The Mathematics of Glider Racing

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Applied Mathematics
= Modeling + Mathematics

L'étude approfondie de la Nature est la source la plus féconde des decouvertes mathématiques. Non seulement cette étude, en offrant aux recherches un but déterminé, a l'avantage d'exclure les questions vagues et les calculs sans issue : elle est encore un moyen assuré de former l'Analyse elle-même, et d'en découvrir les éléments qu'il nous importe le plus de connaître, et que cette science doit toujours conserver : ces éléments fondamentaux sont ceux qui se reproduisent dans tous les effets naturels.

J. Fourier,1807

Understand particular phenomenon
and extract general mathematical principles

This example

I. Modeling

Theme: Decision-making under uncertainty
Physical example: Glider racing
Modeling challenge: Certain future *vs.* uncertain

II. Mathematics

Math difficulty: Non-smooth solutions
Solution technique: viscosity solutions
Common features with other problems

Decision making under uncertainty

- Finance (investment, hedging, …)
  market motions are uncertain
  "efficient market hypothesis":
  completely unpredictable
- Sport (sailboat and glider racing)
  atmospheric conditions uncertain
  partial information


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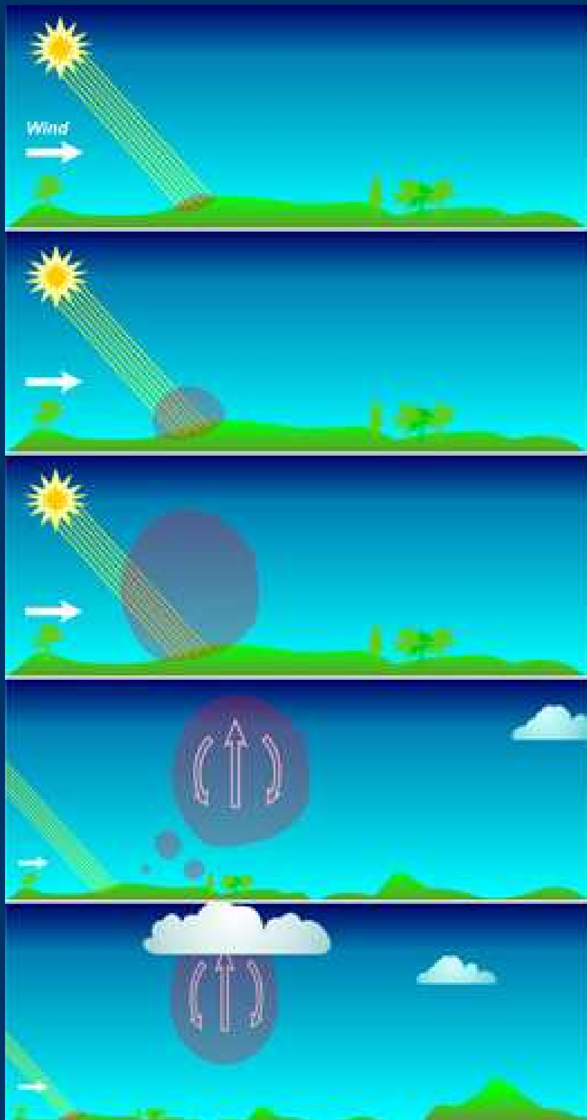
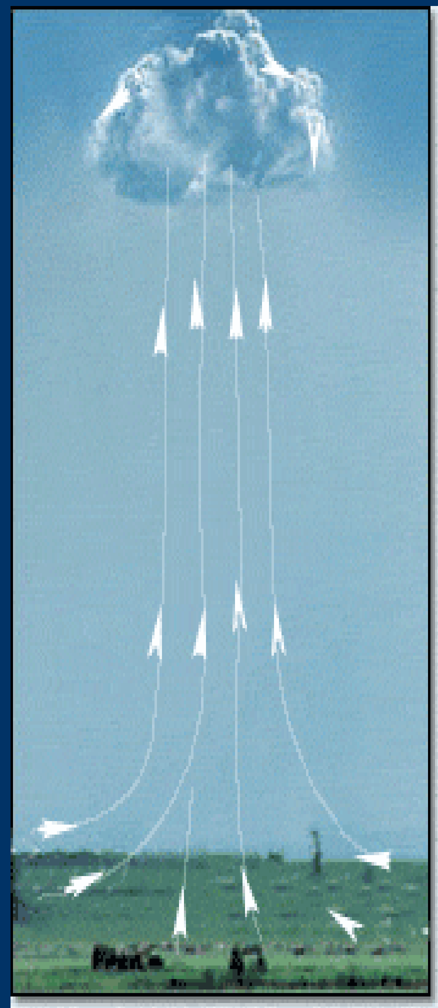
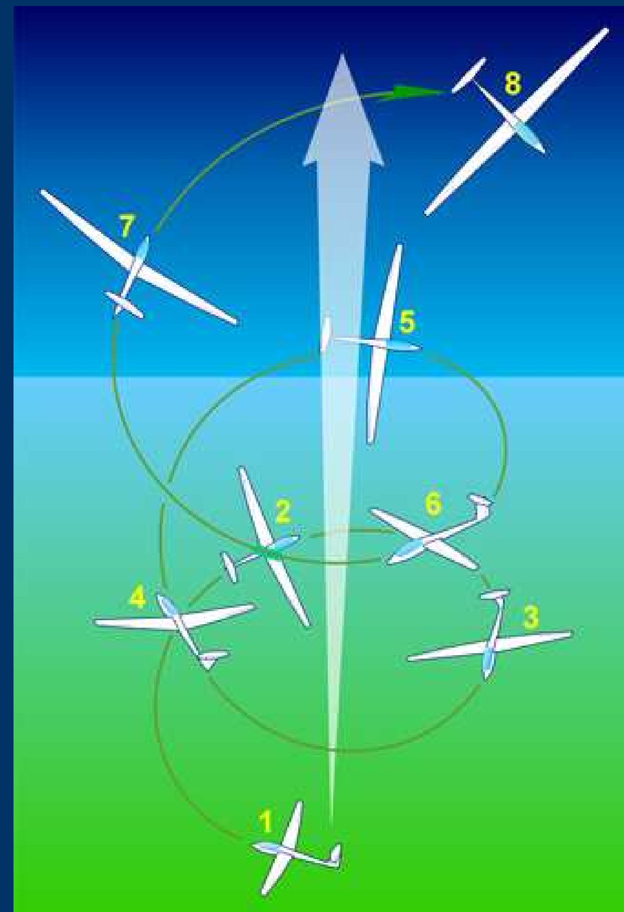
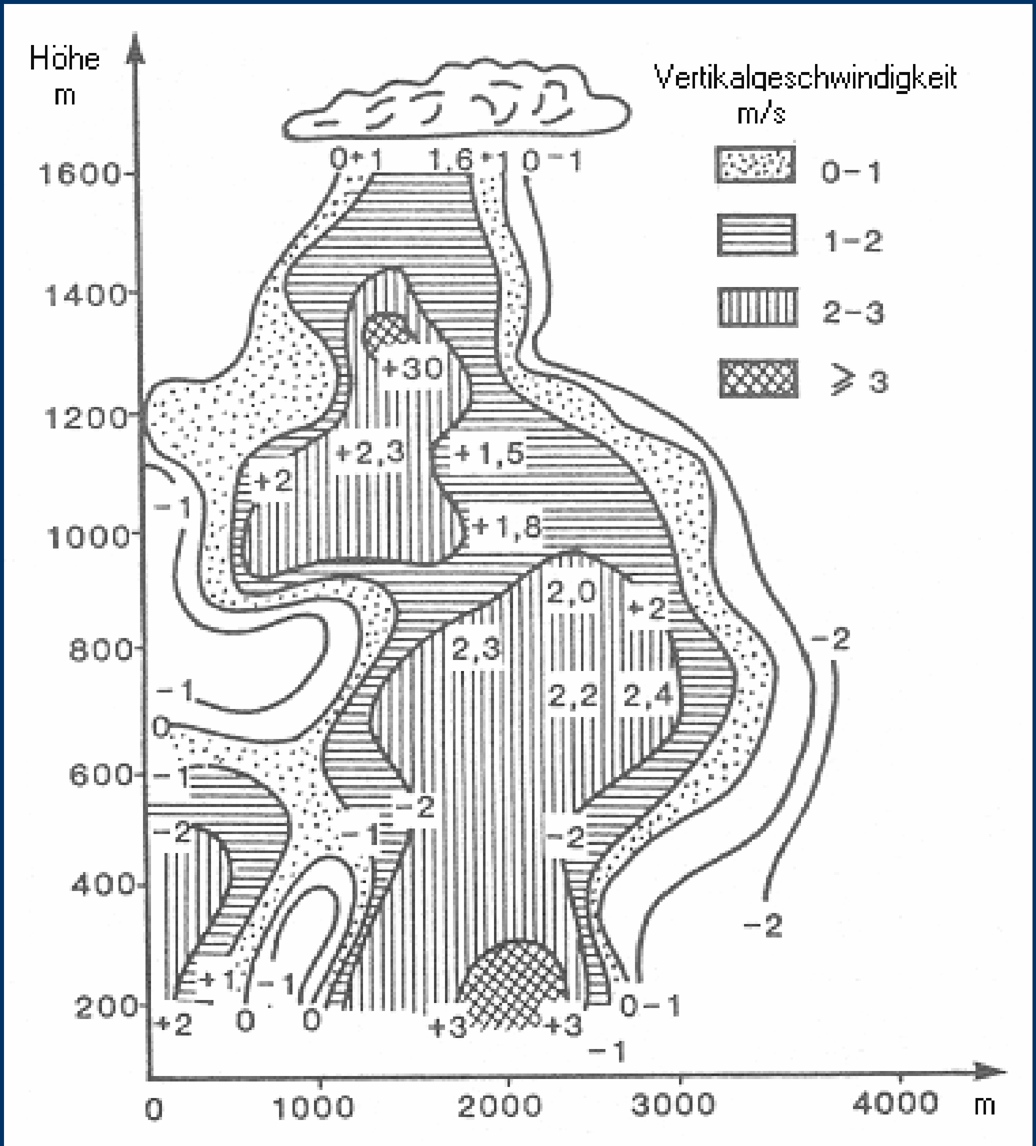
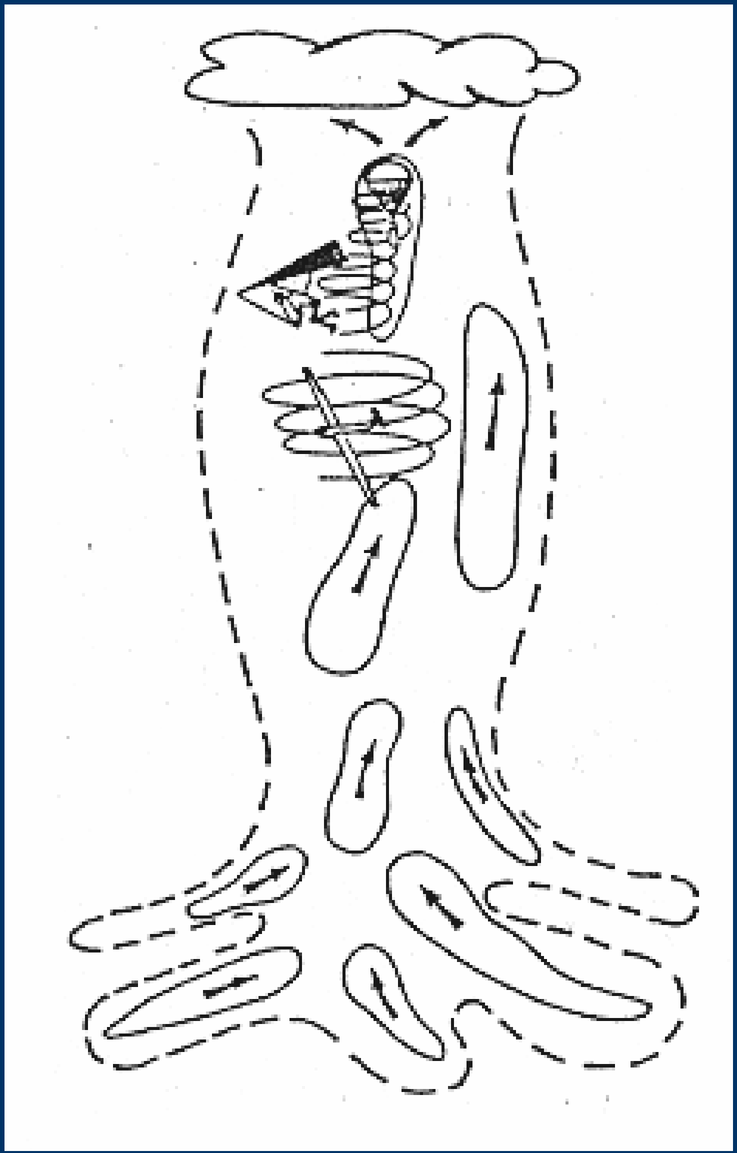
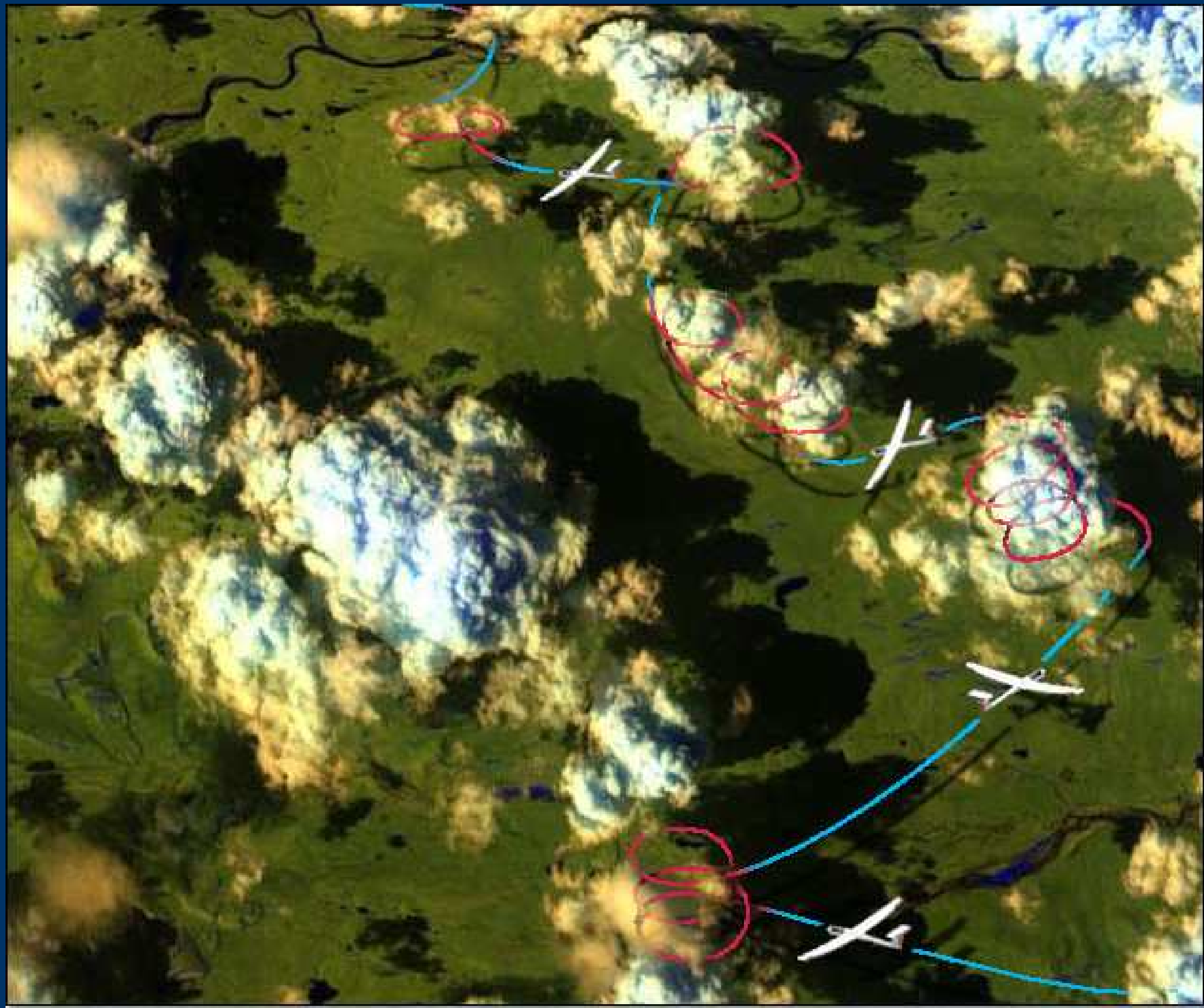

