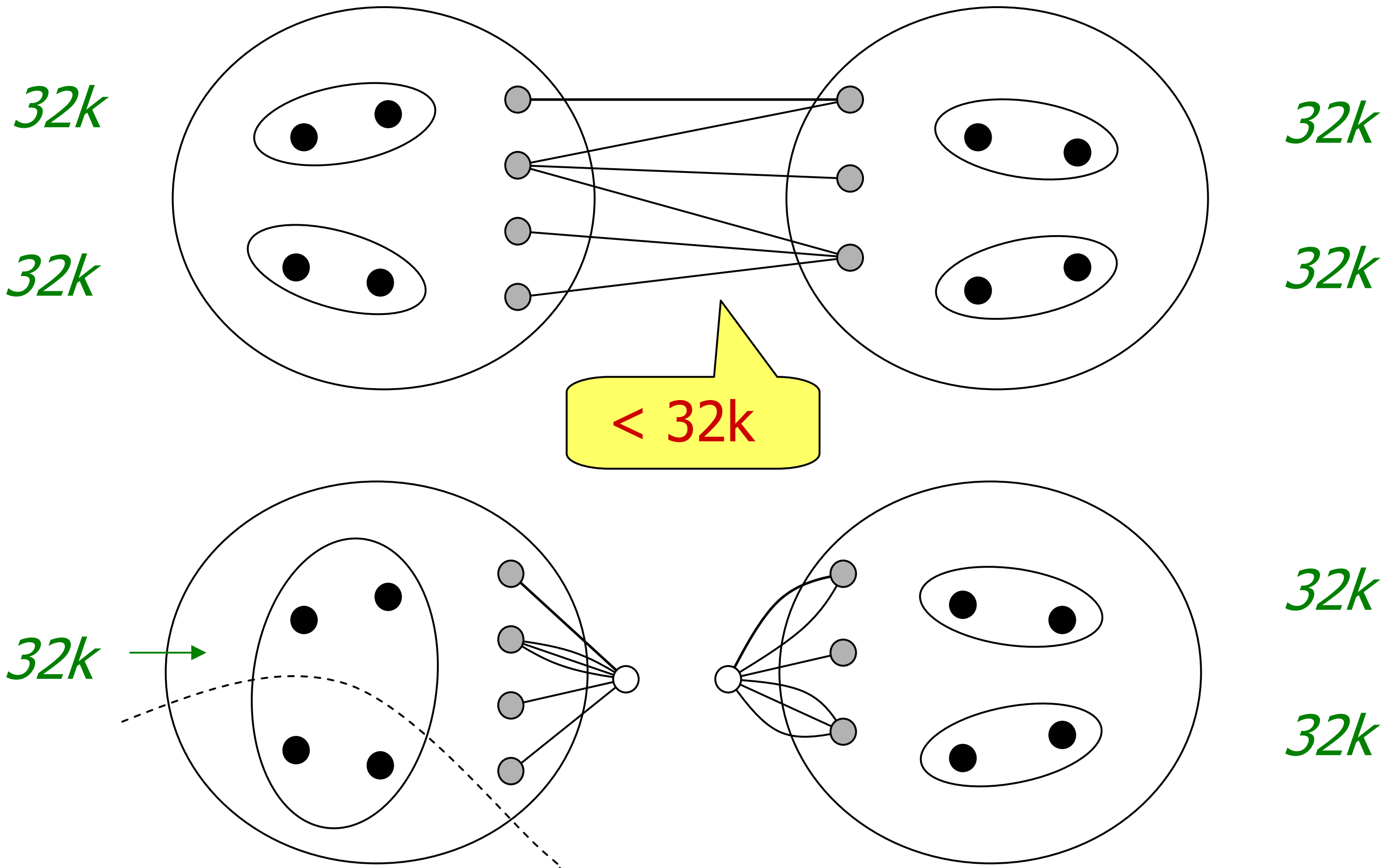

