Gradient-Flow Structure and Stability of Selfsimilar Solutions of Nonlinear Parabolic PDE's

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Gradient Flows

An equation

$$u_t = F(x, t, u, Du, D^2u, \dots)$$

is the *gradient flow* of the energy

$$E(u) = \int G(x, u, Du, \dots)$$

with respect to inner product $< \cdot, \cdot >$ if

$$< u_t, v >= -\frac{\delta E}{\delta u}[v] \qquad \forall\, v - \text{ test function.}$$

Example. Heat equation

$$u_t = \Delta u$$

is the gradient flow of the energy

$$E(u) = \frac{1}{2}\int |\nabla u|^2$$

with respect to L^2 inner product

$$< v_1, v_2 >= \int v_1\, v_2.$$

Gradient flows wrt Wasserstein metric

Otto, Denzler and McCann, Carlen and Gangbo, Carrillo, Villani...

Example. Porous medium/ fast diffusion equation

$$u_t = \Delta u^m \qquad \text{on } \mathbb{R}^d \times [0,\infty) \text{ for } m > \frac{d}{d+2}$$

is a gradient flow of the energy

$$E(u) = \begin{cases} \frac{1}{m-1}\int u^m & \text{if } m \neq 1 \\ \int u \ln u & \text{if } m = 1 \end{cases}$$

with respect to metric that for zero mean functions v_1 and v_2 is given by

$$< v_1, v_2 >_u = \int u \nabla p_1 \cdot \nabla p_2$$

where p_i solves

$$-\nabla \cdot (u \nabla p_i) = v_i \qquad \text{for } i = 1, 2.$$

- Functions u belong to

$$\mathcal{M} = \{u \,:\, u \geq 0, \int u = M > 0, \int x^2 u < \infty\}.$$

- $(\mathcal{M}, < \cdot, \cdot >_u)$ is a manifold.

- The induced distance is the Wasserstein distance:

$$d(u_1, u_2)^2 = \inf_{\Phi_{\#} u_1 = u_2} \int |\Phi(x) - x|^2 u_1(x) dx$$

Long-time asymptotics

Friedman and Kamin, Vazquez, Carrillo, Markowich, Lederman, Toscani, Del Pino and Dolbeault, and using gradient flow approach Otto, Denzler and McCann...

The equation

$$u_t = \triangle u^m$$

possesses selfsimilar (Barenblatt) solutions when $m > \frac{d-2}{d}$.
In particular

$$U(x,t) = t^{-d\alpha}\,\rho(xt^{-\alpha})$$

where $\alpha = 1/(dm - d + 2)$ and

$$\rho(x) = \left(C + a\frac{1-m}{2m}|x|^2\right)_+^{1/(m-1)}.$$

Barenblatt solutions describe the long-time shape of solutions of the equation in the sense that

$$\|u(\,\cdot\,,t) - U(\,\cdot\,,t)\|_{L^1(\mathbb{R}^d)} \longrightarrow 0 \qquad \text{as } t \to \infty.$$

Q: What is the rate of convergence?

Optimal rate of convergence

Similarity variables: Rescale the horizontal and vertical directions so that selfsimilar solution becomes a stationary one.

$$u(x,t) = t^{-d\alpha} w(xt^{-\alpha}, \alpha \ln t)$$

It satisfies the equation

$$w_t = \Delta w^m + \nabla \cdot (xw)$$

which is a gradient flow of the energy

$$\tilde{E}(w) = \int \frac{1}{m-1} w^m + \frac{1}{2} x^2 w$$

- For $m > \frac{d-1}{d}$ this functional is geodesically convex (McCann) on $(\mathcal{M}, < \cdot, \cdot >)$, meaning that for every arc-length parameterized geodesic $\gamma(s)$, $\tilde{E}(\gamma(s))$ is a convex function.

  Otto used the measure of convexity of $\tilde{E}$ to obtain optimal rates of convergence of w toward ρ.