Wind

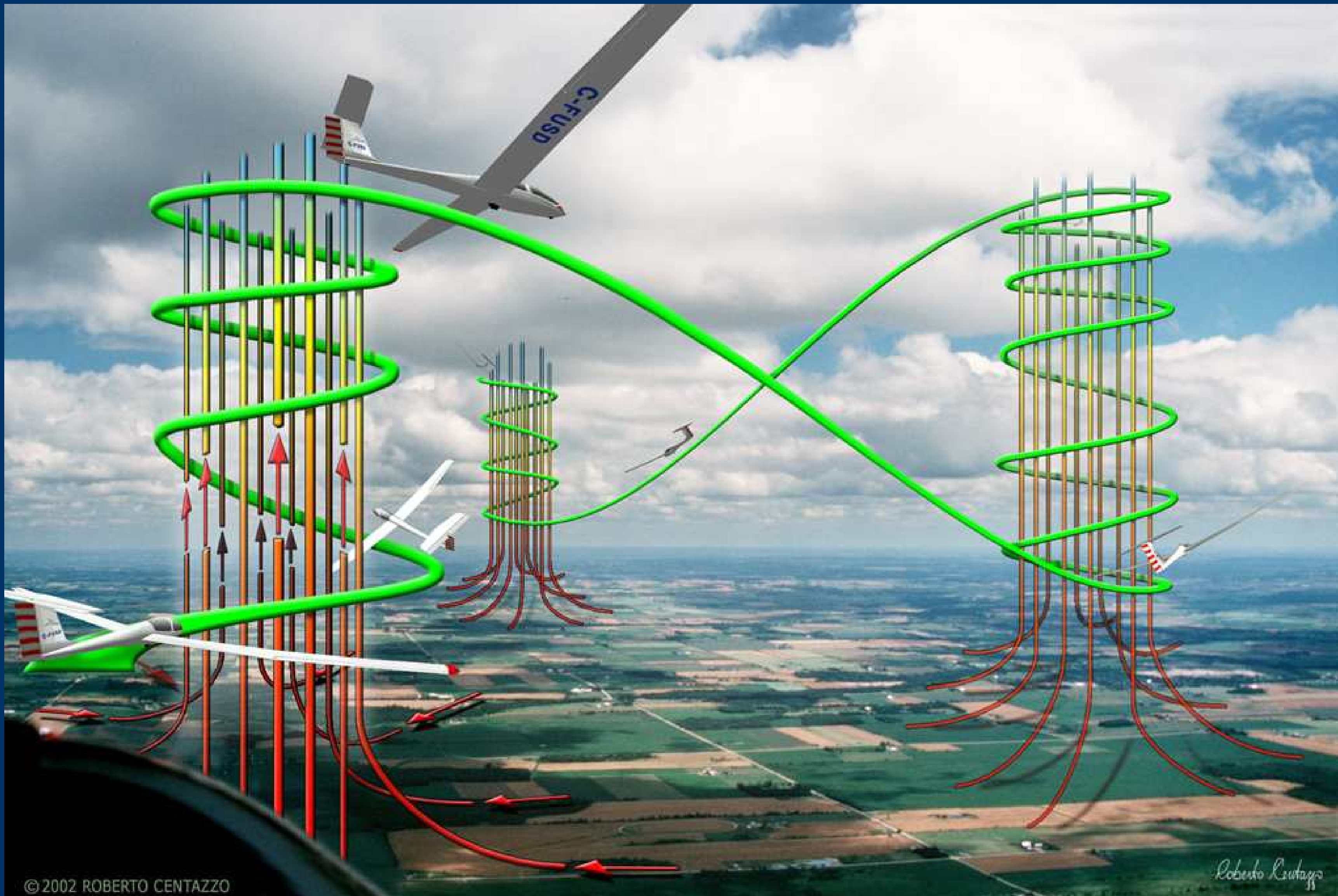



8
7
5
2
6
4
3
1


Höhe
m
Vertikalgeschwindigkeit
m/s
0−1
1−2
2−3
⩾ 3
1600
1400
1200
1000
800
600
400
200
0
1000
2000
3000
4000
m