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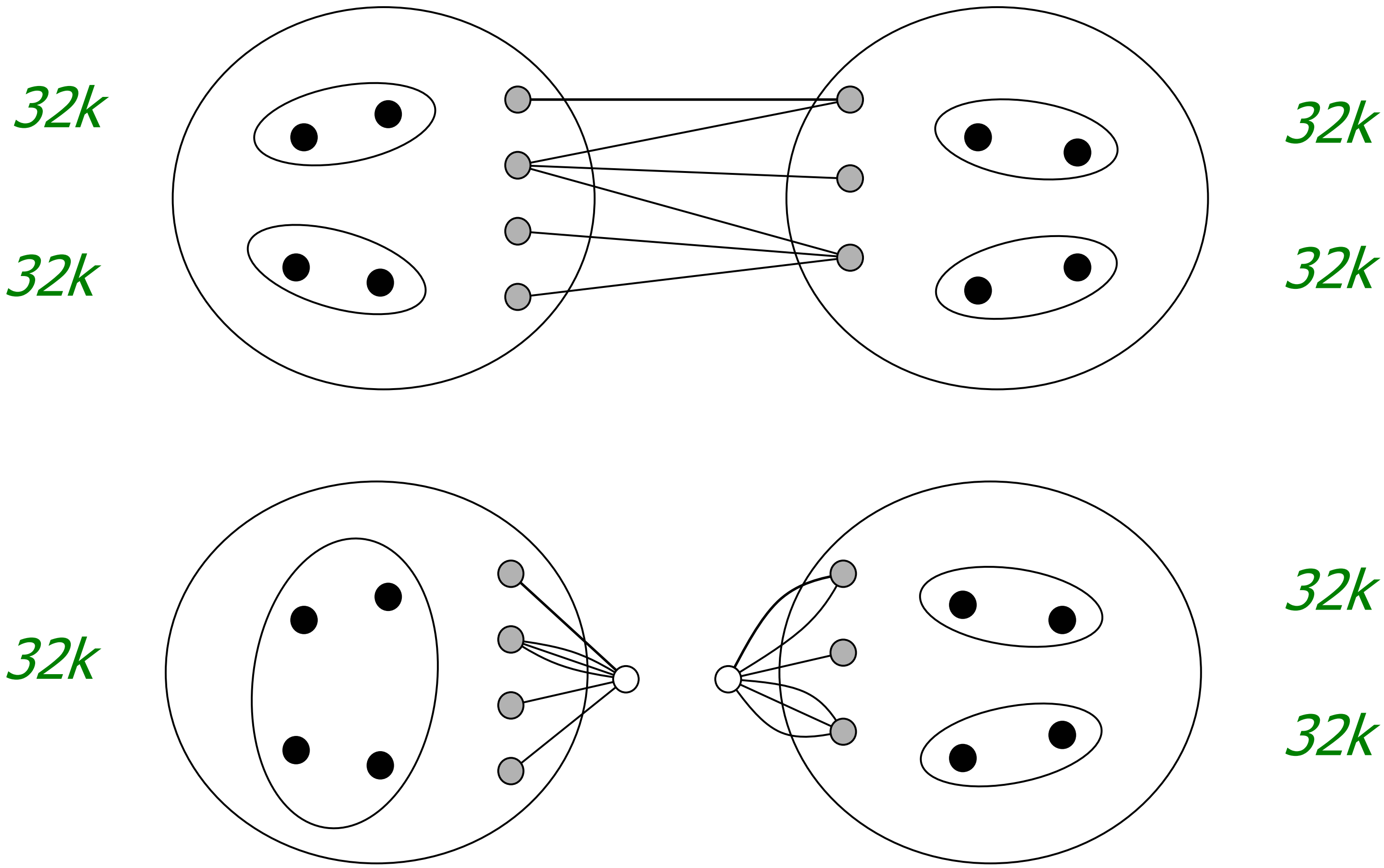

