# Renormalization of Isoenergetically Degenerate Hamiltonian Flows

Hans Koch and Denis Gaydashev, 2003

• **One of the central issues of dynamical systems**

*Existence, persistence and break-up of invariant tori for flows, on which motion is conjugate to a rigid rotation.* **Renormalization** *is a non-perturbative technique which allows to study these questions.*

– *Diophantine property* on the rotation vector:

$$|w \cdot \nu| \geq \frac{\text{const}}{|\nu|^k} \text{ for all } \nu \in \mathbb{Z}^d - \{0\}, \quad k \geq 1,$$

and *isoenergetic non-degeneracy*:

$$\left| \det \begin{bmatrix} \partial_i \partial_j H & \partial_i H \\ \partial_j H & 0 \end{bmatrix} \right| \geq \kappa > 0 \quad \text{on the torus,}$$

appear as conditions for robustness of invariant tori.

– Case of isoenergetically (non-twist) degenerate Hamiltonian flows with shearless tori is notoriously harder.

- **Observation of universality in non-twist area-preserving maps**

(*P. Morrison et al.*)

Area-preserving $f : \mathbb{T} \times \mathbb{R} \mapsto \mathbb{T} \times \mathbb{R}$

$\partial_p q' = 0$, where $(q', p') = f(q, p)$.

*Example*: non-twist Arnold two-parameter family $T_{\omega,\epsilon}$

$$q_{n+1} = q_n + \omega(1 - p_{n+1}^2)$$
$$p_{n+1} = p_n - \epsilon \sin 2\pi q_n$$

The twist condition is violated along the curve $p = -\epsilon \sin 2\pi q$.

– *Bifurcations of invariant tori in two-parameter families of maps*:

**Claim 1.** (*A. Delshamps, R. De la Llave (2000)*)

*Let $\omega_0 = (\sqrt{5} - 1)/2$. Then, for $|\epsilon| << 1$ there is a smooth curve $\omega(\epsilon)$ with $\omega(0) = \omega_0$, such that*

*a) if $\omega > \omega(\epsilon)$, then $T_{\omega,\epsilon}$ admits two invariant circles with rotation number $\omega_0$,*

*b) if $\omega < \omega(\epsilon)$, then $T_{\omega,\epsilon}$ admits no invariant circles with rotation number $\omega_0$,*

*c) if $\omega = \omega(\epsilon)$, then $T_{\omega,\epsilon}$ admits one shearless invariant circle with rotation number $\omega_0$.*

This claim has been proved for a class of families, "similar" to $T_{\omega,\epsilon}$.

– *Phase space universality*: $\frac{a_k}{b_k}$-periodic orbits accumulate on the golden invariant curve at the break up at a geometric rate:

$$\mathrm{dist}(\gamma_\mathrm{k} - \gamma^*) \asymp \alpha^{-\mathrm{k}},\ \alpha > 1$$

– *Parameter space universality*: parameter values $(\omega_k, \epsilon_k)$ for which $\frac{a_k}{b_k}$-periodic orbits of a fixed stability are on the verge of destruction accumulate on $(\epsilon_\infty, \mu_\infty)$ geometrically with two rates: $\delta_1 = 2.678$ and $\delta_2 = 1.583$.

- **Renormalization explanation of universality**
  - $\mathcal{A}$, a Banach space;
  - $B \subset \mathcal{A}$, open;
  - $\mathcal{Y}$, $\dim(\mathcal{Y}) = n$;
  - $F : B \mapsto \mathcal{Y}$, an observable;
  - a "natural" map $r$ on a countable $Y \subset \mathcal{Y}$, with an orbit $\{y_k\}_0^\infty \subset Y$.
  - Level sets of $F$ - equivalent objects.
  - $\exists$ a submnfd $\mathcal{W} \in \mathcal{A}$, s.t. $F^{-1}(y_k) -> \mathcal{W}$ at a geometric rate.
  - Construct $\mathcal{R} : B \mapsto \mathcal{A}$ with a hyperbolic fixed point and $\mathcal{W}$ as its stable mnfd, s. t.

$$F(\mathcal{R}(h)) = y, \quad \text{whenever} \quad F(h) = r(y),$$

  - *The convergence of* $F^{-1}(y_k)$ *is governed by the largest eigenvalue of* $\mathcal{R}$.

- **Renormalization of Hamiltonian Flows** (*H. Koch et al.*)
  - *Objects of renormalization*: Hamiltonian functions $H(q,p)$ in $2d$ variables, analytic on a subset of $\mathbb{C}^{2d}$, generating a flow through

$$i_v\omega_2 = -dH \quad \text{or} \quad \frac{\partial}{\partial t}(q(t),p(t)) = \mathbb{J}(\nabla H)(q(t),p(t)),$$

where $\omega_2$ is the sympl. form in $\mathbb{T}^d \times \mathbb{R}^d$ and $\mathbb{J} = \left[\begin{smallmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{smallmatrix}\right]$.