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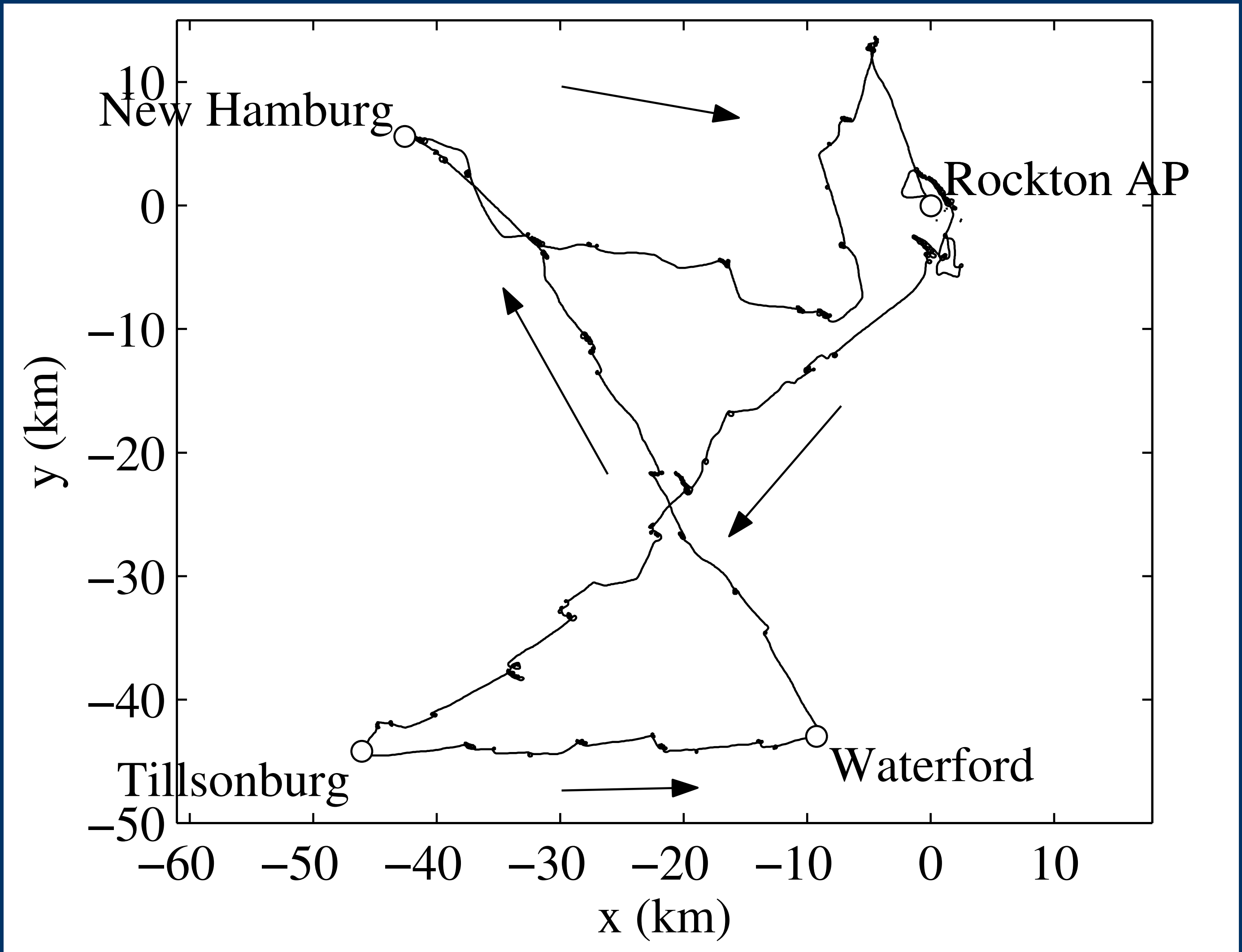
Roberto Centazzo