Proof Idea

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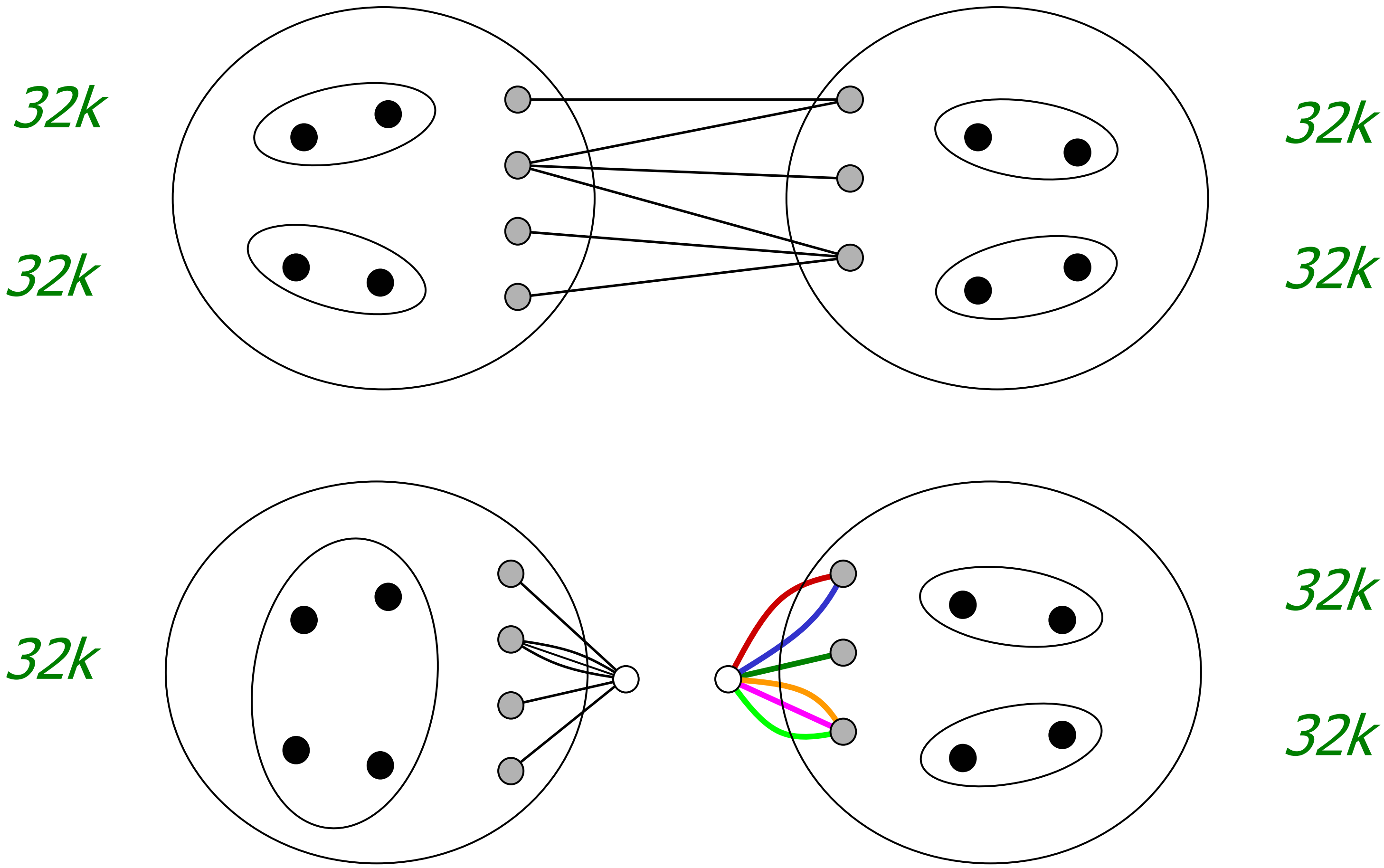

