Modern homological algebra:
- Connected to associative algebra via tilting theory, derived Morita equivalence
- Quantum groups (Drinfel'd)
- Riemann-Hilbert correspondence - "perverse sheaves"
  - → crystal bases, quiver varieties, representation theory of symmetric group
  - → Lie algebras
- Cluster algebras
- Ringel-Hall algebras

Big themes leading development:
- non-commutative desingularization (McKay correspondence)
- homological mirror conjecture
- understanding $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$
- → Motives - ideally, the essence of algebraic geometry

Category theory: We are not interested in the objects themselves, but in the relations between them (i.e. morphisms)

Def. A <u>Category</u> C consists of
(1) A collection $\mathrm{Ob}\,C$, called the <u>objects</u> of C
(2) A collection of disjoint sets $C(X,Y)$ for each ordered pair X, Y of objects in C, called the <u>morphisms</u>
(3) Maps $\circ$: $\begin{cases} C(Y,X) \times C(Z,Y) \longrightarrow C(Z,X) \\ (f,g) \longmapsto f \circ g \end{cases}$

for each ordered triple (X,Y,Z) in $\mathrm{Ob}\,C$.

These data are subject to the following:
(i) (Identity morphisms) For each object X in C there is a morphism $\mathrm{id}_X \in C(X,X)$, the identity on X such that for every morphism $f: X \to Y$: $f \circ \mathrm{id}_X = f = \mathrm{id}_Y \circ f$
(ii) (Associativity of composition): $h: X \to Y$, $g: Y \to Z$, $f: Z \to W$
$(f \circ g) \circ h = f \circ (g \circ h)$

Shortcuts:

$X \in C$ instead of $X \in \mathrm{Ob}\, C$

$(f: X \to Y) \in C$ instead of $f \in C(X,Y)$

NB. • Axiom 2: "disjoint morphism sets" $\Rightarrow$ each morphism has a unique source/domain and target/codomain

• Identities are unique (exercise)

• Write $\mathrm{Hom}_C(X,Y) := C(X,Y)$, $\mathrm{End}_C(X) := C(X,X)$

$\mathrm{End}_C(X)$ is an associative monoid

Def. A category C is <u>small</u> if $\mathrm{Ob}\, C$ is a set.

Eg. • Directed graph $\to$ path category: vertices are objects, paths are morphisms

• Posets = pre-ordered sets

Lemma. Posets $\overset{1:1}{\longleftrightarrow}$ small categories where each Hom set has at most 1 element

E.g. • X a topological space, $\mathrm{Open}\, X$ is the category associated to the poset $U \subseteq V \subseteq X$, U, V open

• Categories with one object are monoids

• G-space X: G a group, X a set

$\mathrm{Ob}(\mathbb{B}_X G) = X$, $\mathrm{Hom}_{\mathbb{B}_X G}(x,y) := \{g \in G \mid gx = y\} \subseteq G$

Def. A morphism $f: X \to Y$ in C is an <u>isomorphism</u> if $\exists g: Y \to X$ such that $g \circ f = id_X$ and $f \circ g = id_Y$.

In $\mathbb{B}_X G$, every morphism is an isomorphism. Such categories are called <u>groupoids</u>

Def. The <u>opposite category</u> C^{op} of a category C is defined by

$\mathrm{Ob}\, C^{op} = \mathrm{Ob}\, C$ $\quad C^{op}(X,Y) := C(Y,X)$

Write X^{op} for X considered as in $\mathrm{Ob}\, C^{op}$ and f^{op} for f in $C^{op}(X,Y)$. $\quad (g^{op} \circ f^{op}) := (f \circ g)^{op}$

Def. A morphism $f: X \to Y$ in C is an <u>epimorphism</u> if for any morphisms $h, g: Y \to Z$, $h \circ f = g \circ f \Rightarrow h = g$
("f is right cancellable")
Equivalently, for each object Z, the map of sets
$\mathrm{Hom}_C(Y, Z) \xrightarrow{\circ f} \mathrm{Hom}_C(X, Z)$ is injective
A morphism $f: X \to Y$ in C is a <u>monomorphism</u> if f^{op} is an epimorphism.
Equivalently, f is left cancellable, i.e.
$f \circ h = f \circ g \Rightarrow h = g$ for all $h, g: U \to X$
or for each object Z, the map of sets
$\mathrm{Hom}_C(Z, X) \xrightarrow{f \circ -} \mathrm{Hom}_C(Z, Y)$ is injective

NB In a general category a map that is a monomorphism and an epimorphism need not be an isomorphism
Consider: • $\mathbb{B}(\mathbb{Z}, \cdot)$
• $H \hookrightarrow G$ inclusion of $\mathbb{Z}/2$ in $G = \mathfrak{S}_3$
• $\mathbb{Z} \hookrightarrow \mathbb{Q}$

Lemma. $\varphi: H \to G$ a group homomorphism, then φ is an epimorphism iff $N_G \varphi(H) = G$.
Cor. Any surjective group homomorphism is an epimorphism.