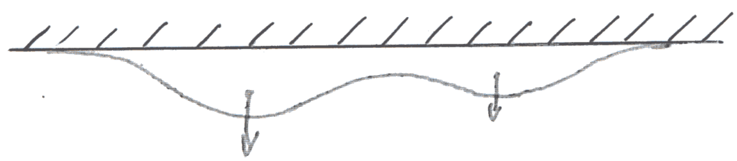
- For $m < 1$, Denzler and McCann have linearized the equation at ρ. The linearized dynamics at a fixed point is governed by the Hessian of E, which is always symmetric! They computed the full spectrum and hence a prediction for optimal rates of convergence.

- For $m \in (\frac{d}{d+2}, \frac{d-1}{d})$ McCann and S. showed that along solution curves the energy is eventually convex and obtained almost optimal rates of convergence.

- For $m \in (\frac{d-2}{d}, \frac{d}{d+2})$ Kim and McCann have established optimal rates of convergence.

Long-wave unstable thin-film equations

(UTFE) $\quad u_t = -(u^n u_{xxx})_x - (u^m u_x)_x \qquad$ on $\mathbb{R} \times [0, T]$



- UTFE describe the evolution of a thin layer of fluid under the effects of destabilizing forces, like gravity.
- UTFE preserves the mass and nonnegativity of initial data.
- UFTE is a gradient flow of the energy

$$E(u) = \int \frac{1}{2} u_x^2 - \frac{1}{(m-n+2)(m-n+1)} u^{m-n+2}$$

with respect to the inner product given by

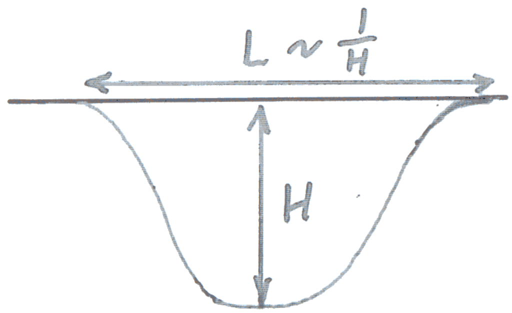
$$< v_1, v_2 >_u = \int u^n \nabla p_1 \cdot \nabla p_2$$

where p_i solves $-\nabla \cdot (u^n \nabla p_i) = v_i$ for $i = 1, 2$.

Dynamics of UTFE

$$u_t = -(u^n u_{xxx})_x - (u^m u_x)_x$$

Q: When are the stabilizing and destabilizing forces in balance?