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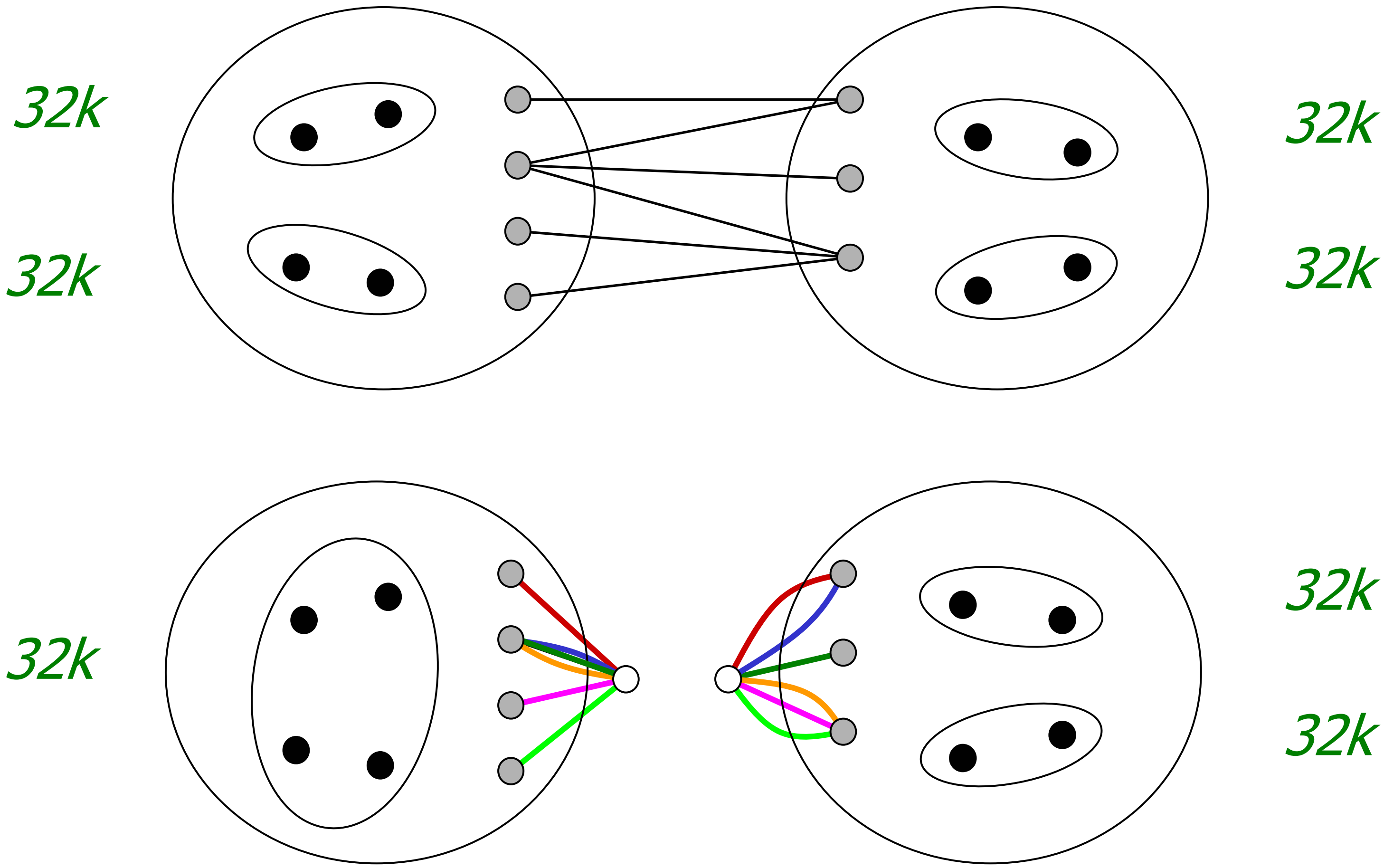