  - *Observable*: normalized rotation vector $w \in \mathbb{R}P^d$.

Given an integral $d \times d$ matrix $T$, such that

a) $T$ has a simple eigenvalue $\vartheta$, $|\vartheta| > 1$, $T\omega = \vartheta\omega$;

b) other eigenvalues are inside the unit circle, simple, non-zero;

c) $\det T = \pm 1$;

starting with $\omega_0 \in \mathbb{R}^2$, $\omega_0 \cdot \omega \neq 0$, one can construct a sequence of normalized "rational" approximates of $\omega$: $\{T^k\omega_0\}_0^\infty$.

– *Renormalization operator*:

$$\mathcal{R}(H) = H \circ T_1 \pmod{\mathcal{G}},$$

where

– $T_\mu(q,p) = (Tq, \mu(T^*)^{-1}p)$, with $\mu \in \mathbb{C}$;

– $\mathcal{G}$ is (ideally) a group of transformations that acts on Hamiltonians, preserving the rotation numbers of the orbits.

**Motivation**: *A Hamiltonian $\mathcal{R}(H)$ has an orbit with a rotation vector $cw$ whenever $H$ has an orbit with $Tw$.*

- **Specifics for d=2** (*H. Koch, D. Gaydashev*)

  – *Objects of renormalization*:

  Let $\omega = (1/\theta, 1)$, where $\theta$ is a quad. irrational, $\Omega = (-1, 1/\theta)$.

  Let $\omega'$ and $\Omega'$ - multiples of $\omega$ and $\Omega$, $\omega \cdot \omega' = 1$ and $\Omega \cdot \Omega' = 1$.

  Given a pair $\rho = (\rho_1, \rho_2)$ of real positive numbers, define

$$\mathcal{D}_1(\rho) = \left\{ q \in \mathbb{C}^2 : |\Im(\omega' \cdot q)| < \rho_1, |\Im(\Omega' \cdot q)| < \rho_2 \right\},$$
$$\mathcal{D}_2(\rho) = \left\{ p \in \mathbb{C}^2 : |\omega \cdot p| < \rho_3, |\Omega \cdot p| < \rho_4, \right\}.$$

  Given a quadruple $\rho = (\rho_1, \rho_2, \rho_3, \rho_4)$ of real positive numbers, define

$$\mathcal{D}(\rho) = \mathcal{D}_1(\rho_1, \rho_2) \times \mathcal{D}_2(\rho_3, \rho_4).$$

  **Definition 2.** *Define $\mathcal{A}(\rho)$ to be the Banach space of all analytic functions on $\mathcal{D}(\rho)$, which extend continuously to the boundary of $\mathcal{D}(\rho)$, of the form*

$$H(q,p) = \sum_{\nu \in \mathbb{Z}^2, k \in \mathbb{N}, n \in \mathbb{N}} H_{(\nu,k,n)} e^{i(\nu \cdot \omega x + \nu \cdot \Omega y)} (\omega \cdot p)^k (\Omega \cdot p)^n,$$

  *for which the norm*

$$\|H\|_\rho = \sum_{\nu \in \mathbb{Z}^2, k \in \mathbb{N}, n \in \mathbb{N}} \left| H_{(\nu,k,n)} \right| e^{|\nu \cdot \omega| \rho_1 + |\nu \cdot \Omega| \rho_2} \rho_3^k \rho_4^n$$

  *is finite.*

– *Elimination of the irrelevant degrees of freedom*

Notice: *the norm of the components of the Fourier-Taylor expansion 'aligned' with $\omega$ grows under $T_1$.:*

**Definition 3.** *Given real positive numbers $\sigma$, $\kappa_3$ and $\kappa_4$ define the* **non-resonant** *and* **resonant** *index sets*

$$
\begin{aligned}
I^{-} &= \{(\nu, k, n) \in \mathbb{Z}^2 \times \mathbb{N}^2 : |\omega \cdot \nu| > \sigma |\Omega \cdot \nu| \, , |\omega \cdot \nu| > \kappa_3 k \\
&\quad \text{and } |\omega \cdot \nu| > \kappa_4 n\} \, , \\
I^{+} &= \mathbb{Z}^2 \times \mathbb{N} \times \mathbb{N} \setminus I^{-} .
\end{aligned}
$$