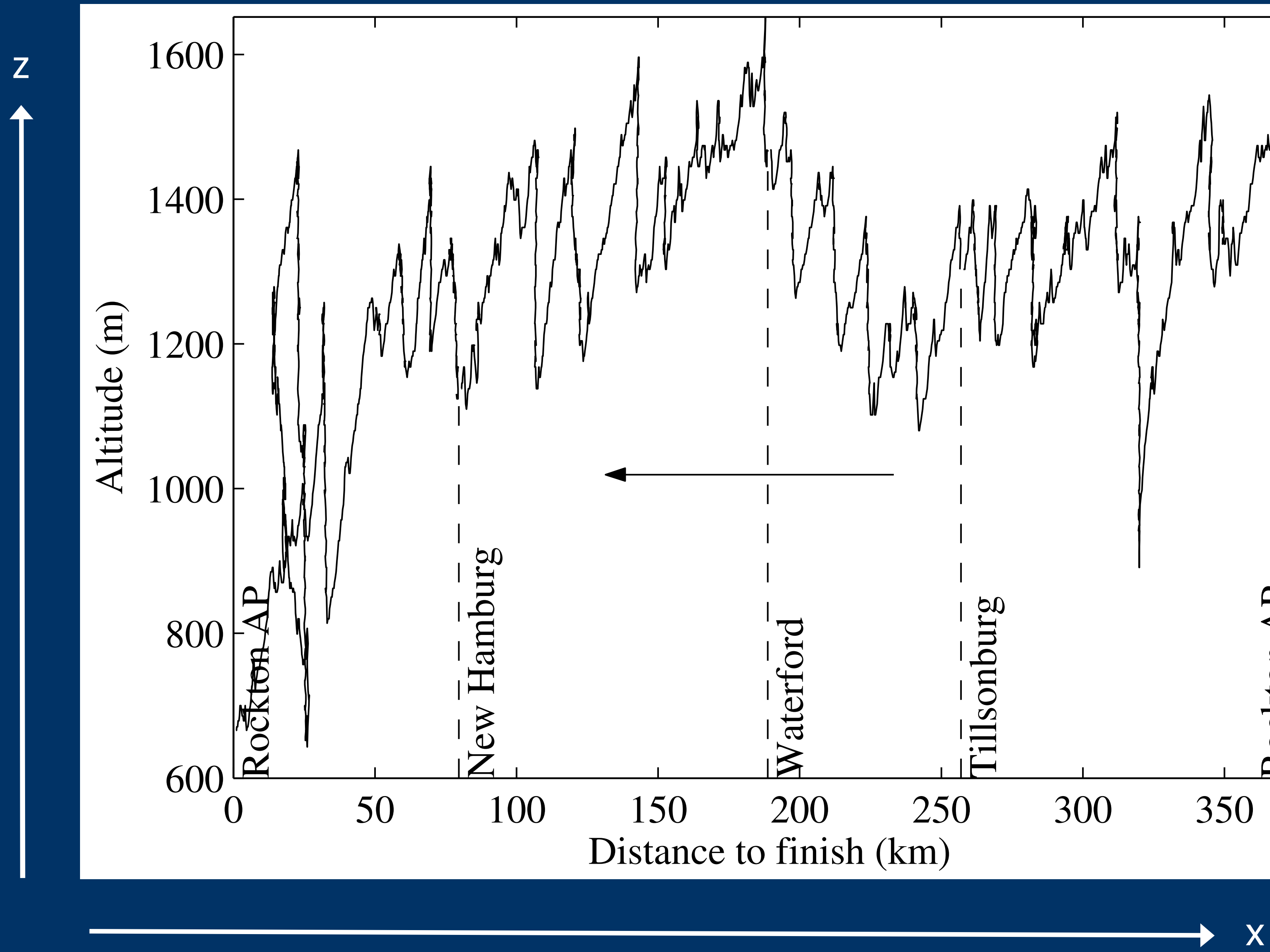
If you don't find a thermal …


computers

Soaring competition



2001 Canadian Nationals
Soaring Association of Southern Ontario – Rockton ON

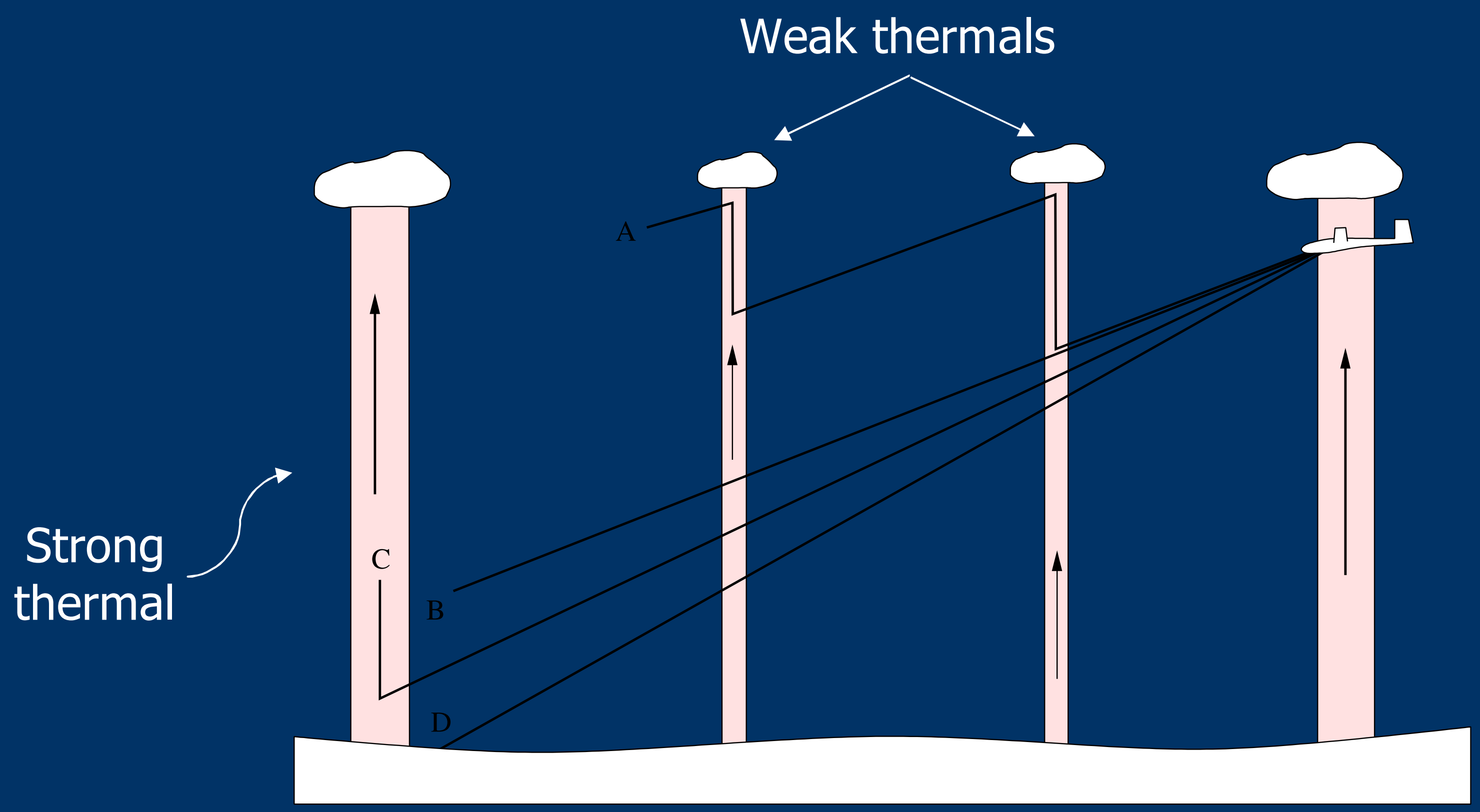

z
1600
1400
1200
1000
800
600
Altitude (m)
0
50
100
150
200
250
300
350
Distance to finish (km)
Rockton AP
New Hamburg
Waterford
Tillsonburg
Rockton AP
x

Need for math modeling

- Distances are far
- Speed is important even if not racing
- On-board computation is available
- History of quantitative modeling

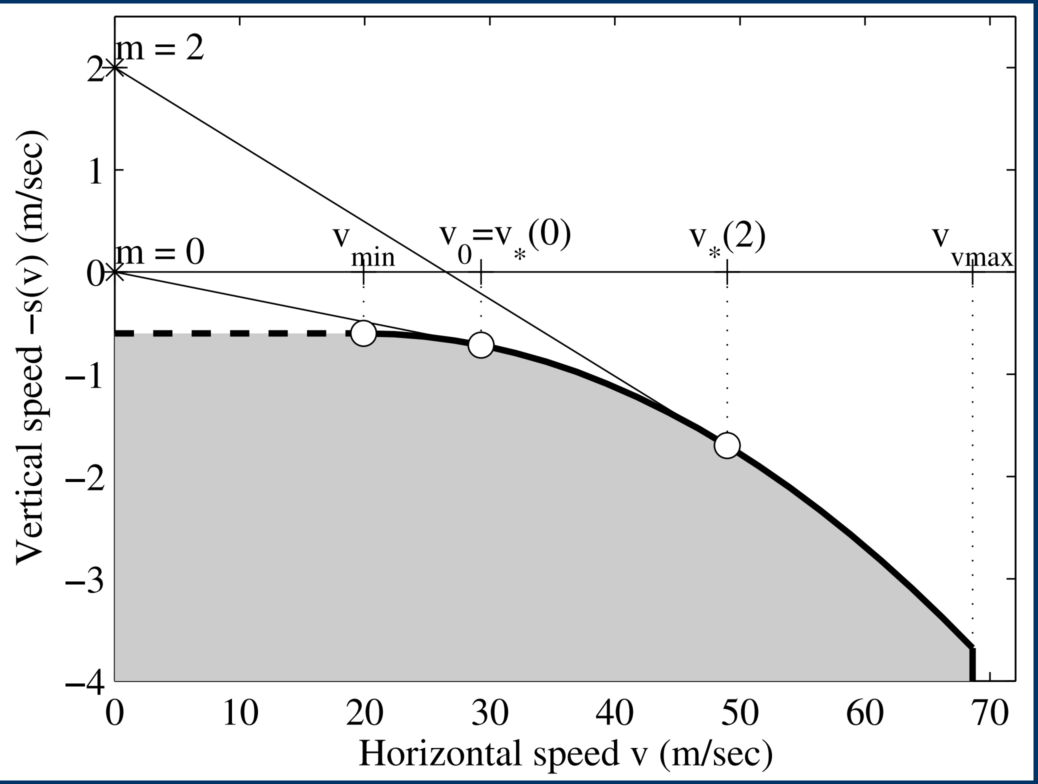
Optimization with known lift

Paul MacCready, *Soaring* 1955



MacCready construction

draw tangent to graph
of aircraft performance



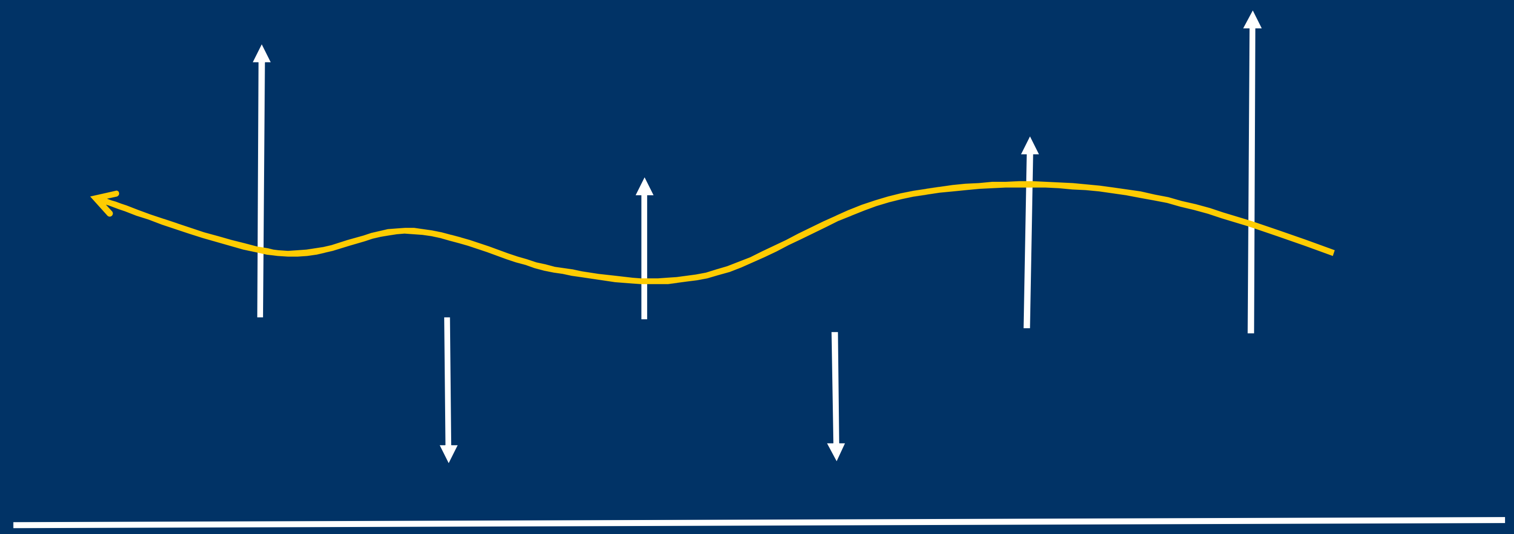
MacCready value

m = strength of **next** thermal in range

- Minimum lift to accept
- Optimal speed in varying lift $l(x)$: $v = v_*(m-l(x))$

  (Fly slow in lift, fast in sink)

m is a control input to flight computer



m = “speed to fly”

optimal speed as function
of m and local lift

A Stochastic Cross-Country or Festina Lente

A.W.F. Edwards, *Sailplane and Gliding* 1963

'A stochastic cross-country? What does "stochastic" mean?'