Proof Idea

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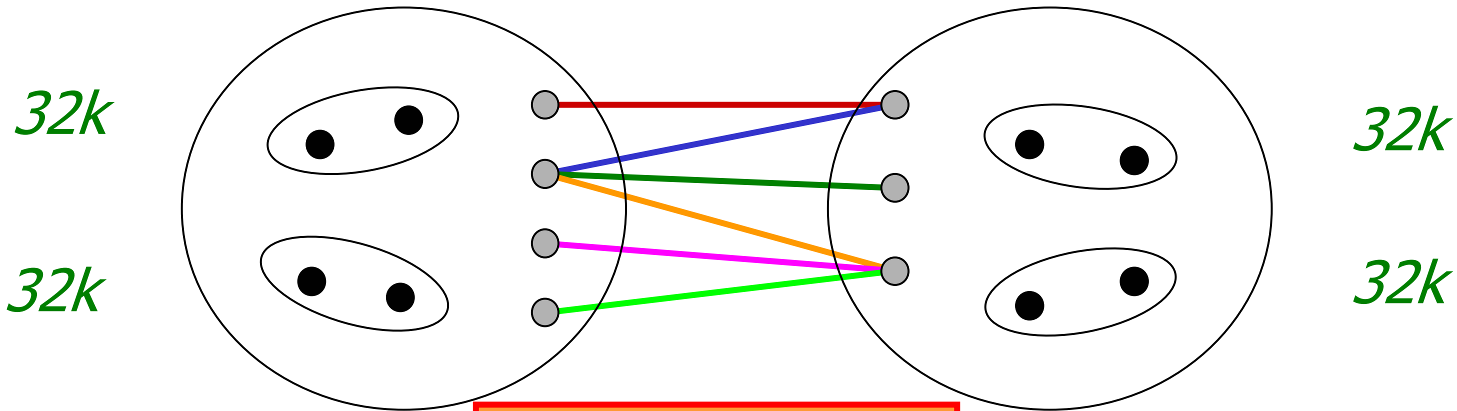
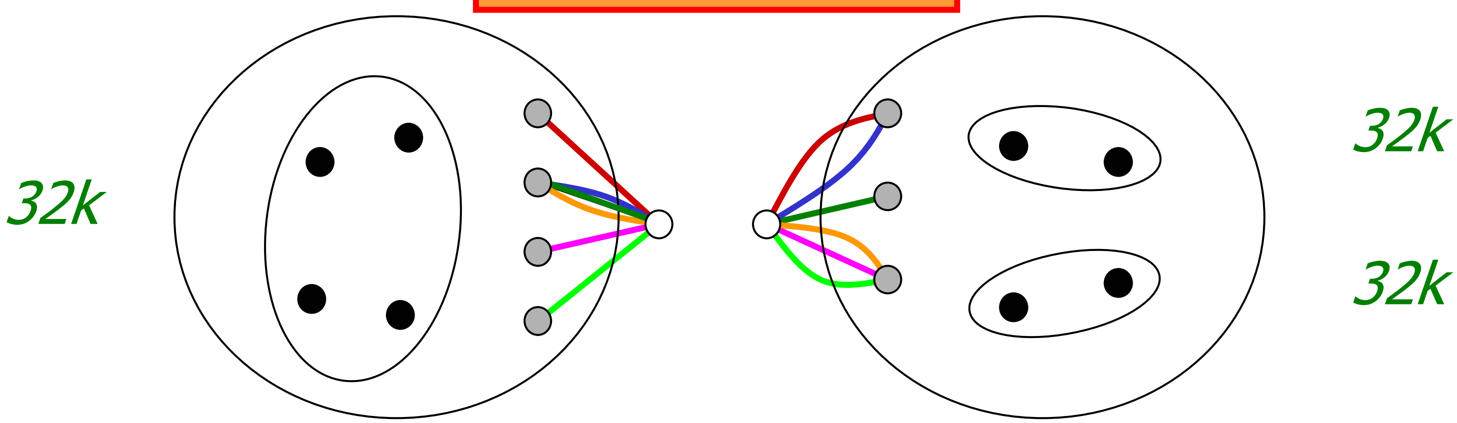
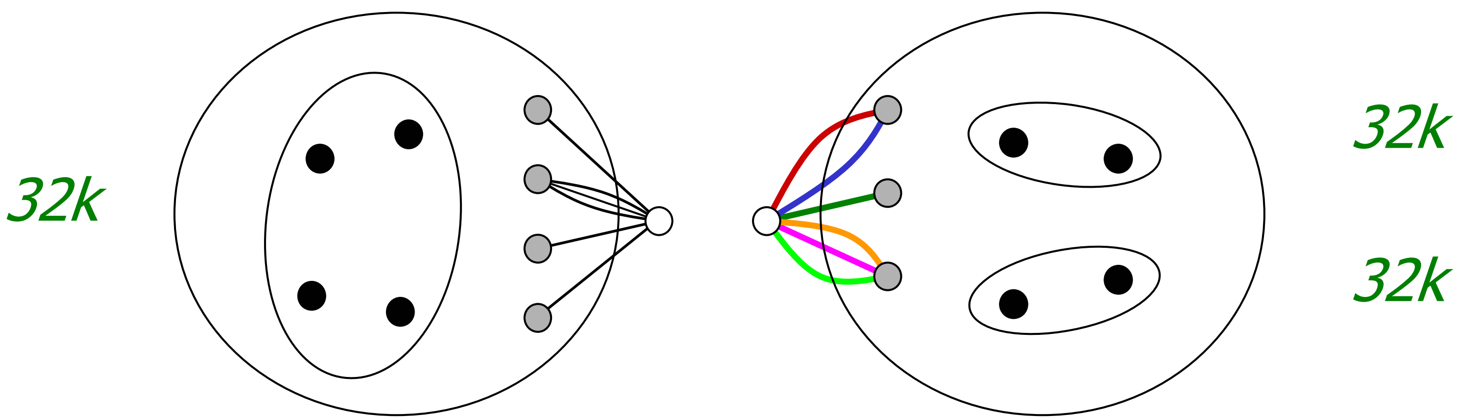