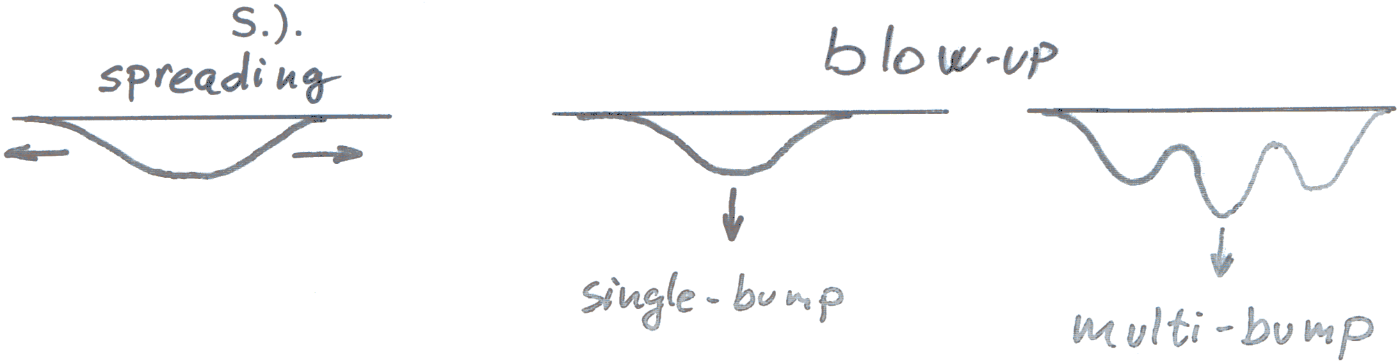


$$(u^n u_{xxx})_x \sim \frac{H^{n+1}}{L^4} = H^{n+5}$$

$$(u^m u_x)_x \sim \frac{H^{m+1}}{L^2} = H^{m+3}$$

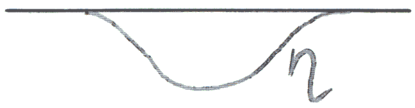
in balance if $\boxed{\text{m=n+2}}$

- If $m < n+2$ Bertozzi and Pugh have shown that weak solutions exist for all time
- If $m = n+2$ selfsimilar source-type (spreading) solutions exist for $0 < n < 3$ (Beretta) and selfsimilar blowup solutions exist when $0 < n < 3/2$ (Pugh and S.).



- If $m > n+2$ it is conjectured (Bertozzi and Pugh) blowup is possible. The conjecture has been proven for $n = 1$

Stability of steady states when $n = 1$



droplet

droplet configuration

- The linear stability can be obtained from Hess E at η. Since the Hessian is symmetric its eigenvectors and eigenfunctions can be determined from the quadratic form $H(v) =$ Hess E(v, v).

- The construction of the inner product on $\mathcal{M}$ suggests the use of particular coordinates on $T\mathcal{M}$.

$$v \longrightarrow f \qquad \text{where} \quad -(\eta f)_x = v$$

  Then the metric on the space of functions f is weighted L^2 inner product $< f_1, f_2 >= \int \eta f_1 f_2$.

- Geodesic in the direction f is $\gamma(s) = (Id + s\, f)_{\#}\eta$.

- $H(f) = \dfrac{E(\gamma(s))''}{< f, f >} = \displaystyle\int \eta^2 f_{xx}^2 - \frac{m-3}{m+1}\eta^{m+1} f_x^2 \, dx \Big/ \int \eta f^2$

- $f = 1$ corresponds to translations; H(1)=0

- $f = x$ corresponds to dilations
  - If $m > 3$ then $H(x) < 0$ – an unstable direction
  - If $m = 3$ then $H(x) = 0$ – a neutral direction
  - If $m < 3$ then $H(x) > 0$ and moreover $H(f) > \lambda > 0$ for all f such that $< f, 1 >= 0$.

Stability of selfsimilar solutions when $n = 1$ and $m = 3$

- All droplet steady states have the same mass, denote it M_c.
- Initial data with mass less than M_c do not blow up.
- Spreading selfsimilar solutions are linearly stable.

Stability of blowup profiles ρ

- In similarity variables the equation becomes:

$$w_t = -\left(w w_{xxx} + w^3 w_x + \frac{xw}{5}\right)_x .$$

- it is a gradient flow of energy

$$\tilde{E} = \int \frac{1}{2} w_x^2 - \frac{1}{12} w^4 - \frac{x^2 w}{10} dx.$$

- Hessian quadratic form is:

$$H(f) = \int \rho^2 f_{xx}^2 - \frac{4}{5} \int_{|x|}^{L} s\rho(s) ds - \frac{1}{5} \rho f^2 dx.$$

- Note that $H(1) < 0$ and $H(x) < 0$.
- For single-bump profiles ρ that have been constructed $H(f) > \lambda > 0$ for all f such that $< f, 1 >= 0$ and $< f, x >= 0$.
- For multi-bump profiles there exist other unstable directions.

Future directions and open problems

On dynamics of UTFE

- Linear stability of selfsimilar solutions when $n \neq 1$.
- Asymptotic stability.
- Show that blowup is possible when $m > n + 2$ (known when $n = 1$)
- Show that blowup is generic when in the critical case $m = n + 2$ when the mass is greater then M_c
- When $m < n+2$ the solutions are expected to converge to a configuration of droplets. Prove and determine the configuration.

On gradient flows wrt Wasserstein distance

- Higher order asymptotics for porous medium and fast diffusion equations
- Convexity of higher order functionals. An interesting example (on the space of periodic functions in $\mathbb{R}$):

$$E(u) = \int_0^1 u^\alpha u_x^2 \, dx$$

with $\alpha \in [-5, -4]$ is geodesically convex.