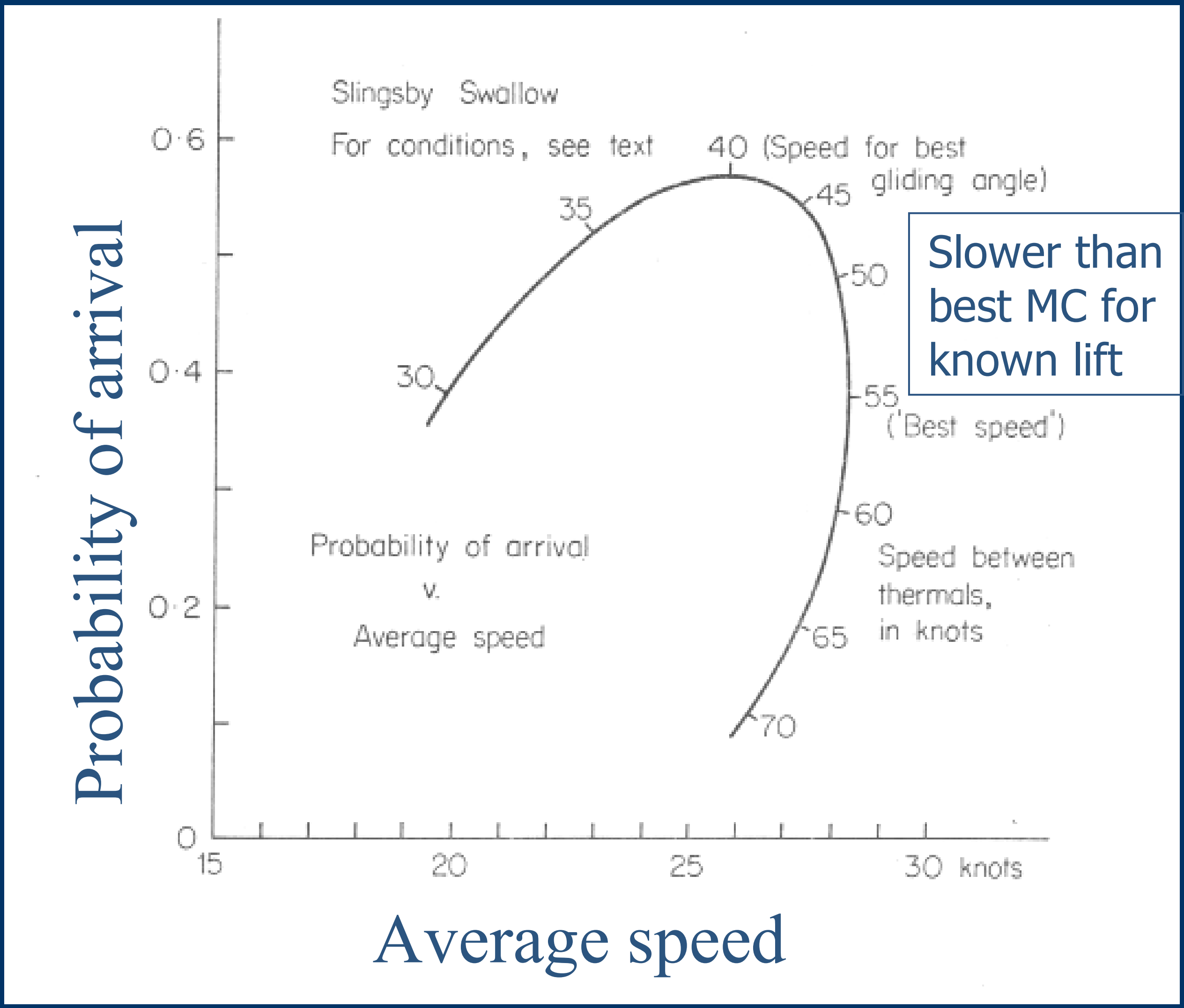
'It means there is an element of chance in the flight:
you might not reach your goal.'

"Every cross-country pilot knows that his primary task is to stay up… Much is known about Stochastic Processes nowadays, and in this article I want to introduce them to gliding in a very simple example: so simple, in fact, as to be rather unrealistic. But one has to start somewhere.

"My rate of climb in thermals will be u ft/sec. No down between thermals…The distance between adjacent thermals is a random variable, x, which is evidently exponentially distributed with probability density (1/d) exp(-x/d)."

The Efficient Frontier of Optimal Soaring

(like efficient frontier of investment)

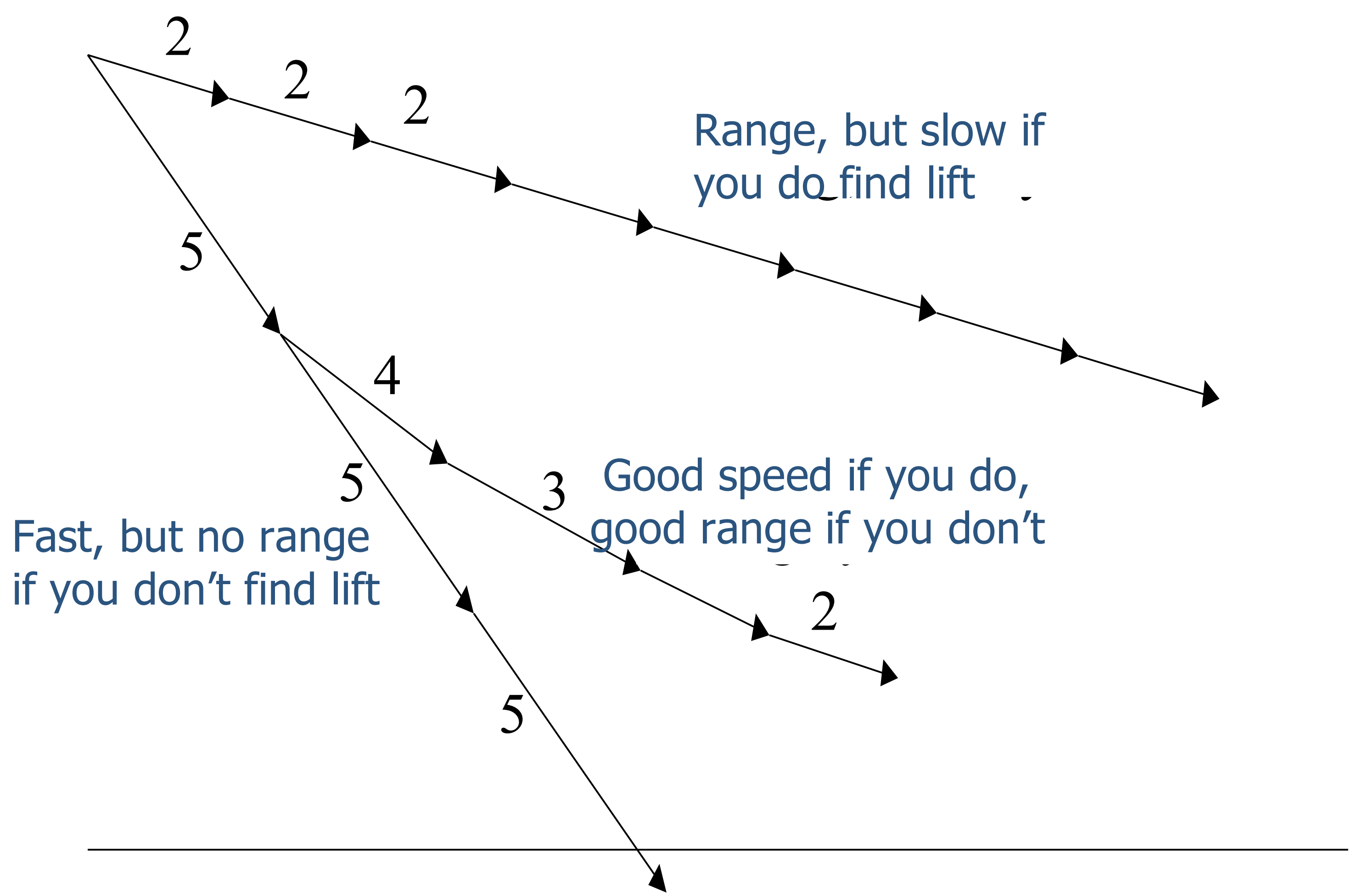


MacCready Theory with
Uncertain Lift and Limited Altitude

John H. Cochrane, *Technical Soaring* 1999

Myron S. Scholes Professor of Finance, University of Chicago

"Given a MacCready ring setting, every textbook tells you how to fly. Much of the mystery, challenge, and art of cross-country thermal flying comes down to a judgement what that setting should be. What setting should you use, given the fact that the strength and position of thermals is uncertain, and you may not have enough altitude to reach them before running into the ground?"



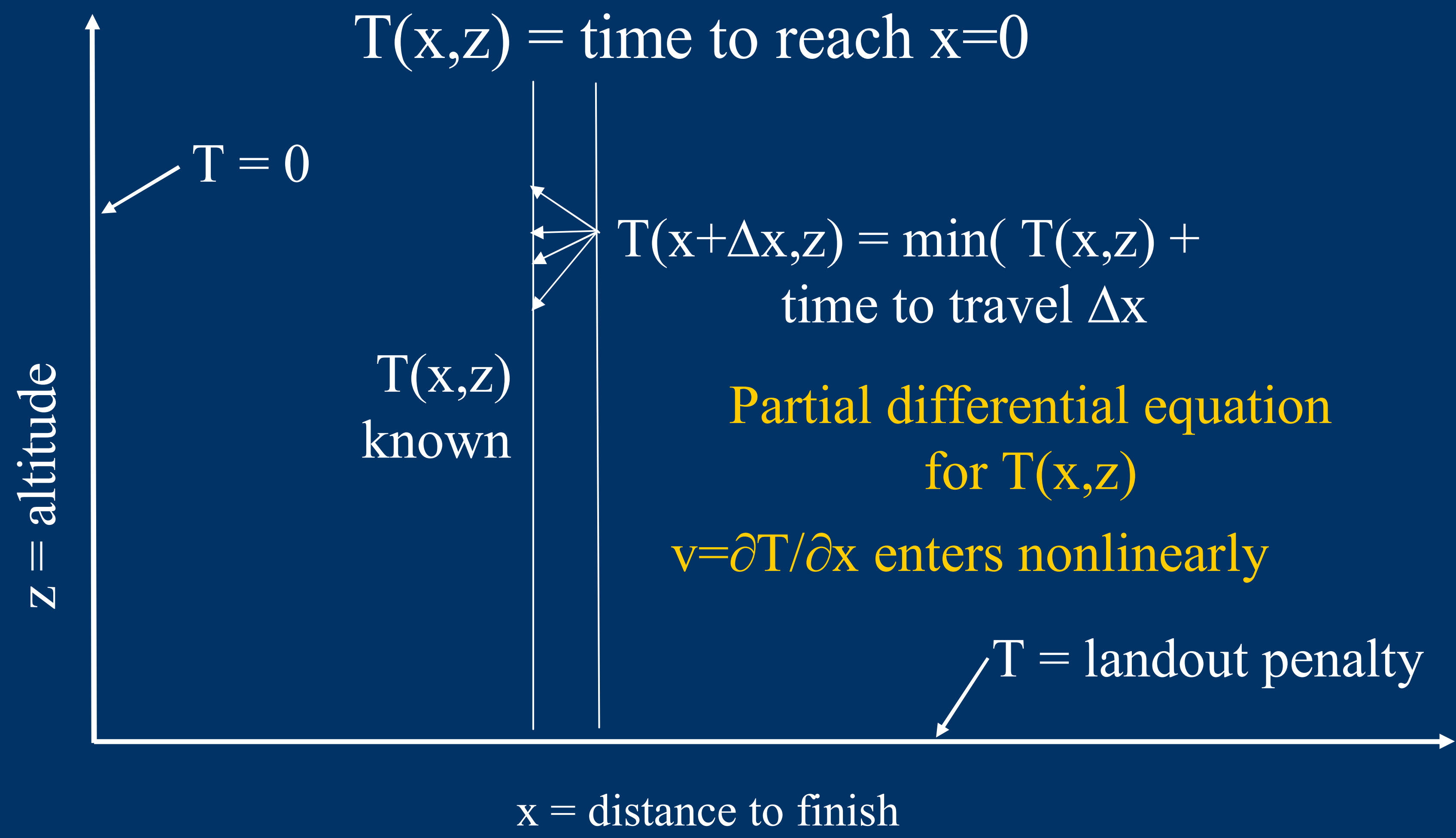
Cochrane: m depends on distance and altitude