Done? Not yet…



A Subtlety



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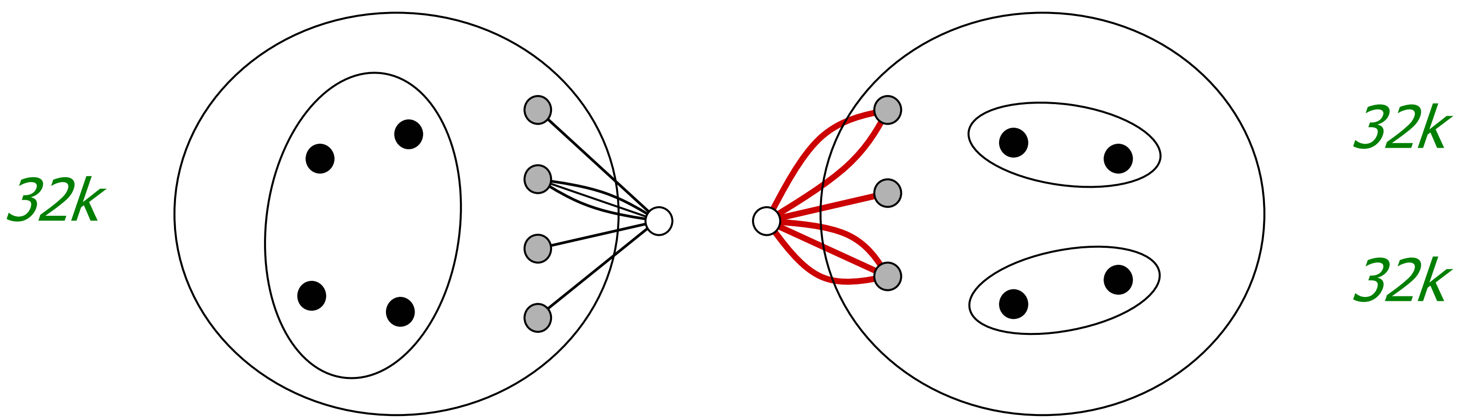
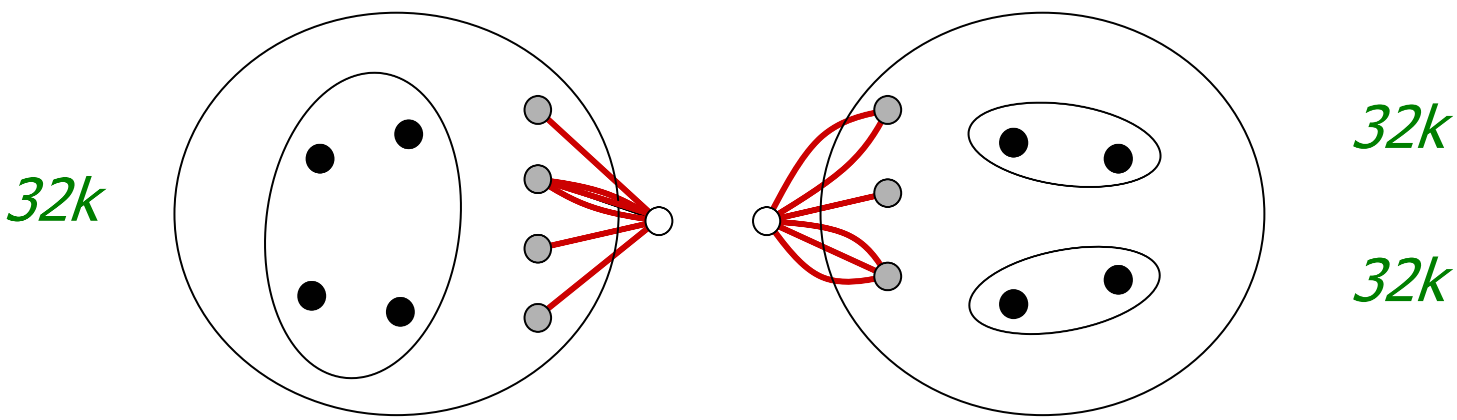


