Additive and Abelian categories

Pre-additive structure, k-linear structure

$$\begin{array}{ccc} & & \exists\, k\text{-mod} \\ & \nearrow & \downarrow \\ \mathrm{Hom}(??): C \times C^{op} & \longrightarrow & \mathrm{Sets} \end{array}$$

$$(X, Y) \longmapsto \mathrm{Hom}_C(X, Y)$$

① Each $\mathrm{Hom}_C(X,Y)$ comes equipped with a k-module structure i.e. commutative addition and scalar multiplication from k

② The pairing $\mathrm{Hom}_C(Y,Z) \times \mathrm{Hom}_C(X,Y) \longrightarrow \mathrm{Hom}_C(X,Z)$ is k-linear

But k-linear structure may not be unique

Lemma A a pre-additive category. For an object $Z \in A$ TFAE

① Z is initial

② Z is terminal

③ Z is a zero object, Moreover $\mathrm{End}_A(Z) = \{0\}$

pf If Z is initial, $\exists!$ morphism $Z \to$ anywhere, "a factor" $Z \to Z$, $0 = id_Z$.

(1) $\Rightarrow$ (3) Given any object $X \in A$, $\exists!\ Z \to X$

$\to \mathrm{Hom}_A(Z,X) = \{0\}$; $\mathrm{Hom}_A(X,X)$ is an abelian group, so $\exists\, 0_X \in \mathrm{Hom}_A(X,X)$

Claim: 0_X factors uniquely through Z

$\mathrm{Hom}_A(X,Z)$ is an abelian group, thus contains some element

If $f \in \mathrm{Hom}_A(Y,Z)$, $0 = id_Z \circ f = f \Rightarrow \mathrm{Hom}_A(X,Z) = \{0\}$

Thus Z is terminal as well!

Rmk. $\mathbb{B}\mathbb{Z} = \{0^2\}$ is pre-additive, but has no zero object

(3) $\Rightarrow$ (1), (2) Obvious $\square$

Direct sums: Given $X, Y \in A$ (pre-additive), a direct sum Z of X and Y consists of $X \underset{p_1}{\overset{i_1}{\rightleftarrows}} Z \underset{p_2}{\overset{i_2}{\rightleftarrows}} Y$ such that

① $p_1 i_1 = id_X$, $p_2 i_2 = id_Y$

② $i_1 p_1 + i_2 p_2 = id_Z$

Prop A preadditive category, X, Y objects. Then TFAE

1. X, Y admit a direct sum
2. X, Y admit a product
3. X, Y admit a coproduct

Moreover, under either condition, the natural morphism $X \amalg Y \to X \times Y$ is an isomorphism and either serves as a direct sum.

Pf Assume Z is a direct sum of X and Y.

Consequence of the axioms:

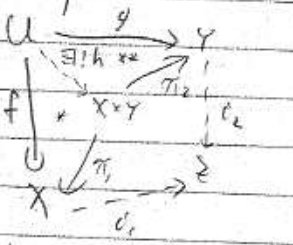
$p_2 i_1 = 0 : X \to Y$, $p_1 i_2 = 0 : Y \to X$

$p_1 i_2 = p_1 \, id_Z \, i_2 \underset{(1)}{=} p_1 (i_1 p_1 + i_2 p_2) i_2 = p_1 i_1 p_1 i_2 + p_1 i_2 p_2 i_2 \underset{(1)}{=} p_1 i_1 + p_2 i_2$

$\therefore p_1 i_1 = 0$

$p_2 i_1 = 0$ proved similarly

Now, we prove that $Z \cong X \times Y$:



Claim: $X \xleftarrow{p_1} Z \xrightarrow{p_2} Y$ is a product

Given $f : U \to X$, $g : U \to Y$, set

$h = i_1 f + i_2 g$

$p_1 h = p_1 i_1 f + p_1 i_2 g = f$, similarly $p_2 h = g$

Thus (*) and (**) commute

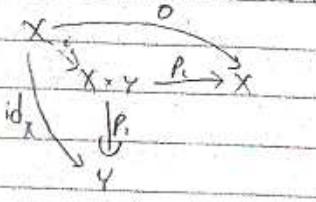
Assume $h' : U \to Z$ satisfies $p_2 h' = g$, $p_1 h' = f$

Want: $h = h'$

$h' = id_Z h' \underset{(1)}{=} i_1 p_1 h' + i_2 p_2 h' = i_1 f + i_2 g = h$ ✓

By duality, $Z \cong X \amalg Y$, proving ②, ①, and Moreover, ...