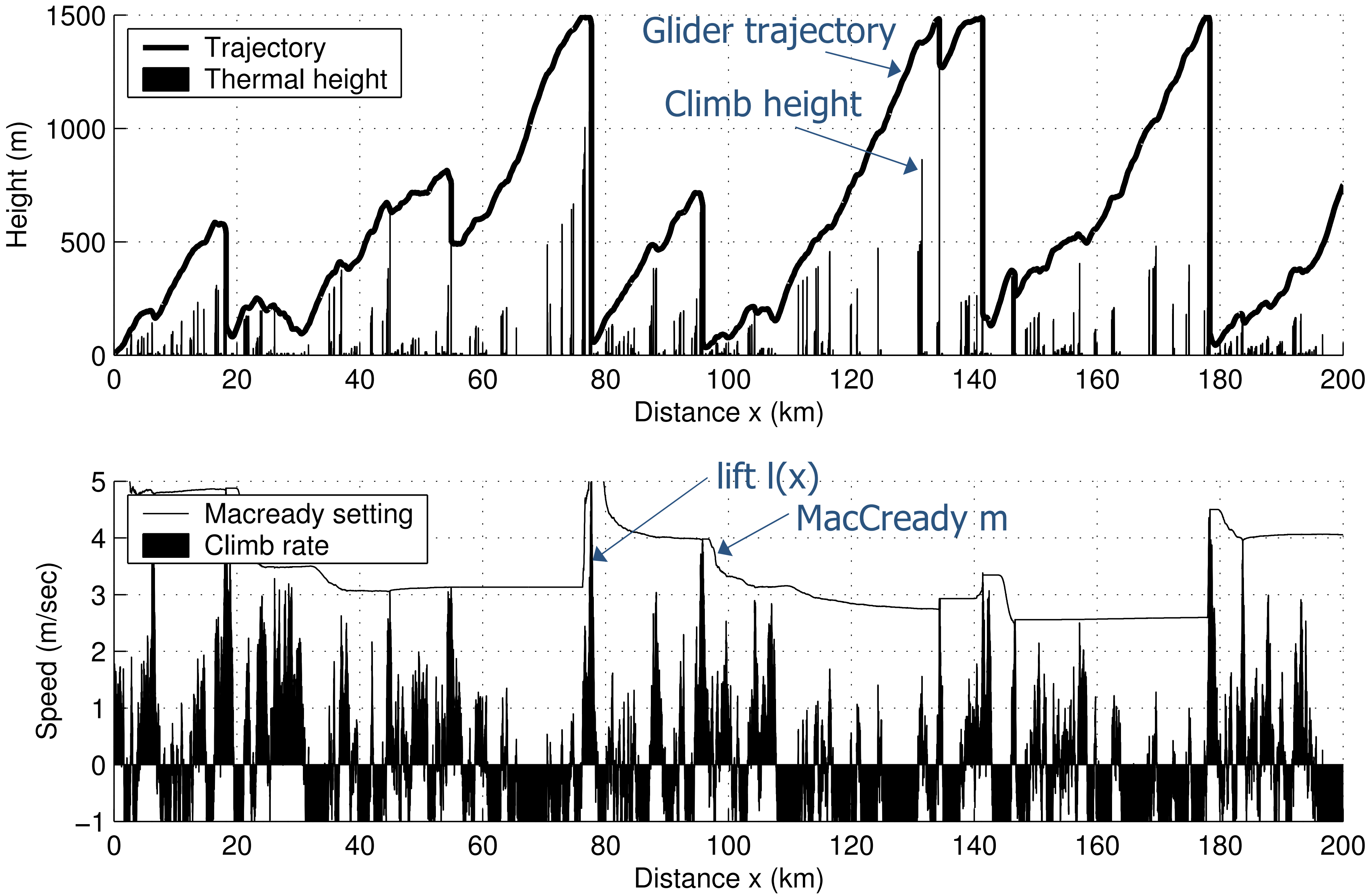
Our model

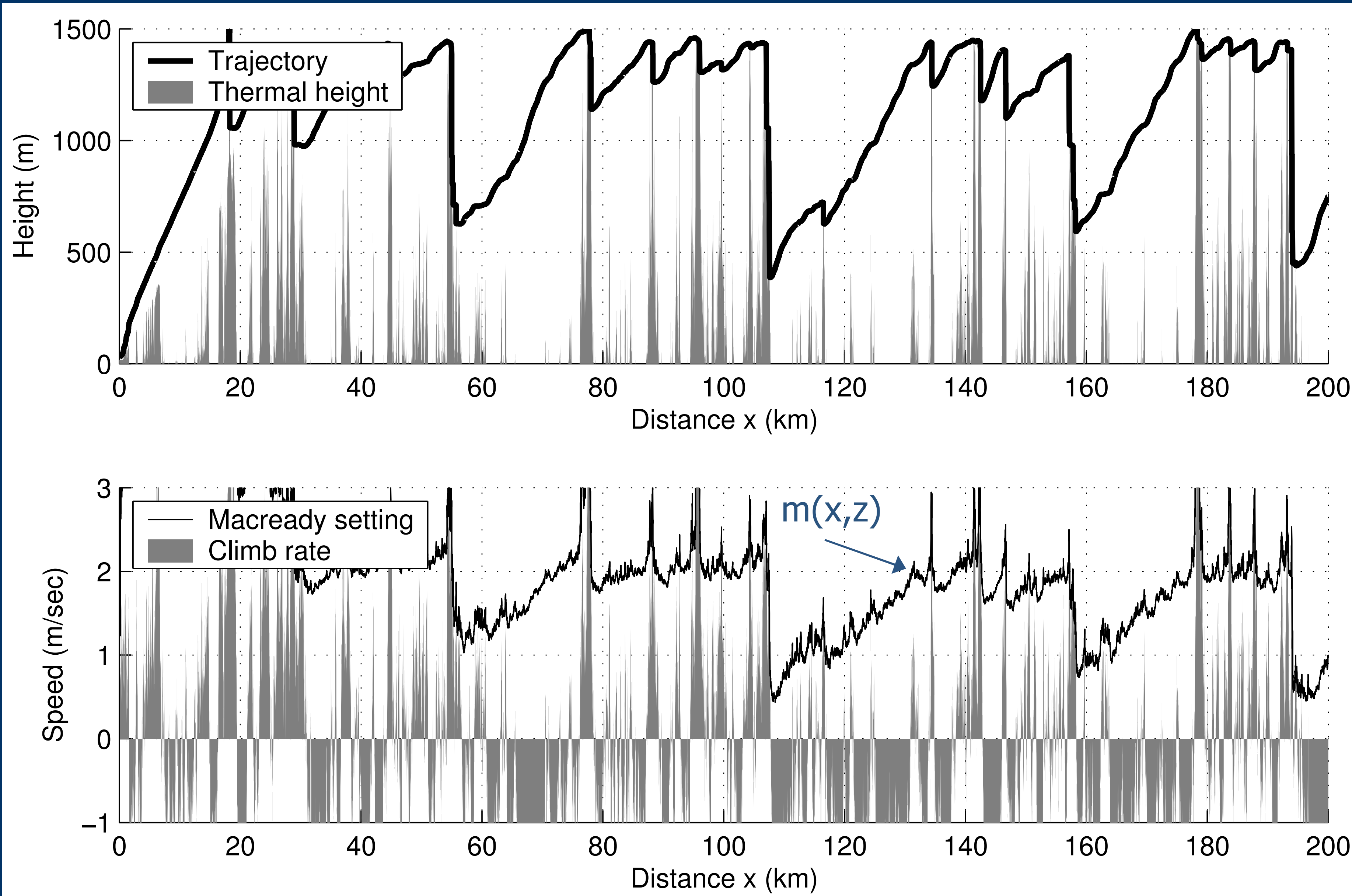
Optimal Soaring via Hamilton-Jacobi-Bellman Equations
Robert Almgren and Agnes Tourin, 2004

- Stochastic model for atmosphere
- Penalty for landout
- Minimize *expected* time to finish