A Subtlety

May not be extendible,

the graph may have a “small” cut

which needs “many” colours

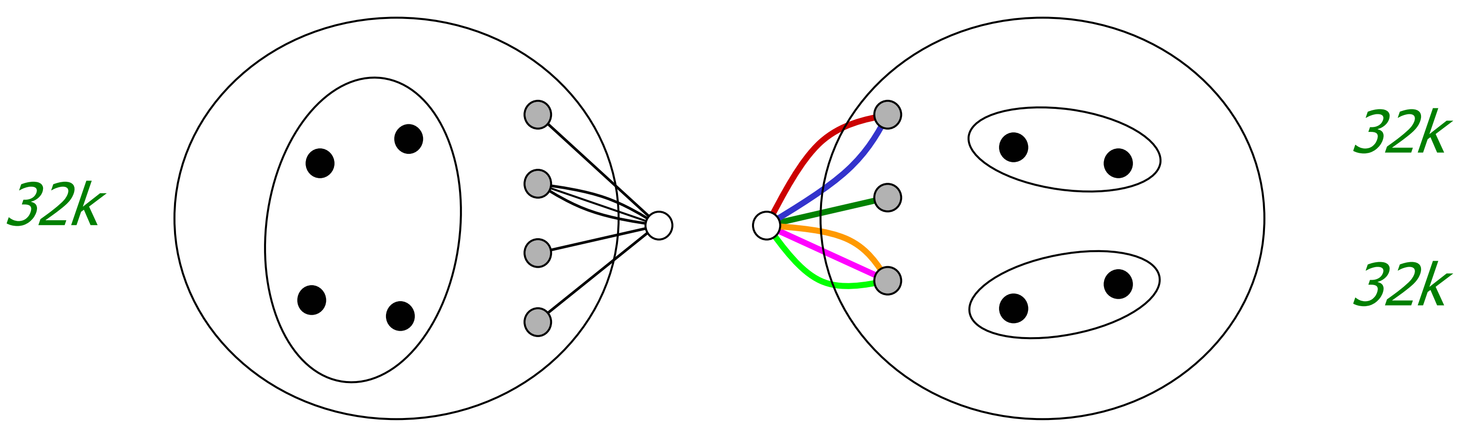


A Subtlety

Low degree: doesn't matter

High degree: additional "colourful" requirement

32k suffices...



Open Problem

An undirected multigraph G,

a connectivity requirement $r(u,v)$ for every pair u,v of $V(G)$.

Steiner Network:

A subgraph of G that there are $r(u,v)$

edge-disjoint u,v-paths for every pair u,v of $V(G)$.

Packing Steiner Networks

A largest collection of edge-disjoint Steiner networks of G.

Conjecture

Conjecture:

There exists a constant C such that

if there are $Ck\ r(u,v)$ edge-disjoint u,v -paths

then there are k edge-disjoint Steiner networks.

Known: If $r(u,v) \in \{0,1\}$, then $C = 32$ suffices.

Conjecture:

$C = 2$ for Eulerian graphs.

Known: If $r(u,v) \in \{0,1\}$, then it is true.