Exercise. Monomorphisms in groups are injective group homomorphisms.
Exercise. ① In category of commutative rings, monomorphisms and injective homomorphisms are the same
② A ring homomorphism $\varphi: S \to R$ is an epimorphism iff $R \otimes_S R \to R$ via $r_1 \otimes r_2 \mapsto r_1 r_2$ is a bijection.

Q. In which categories are isos the same as epi + mono? "balanced"
This means that any map can be factored uniquely into epi + mono
This would allow definition of the image of a map

Small categories as algebraic theory

Def. A magma is a set S together with a binary operation
$-\circ-: S\times S \longrightarrow S$
- $(S,\circ)$ is an associative magma if $-\circ-$ is associative

Def. An element $e\in S$ is
(a) idempotent if $e\circ e=e$
(b) a zero element if $e\circ x=e=x\circ e \ \forall x\in S$
(c) a unit element if $e\circ x=x=x\circ e \ \forall x\in S$

N.B. Zero, unit elements are unique if they exist and are idempotent

Def. Let $(M,\cdot,\circ)$ be a magma with 0. A subset $E\subseteq M$ is a distinguishing set of idempotents if
(i) Each $e\in E$ is a nonzero idempotent
(ii) If $e_1, e_2\in E$ and $e_1\circ e_2\neq 0$, then $e_1=e_2$
(iii) For each $m\in M$, there exist $e_1, e_2\in E$ such that $e_1\circ m=m=m\circ e_2$

Lemma/Def. $(M,\cdot,\circ)$ an associative magma with 0, such that E is a D.S.o.I.
(1) If $m\neq 0$, then $\exists!\ s(m)\ t(m)\in E \ni t(m)m=m=ms(m)$
s is the source and t is the target
(2) If $m, m'\in M$ with $mm'\neq 0$, then
$s(m)=t(m')$, $s(mm')=s(m')$, $t(mm')=t(m)$
(3) In particular, for $e\in E$, $s(e)=e=t(e)$

Prop. An associative magma with zero contains at most one distinguishing set of idempotents.

Prop. The following association C: a small category $\longmapsto M:=\coprod_{X,Y\in C} C(X,Y)\cup\{0\}$
where $mm'=\begin{cases} m\circ m' & \text{if } m: X\to Y,\ m': Y\to Z \\ 0 & \text{otherwise}\end{cases}$

is an associative magma with 0 and a D.S.o.I.

(cont) Conversely, if $(M, \cdot, 0)$ is an associative magma w/ 0 that admits a D.S.o.I. E, then $Ob(\mathbb{B}M) := E$, $e, e' \in E$;
$Hom_{\mathbb{B}M}(e, e') := \{m \in M \mid e'm = m = me\}$, $m \circ m' := mm'$ yields a category and $\mathbb{B}Mg(C) = C$, $Mg\,\mathbb{B}(M, \cdot, 0, E) = M$.

NB Gabriel's thm assumes $Hom_C(X, Y)$ forms an abelian group for any X, Y in C. Then $C \mapsto RC := \bigoplus_{X,Y} Hom_C(X, Y)$, a ring without unit.

Exercise. Given the connection of small categories with magmas, what is the proper notion of homomorphism of categories?

lecture 24.1 NB No class 7 February

Correction: In Gps, morphism is an epimorphism iff it is surjective.

NB. • If g, h are epi, then gh is epi
• If gh is epi, then h is epi
• If g has a "section", i.e. $\exists\, h : gh = id$, then g is an epi
In this case, g is a <u>split epimorphism</u>
E.g. In Sets, every epi is split
In k-v.s. every epi or mono is split