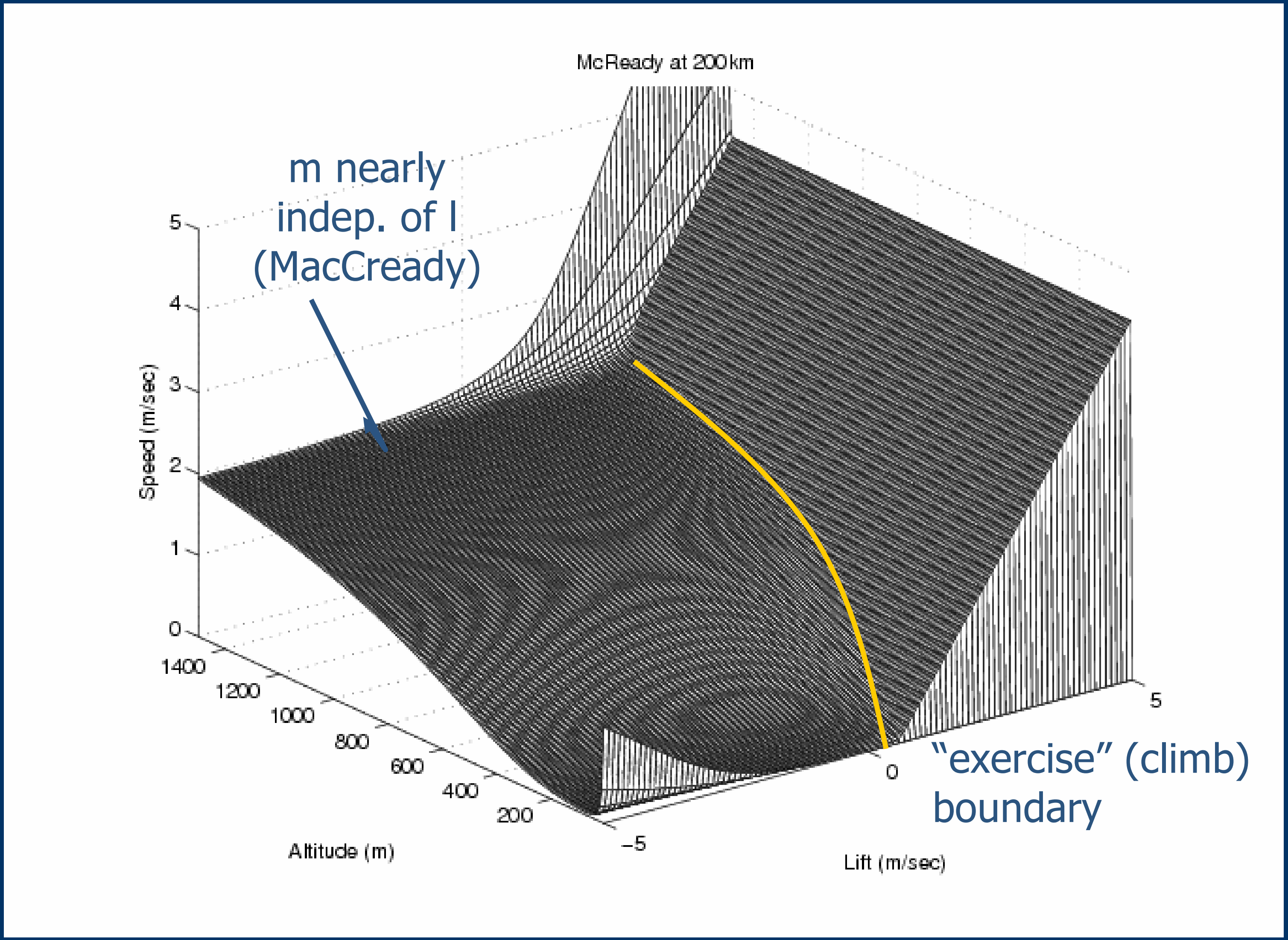
Dynamic Programming




Trajectory
Thermal height
Glider trajectory
Climb height
Height (m)
Distance x (km)
Macready setting
Climb rate
lift l(x)
MacCready m
Speed (m/sec)
Distance x (km)



Trajectory
Thermal height
Height (m)
Distance x (km)
Macready setting
Climb rate
m(x,z)
Speed (m/sec)
Distance x (km)

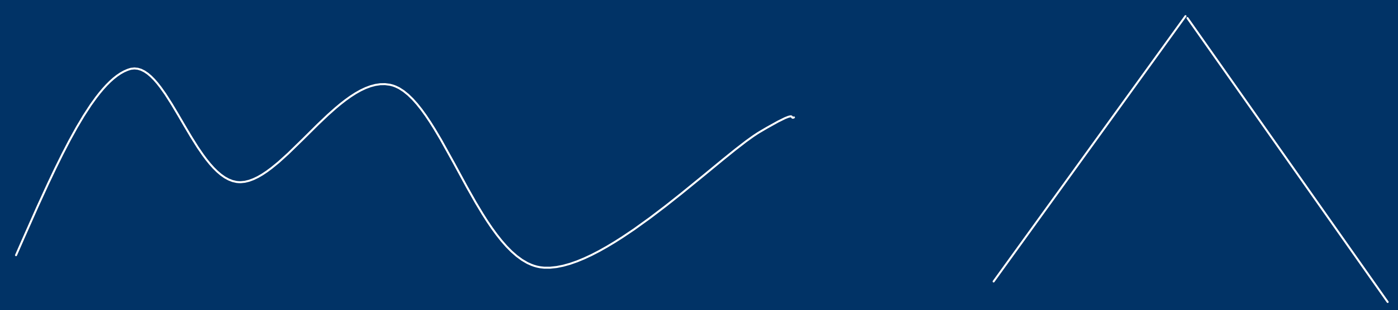

McReady at 200km
m nearly
indep. of l
(MacCready)
"exercise" (climb)
boundary
Speed (m/sec)
5
4
3
2
1
0
1400
1200
1000
800
600
400
200
Altitude (m)
−5
0
5
Lift (m/sec)

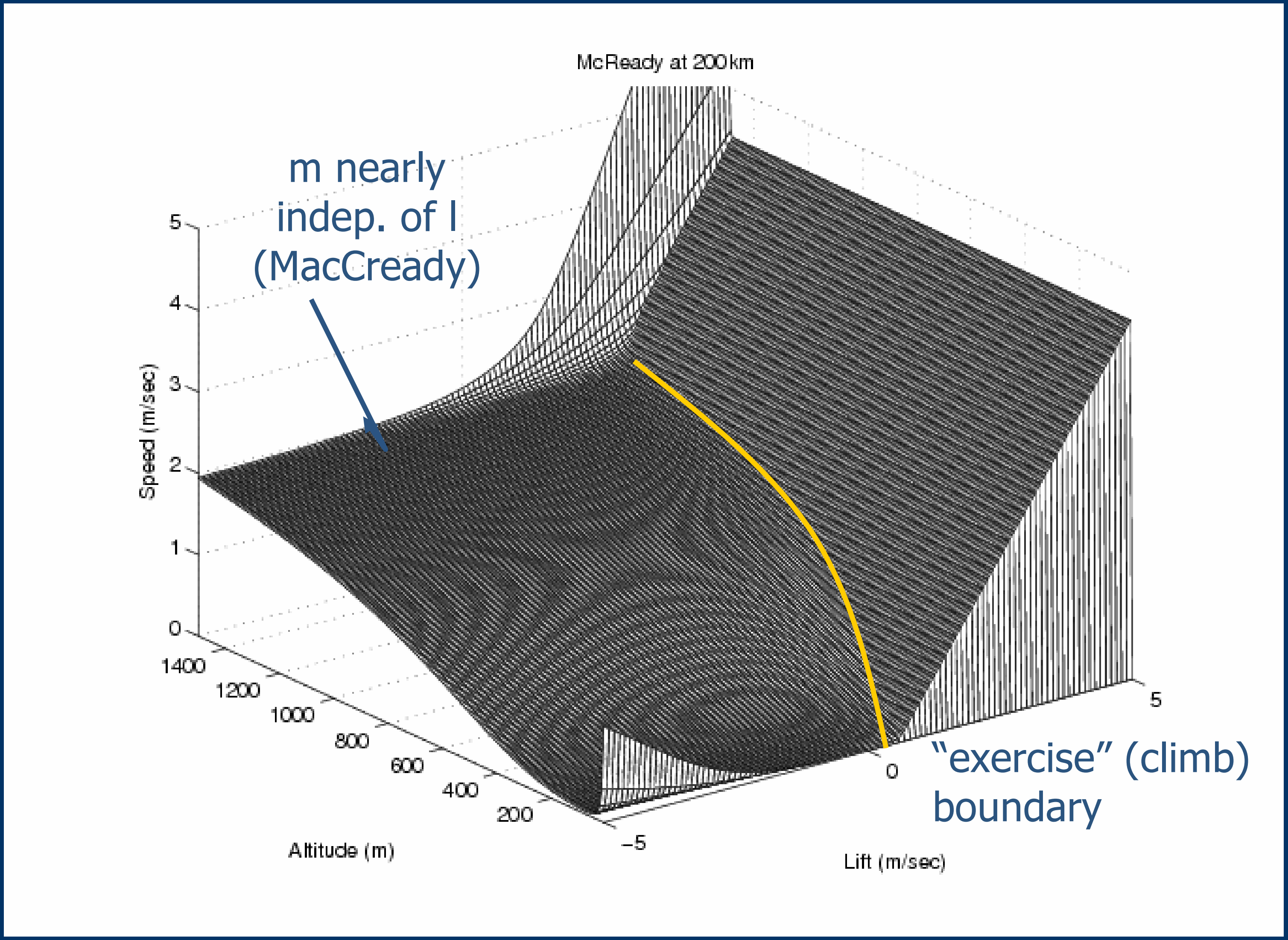
II. Mathematics

- Why is this problem challenging?

The solution is not smooth

The differential equation does not hold!




McReady at 200km
m nearly
indep. of l
(MacCready)
Speed (m/sec)
5
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0
1400
1200
1000
800
600
400
200
Altitude (m)
-5
0
5
Lift (m/sec)
“exercise” (climb)
boundary

The machinery of viscosity solutions

- It has been developed in the past 20 years by a group of researchers.
- The theory is based on an “order-perserving” principle.
- In the glider problem:

The bigger the penalty is, the bigger the time to finish!

Giving a meaning to the equation

- The theory provides a notion of solution.
- It turns out that this solution is indeed the correct physical solution.
- Last but not least, this solution can be computed!

The algorithm

- A naïve algorithm fails: it is unstable or does not pick the correct solution.
- Our algorithm satisfies the same order-preserving property as the equation.