**Theorem 4.** *Given a quadruple $\rho$ there exists a choice of $\rho' < \rho$, $\sigma$, $\kappa_3$ and $\kappa_4$, a non-empty set $\mathcal{H} \in \mathcal{A}(\rho)$ containing $H^0 = \omega \cdot p$, and a map $\mathcal{N}$ that assigns to each $H \in \mathcal{H}$ a* **canonical transformation** *$\mathcal{U}_H$ from $\mathcal{D}(\rho')$ to $\mathcal{D}(\rho)$, such that*

$$
H \circ \mathcal{U}_H \in \mathcal{A}(\rho'), \quad \mathbb{I}^{-} H \circ \mathcal{U}_H = 0.
$$

*The map $\mathcal{N}$ is analytic from $\mathcal{H}$ to $\mathcal{A}(\varrho')$. Furthermore, if $H \in \mathcal{H}$ satisfies $\mathbb{I}^{-} H = 0$.*

– *Compactness*

**Theorem 5.** *There exists a choice of renormalization parameters for which* $H \mapsto H \circ T_\mu$ *is a compact linear map from* $\mathbb{I}^+\mathcal{A}(\rho')$ *to* $\mathcal{A}(\rho)$ *of norm less or equal to one.*

– *Renormalization operator*

**Definition 6.** *Given a four-vector* $\rho$*, define*

$$\begin{aligned} \mathfrak{R}(H) &= \frac{\vartheta^3}{\xi\eta}\left(\hat{H} \circ T_{\vartheta^{-2}} \circ S_\eta - \epsilon\right), \\ \hat{H} &= H \circ \mathcal{U}_H \circ R_t, \\ R_t(q,p) &= (q, p + t\Omega), \\ S_\eta(q,p) &= (q, \eta p). \end{aligned}$$

*where* $\xi$*,* $\eta$*,* $t$ *and* $\epsilon$ *are chosen from the normalization conditions:*

$(\mathfrak{R}(H))_{(0,0,1,0)} = 1,$
$(\mathfrak{R}(H))_{(0,0,0,3)} = \pm\gamma,\ \gamma > 0$
$(\mathfrak{R}(H))_{(0,0,0,2)} = 0,$
$(\mathfrak{R}(H))_{(0,0,0,0)} = 0.$

– *Fixed points*

$\mathfrak{R}$ has a period two simple fixed point

$$(H_+^3 = \omega \cdot p + \gamma(\Omega \cdot p)^3, H_-^3 = \omega \cdot p - \gamma(\Omega \cdot p)^3).$$

The rotation vectors of these integrable flows are given by

$$w = \frac{1 \pm 3\gamma(\Omega \cdot p)^2 \vartheta}{\vartheta \mp 3\gamma(\Omega \cdot p)^2}$$

– *Analyticity and compactness*

**Theorem 7.** *$\exists$ nbhds $\mathcal{B}_\pm \subset \mathcal{A}(\rho)$ of $H_\pm^3$, such that the operator $\mathfrak{R}$ is well-defined, analytic and compact as a map from $\mathcal{B}(b)_\pm$ to $\mathcal{A}(\rho)$.*

– *Linear analysis at the simple fixed point*

$\mathfrak{R}$ is hyperbolic at $H_\pm^3$ with one unstable direction spanned by $\omega \cdot p$ and has a stable manifold $\mathcal{W}_s^3$ of codimension one.

- **Bifurcations and shearless invariant tori** (*H. Koch, D. Gaydashev*)

Consider the unstable family:

$$F^*(\alpha) = H_+^3 + \alpha(\Omega \cdot p), \quad w(\alpha) = \frac{1 + (\alpha + 3\gamma(\Omega \cdot p)^2)\vartheta}{\vartheta - (\alpha + 3\gamma(\Omega \cdot p)^2)}$$

A family $F : \mathbb{C} \mapsto \mathcal{A}(\rho)$ is said to be **real** if the map $(\alpha, q, p) \mapsto F(\alpha)(q, p)$ takes real values when restricted to real arguments. Similarly for a torus map $\Gamma_H : \mathcal{D}_1(\rho_1, \rho_2) \mapsto \mathcal{D}(\rho)$.

**Theorem 8.** *$\exists\, \mathcal{D}_\delta \ni 0$ in $\mathbb{C}$, a nbhd $\mathcal{U}$ of $F^*$ and an analytic $F \mapsto \alpha_F$ from $\mathcal{U}$ to $\mathcal{D}_\delta$, such that the following holds for every family $F \in \mathcal{U}$ and $\alpha \in \mathcal{D}_\delta$.*

a) *$\exists$ a complex nbhd $\mathcal{D}$ of $\mathbb{T}^2$, and two torus maps $\Gamma^0_{F(\alpha)}$ and $\Gamma^1_{F(\alpha)}$, analytic on $\mathcal{D}$, and satisfying*

$$(\mathbb{J}\nabla F(\alpha)) \circ \Gamma^i_{F(\alpha)} = c\,\omega \cdot \nabla_q \Gamma^i_{F(\alpha)}. \tag{1}$$