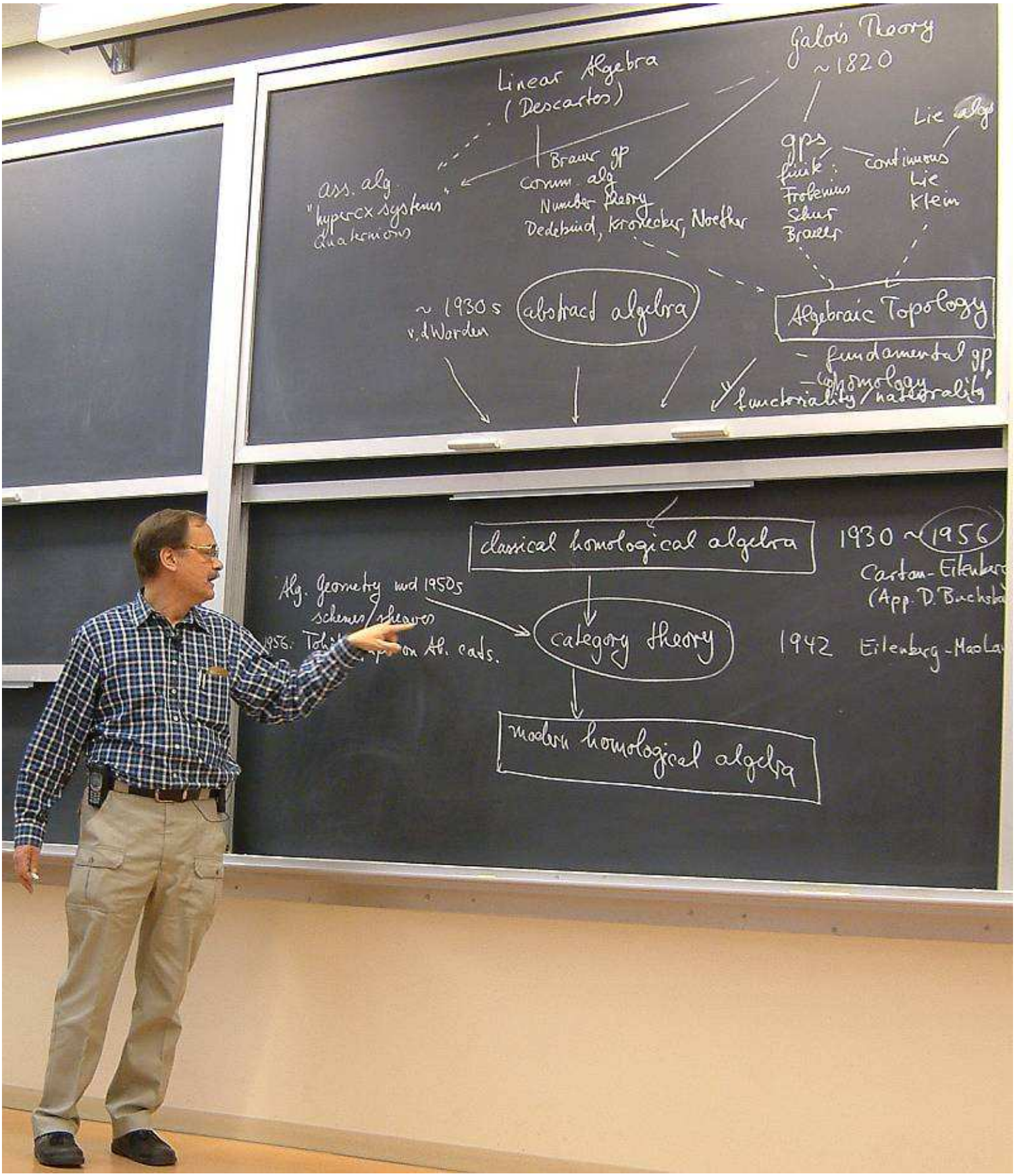
f is iso $\Leftrightarrow$ f is split mono $\Rightarrow$ f is mono
$\Leftrightarrow$ f is split epi $\Rightarrow$ f is epi

Def. $f : X \to Y$ is a <u>strong epimorphism</u> if in every diagram

$$\begin{tikzcd} & Y \arrow[dr, "\alpha"] & \\ X \arrow[ur, "f"] \arrow[dr, "b"'] & & V \\ & U \arrow[ur, "k"'] \arrow[uu, dashed, "\exists\alpha"] & \end{tikzcd}$$

with k a monomorphism $\exists\, \alpha : Y \to U$ such that $k\alpha = a$.

NB It follows that $\alpha f = b$ and α is necessarily unique.


Galois Theory
~1820
Linear Algebra
(Descartes)
Lie alg
gps
finite:
Frobenius
Schur
Brauer
continuous
Lie
Klein
Brauer gp
comm. alg
Number theory
Dedekind, Kronecker, Noether
ass. alg.
"hypercx systems"
quaternions
~1930s
v.d Warden
abstract algebra
Algebraic Topology
fundamental gp
cohomology
functoriality/naturality
classical homological algebra
1930 ~ 1956
Cartan-Eilenberg
(App. D. Buchsbaum)
Alg. Geometry mid 1950s
schemes/sheaves
category theory
1942 Eilenberg-MacLane
modern homological algebra

modern homological algebra

tilting theory

ass. alg

"derived Morita equiv." Rickard

Drinfel'd ICM '86

quantum gps

Riemann-Hilbert correspondence. "perverse sheaves" crystal bases quiver varieties repr. of symm gp

Lie algebras

cluster algebras
Ringel-Hall algebras
(M. Kapranov)

Knot theory
"categorification"

- moduli stacks
- desingularization

"non-commutative desingularization"
McKay correspondence
(Bridgeland-King-Reid JAMS 2002)
M. v.d. Bergh
D. Smith, Zhang, Artin
Rogalski ...

- String Theory
- Mirror Symmetry

Homological Mirror Conjecture

Motives

- Knot modules
(Faltings / Gabber)

$Gal(\bar{\mathbb{Q}}/\mathbb{Q})$

Langlands correspondence
alg / geom form