Other applications

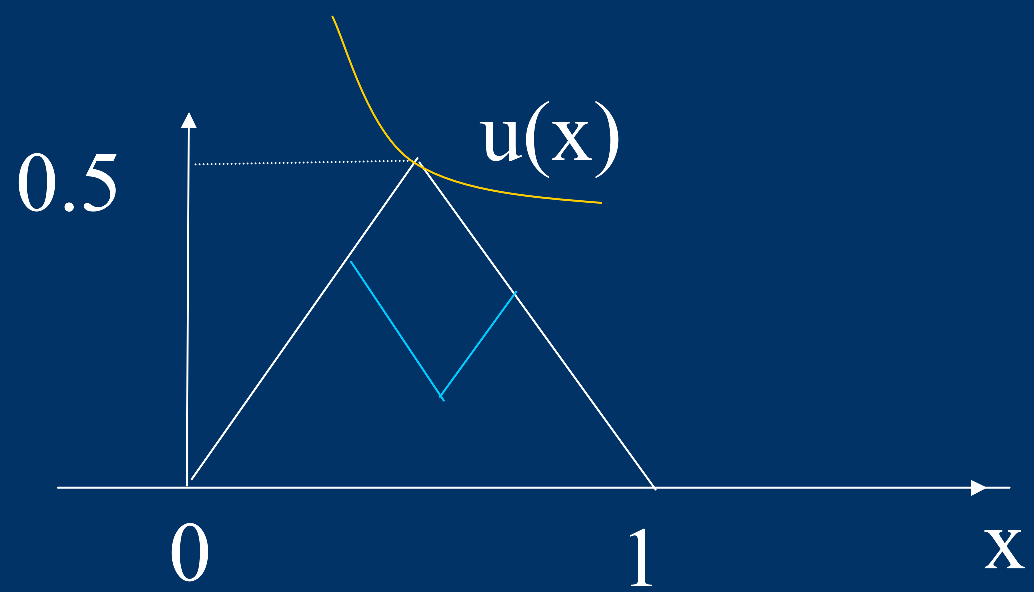
- The theory of viscosity solutions has a very broad range of applications.
- Examples: finance, economics, turbulent flame propagation, image processing.

Common features for all these problems

- The order-preserving principle
- Nonlinear equation in the gradient
- Toy example: the eikonal equation

The Eikonal equation

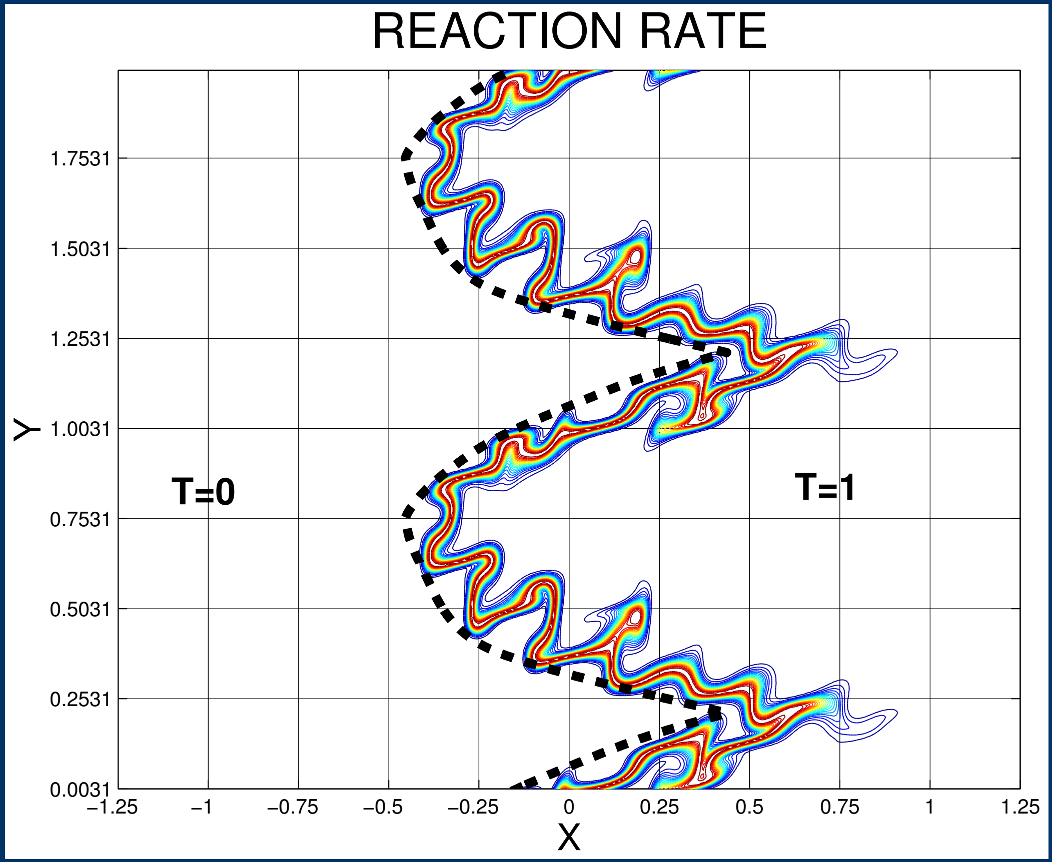
$$|u'(x)| = 1 \text{ on } [0,1]$$
$$u(0) = u(1) = 0.$$



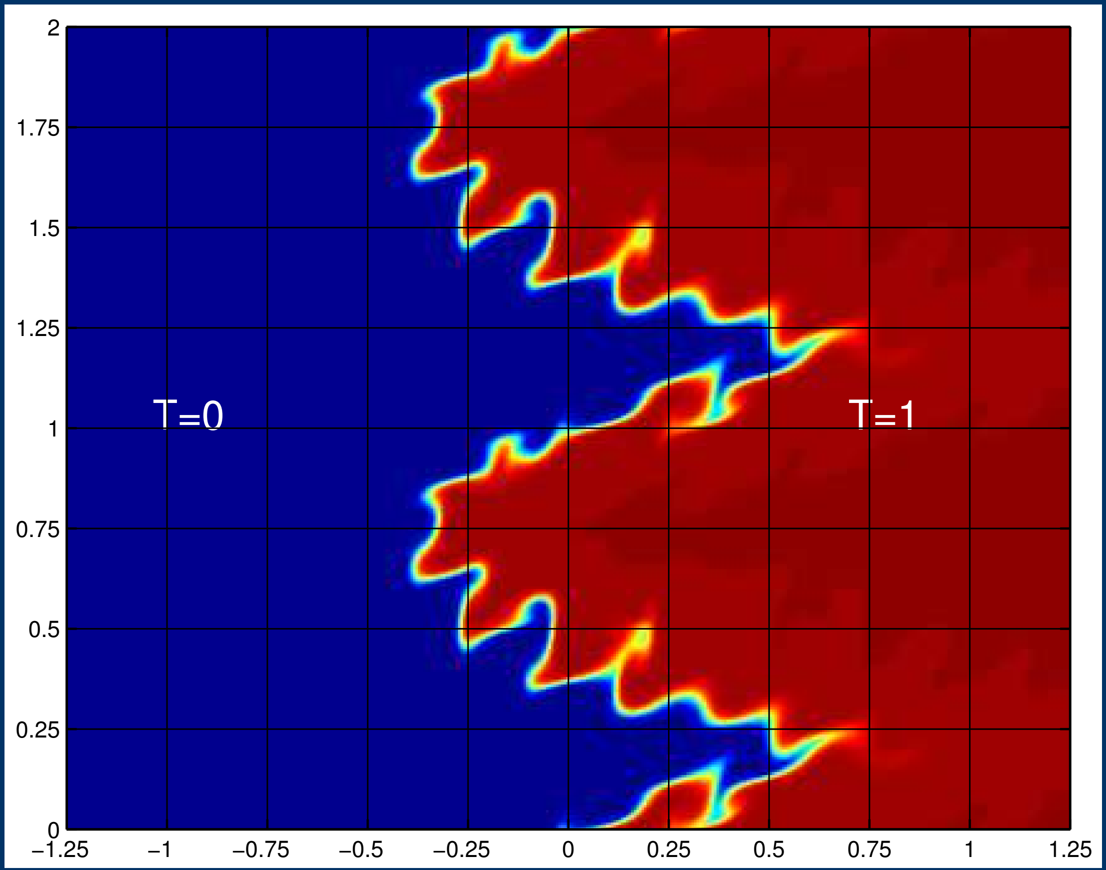
The Eikonal equation

- Solution u(x) is the distance to the boundary of the interval [0,1] (distance to 0 or 1, depending on which one is smaller)

Turbulent flame propagation
Anne Bourlioux (Univ. Montréal)



Turbulent flame propagation

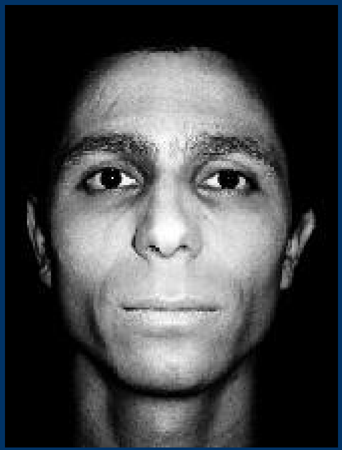


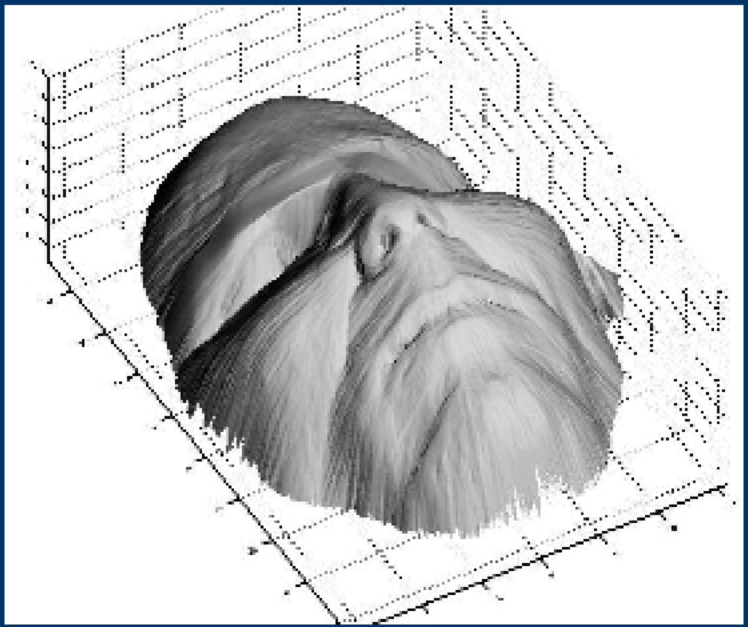
Shape from shading

Elisabeth Rouy, Olivier Faugeras, Emmanuel Prados (France)

They reconstruct a three-dimensional surface from a single two-dimensional image.

Reconstruction of a face from a real photograph





Conclusions

- We solved a problem coming from the real world with sophisticated mathematical techniques developed in the last 20 years.
- The two remaining difficulties for treating this type of problem:

No "ready to use" tool available.

The curse of dimensionality