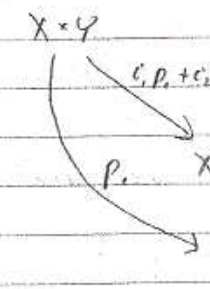
Assume $X \times Y$ exists. Then $X \underset{i_1}{\overset{p_1}{\rightleftarrows}} X \times Y \underset{i_2}{\overset{p_2}{\rightleftarrows}} Y$



Define i_1 as the unique morphism $X \to X \times Y$ such that $p_1 i_1 = id_X$, $p_2 i_1 = 0$

i_2 defined similarly.

(cont.) To check (3):



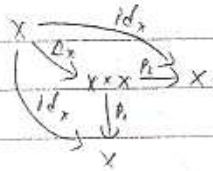
$$p_1(i_1p_1 + i_2p_2) = \underbrace{p_1 i_1}_{id_X} p_1 + \underbrace{p_1 i_2}_{0} p_2$$

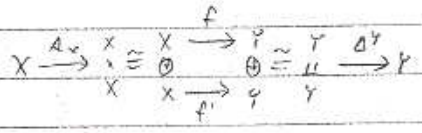
$$= p_1$$

Claim follows by symmetry. □

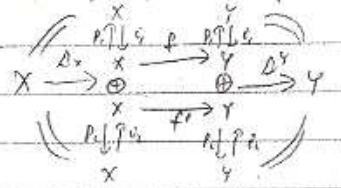
Prop. For morphisms $f, f' : X \to Y$, one has

$$f + f' = \nabla^Y \circ (f \oplus f') \circ \Delta_X$$





Pf. $\nabla^Y \circ (f \oplus f') \circ \Delta_X = \nabla^Y \circ (f \oplus f') \circ (i_1p_1 + i_2p_2) \circ \Delta_X$



$$= \nabla^Y \circ (i_1 f + i_2 f') = f + f' \quad \square$$

Cor. If A is a preadditive category in which any pair of objects admits a product (equivalently, a coproduct), then the pre-additive structure is unique.

Assume A, B preadditive with direct sums (of two objects) and $F : A \to B$ is a functor. Then TFAE (F is an additive functor):

1. For any objects $X, Y \in A$, $F : \mathrm{Hom}_A(X, Y) \to \mathrm{Hom}_B(FX, FY)$ is a group homomorphism
2. F preserves products of two objects
3. F preserves coproducts "
4. F preserves direct sums "

Pf. Simply apply F to $f+f' = \Delta^{\vee} \circ (f \oplus f') \circ \Delta_X$ $\square$

Gabriel's realization of small pre-additive categories:

If A is a small pre-additive category, then the "Gabriel Rng" is

$$R(A) := \bigoplus_{X,Y \in A} \mathrm{Hom}_A(X,Y), \quad \text{component-wise addition}$$

$$f * g = \begin{cases} f \circ g & \text{if } \mathrm{domain}(f) = \mathrm{codomain}(g) \\ 0 & \text{otherwise} \end{cases}$$

Some properties:

- $R(A)$ has a multiplicative unit only if $\mathrm{Ob}\, A$ is finite
- There is a family of orthogonal idempotents $\{e_X := id_X\}_{X \in A}$ such that $\forall a \in R(A)$ the sets

  $\text{left support}(a) := \{e_X \mid e_X a \neq 0\}$

  $\text{right support}(a) := \{e_Y \mid a e_Y \neq 0\}$

  are finite

  Pf. $a \in R(A) \Rightarrow a = \sum (f_i : X_i \to Y_i)$

  $a e_X = \sum_{f_i : X \to Y_0} f_i \qquad e_Y a = \sum_{f_i : X_i \to Y} f_i$ $\square$

- $R(A)$ does not determine the category in general

  E.g. $R = \mathbb{Z} \times \mathbb{Z} = R(1B(\mathbb{Z} \times \mathbb{Z}))$ and

  $R = R(\circlearrowright \mathbb{Z} \quad \circlearrowright \mathbb{Z})$

$(S, \leq)$ partially ordered set

$C = 1B(S, \leq)$: $|\mathrm{Hom}_C(X,Y)| \leq 1$

To make this pre-additive: replace $\mathrm{Hom}_C(X,Y)$ with $\mathbb{Z}^{\mathrm{Hom}_C(X,Y)}$, the free abelian group generated by the set $\mathrm{Hom}_C(X,Y)$

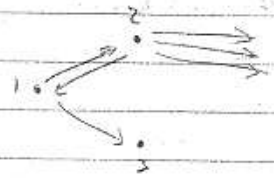
If $(S, \leq)$ is finite, then can refine to a linear ordering $S \leq S'$ such that it is linear and extends $\leq$. Picture of corresponding Gabriel rng:

$S_1 \leq S_2 \leq \cdots \leq S_n$ $\quad *$ if $\mathrm{Hom}(S_i, S_j) \neq 0$

$$\begin{array}{c} S_1 \\ \leq S_2 \\ \leq \\ \vdots \\ S_n \end{array} \begin{pmatrix} * & * & & & \\ * & 0 & * & * & \\ & & & & \\ 0 & & & 0 & \\ & & & & * \end{pmatrix} \quad 0 \text{ else}$$

Element is matrix w/ entries from k with specified 0's

Small category C, $\mathrm{Hom}(C, K\text{-mod})$ K a commutative ring



Functor sends vertex i to K-mod V_i

oriented edge $\alpha \longmapsto$ K-linear map $V(\alpha): V_{\text{tail}_\alpha} \to V_{\text{head}_\alpha}$

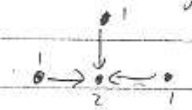
If morphisms in C satisfy relations, functor must preserve those relations (K-linear)

This is a quiver representation

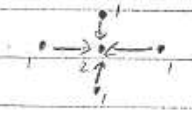
A morphism between two such functors is a morphism between quiver representations

- Given by family of linear maps $\varphi_i: V_i \to W_i$ making all relevant diagrammes commute

Exercise: Classify all isomorphism classes of quiver representations into v.s. of the following types:



dimension vector: $(\dim V_i)_{i \in \text{vertices}}$



Kronecker quiver: $\bullet \rightrightarrows \bullet$

Exercise: $A, B: V \to W$. Is there a simultaneous normal form for these matrices? Solved by Kronecker.

Representation of Kronecker quiver is the same as the theory of coherent sheaves on $\mathbb{P}^1$.

Suppose $C \underset{L}{\overset{R}{\rightleftarrows}} D$. R, L called adjoint functors (R right adjoint, L left) if $\forall X \in C, Y \in D$,

$$\mathrm{Hom}_D(Y, RX) \cong \mathrm{Hom}_C(LY, X)$$

as functors on $D^{op} \times C$ with values in Sets.

Eg. • $R = \text{forget} : \underline{Ab} \longrightarrow \underline{Sets}$

Set S $(=Y)$, abelian gp G $(=X)$

Want $L : \underline{Sets} \longrightarrow \underline{Ab}$ such that $\mathrm{Hom}_{\underline{Sets}}(S, RG) = \mathrm{Hom}_{\underline{Ab}}(LS, G)$

$L = \mathbb{Z}[S]$ satisfies the property

• $R = \text{forget} : \underline{Rings} \longrightarrow \underline{Sets}$

$L :=$ tensor algebra over $\mathbb{Z}$ on S is left adjoint

• k-algebras $\longrightarrow$ k-Lie algebras

$k \longmapsto Z(A) \longmapsto (A; [a,b] := ab - ba)$

$L :=$ universal enveloping algebra

$\mathrm{Hom}_{k\text{-Lie alg}}(\mathfrak{g}, (A, [\cdot,\cdot])) \cong \mathrm{Hom}_{k\text{-Alg}}(U(\mathfrak{g}), A)$

• forget : $\underline{Ab} \longrightarrow \underline{Gps}$

L is abelianization

Assume $f : A \to B$ is a ring homomorphism.

$f_* : B\text{-mod} \longrightarrow A\text{-mod}$

${}_B N \longmapsto {}_A N$, i.e. $a \cdot n := f(a)\, n$.

Sometimes called restriction of scalars along f

There exists a left adjoint f^* and a right adjoint $f^!$

$f^* M := B \otimes_A M$, M A-module

$\mathrm{Hom}_A(M, f_* N) \cong \mathrm{Hom}_B(B \otimes_A M, N)$ N B-module

$\varphi \longmapsto \tilde{\varphi}(b \otimes_A m) := b\, \varphi(m)$

inverse $\Psi : B \otimes_A M \to N$

$\tilde{\Psi} : M \to N$, $\tilde{\Psi}(m) := \Psi(1 \otimes_A m)$

$f^! M' := \mathrm{Hom}_A(B, M')$, M' A-module

$\mathrm{Hom}_A(f_* N, M') \cong \mathrm{Hom}_B(N, \mathrm{Hom}(B, M'))$ N B-module

$\varphi : N \to M' \longmapsto \tilde{\varphi} : N \to \mathrm{Hom}(B, M')$

$\tilde{\varphi}(n) = (b \mapsto b\, \varphi(n))$

$\Psi : N \to \mathrm{Hom}(B, M') \longmapsto \tilde{\Psi} : N \to M'$

$\tilde{\Psi}(n) = [\Psi(n)](1)$

Exercise: $A = \mathbb{Z}$, G finite group, $B = \mathbb{Z}G$, $f: A \to B$

Show each of $f_*, f^*, f^!$ again has both left and right adjoints.

What is a property of f that makes this possible?

Think of at least five more examples of left and right adjoint functors