*The torus maps are distinct if $\alpha \neq \alpha_F$ and coincide if $\alpha = \alpha_F$.*

b) *The torus $\Gamma^0_{F(\alpha_F)}$ is shearless (in a proper sense).*

c) *If $F$ is real and $\alpha$ belongs to the interval $(-\delta, \alpha_F]$ the torus maps $\Gamma^i_{F(\alpha)}$ are real.*

d) *If $F$ is real and $\alpha \in (\alpha_F, \delta)$ the torus maps $\Gamma^i_{F(\alpha)}$ are not real. For every such $F$ and $\alpha$ $\exists$ a nbhd $\mathfrak{D} = \mathfrak{D}(\alpha, F)$ of $\mathbb{T}^2$ in $\mathbb{C}^4$, such that the range of both $\Gamma^0_{F(\alpha)}$ and $\Gamma^1_{F(\alpha)}$ is contained in $\mathfrak{D}$ and there exist no real torus map satisfying the equation (1) with the range in $\mathfrak{D}$.*

- **Renormalization at criticality and break-up of the invariant $\omega$-tori** (*H. Koch, D. Gaydashev*)

  – *Numerical implementation of renormalization*

$$H(q,p) = \omega \cdot p + \sum_{(\nu,k)\in I} H_{\nu,k} e^{i\nu\cdot q}(\Omega \cdot p)^k,$$

$$\omega = (1/\vartheta, 1),\ \Omega = (-1, 1/\vartheta) \text{ and } \vartheta = \frac{\sqrt{5}+1}{2},$$

$$T = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix},\ T\omega = \vartheta\omega,\ T\Omega = -\frac{1}{\vartheta}\Omega$$

  – *"Obvious" symmetries*:

$$\mathcal{J}H = -H, \text{ here } \mathcal{J}H(q,p) = H(-q,-p).$$

  – *Cut-offs*:

$$|\nu_1| + |\nu_2| \leq N_1\,,\ k \leq N_2\,,\ |\nu_1| + |\nu_2| + k \leq N_3\,,\ N \in \mathbb{N}^3\,.$$

  – *Period six*: for some of the better Hamiltonians on the critical stable manifold, $\mathcal{R}^{n+6}(H)$ was noticeably similar to $\mathcal{R}^n(H)$, modulo a translation $\mathcal{J}_\alpha$, where

$$\mathcal{J}_\alpha H = H \circ J_\alpha\,, \qquad J_\alpha(q,p) = (q + \pi\alpha, p)\,,$$

  with $\alpha$ a vector from $A = \{(1,0),(0,1),(1,1)\}$. Hence, $\mathcal{R}^{12}$ was substituted by $\mathcal{J}_\alpha \circ \mathcal{R}^6$.

  – *Symmetries*: $H_n^*$ is invariant under $\mathcal{S}_{-1} \circ \mathcal{J}_\alpha$.

– *Critical scaling parameter*: $\bar{\mu}^6 = \mu(H_0)\mu(H_1)\cdots\mu(H_5)$.

| $N$ | $\bar{\mu}$ | $\delta_1^{1/6}$ | $\delta_2^{1/6}$ |
|---|---|---|---|
| (7, 7, 9) | 0.36546 | 2.6620 | 1.5850 |
| (12, 8, 14) | 0.36594 | 2.6613 | 1.5850 |
| (16, 8, 18) | 0.36589 | 2.6612 | 1.5850 |

Eigenvalues describe the accumulation of the bifurcation points of orbits in two parameter families of Hamiltonians.

– *Eigenvalues of the scaling transformation at the symmetry point* $(0,0)$: $(\vartheta/\bar{\mu})^{12} H_n^* \circ \Lambda_n = H_n^*$, $\Lambda_n = T_{\mu(H_n)} \circ U_{H_n} \circ \ldots \circ T_{\mu(H_{n+11})} \circ U_{H_{n+11}}$.

| $N$ | $\lambda_1^{1/12}/\vartheta - 1$ | $\lambda_2^{1/12}$ | $(\lambda_2\lambda_3)^{1/12}/\bar{\mu} - 1$ | $\vartheta\lambda_4^{1/12}/\bar{\mu} - 1$ |
|---|---|---|---|---|
| (7, 7, 9) | $5.2 \times 10^{-12}$ | 0.65643 | 0.016 | $-4.1 \times 10^{-11}$ |
| (12, 8, 14) | $5.2 \times 10^{-12}$ | 0.65704 | 0.017 | $-2.2 \times 10^{-11}$ |
| (16, 8, 18) | $5.2 \times 10^{-12}$ | 0.65694 | 0.017 | $5.6 \times 10^{-11}$ |

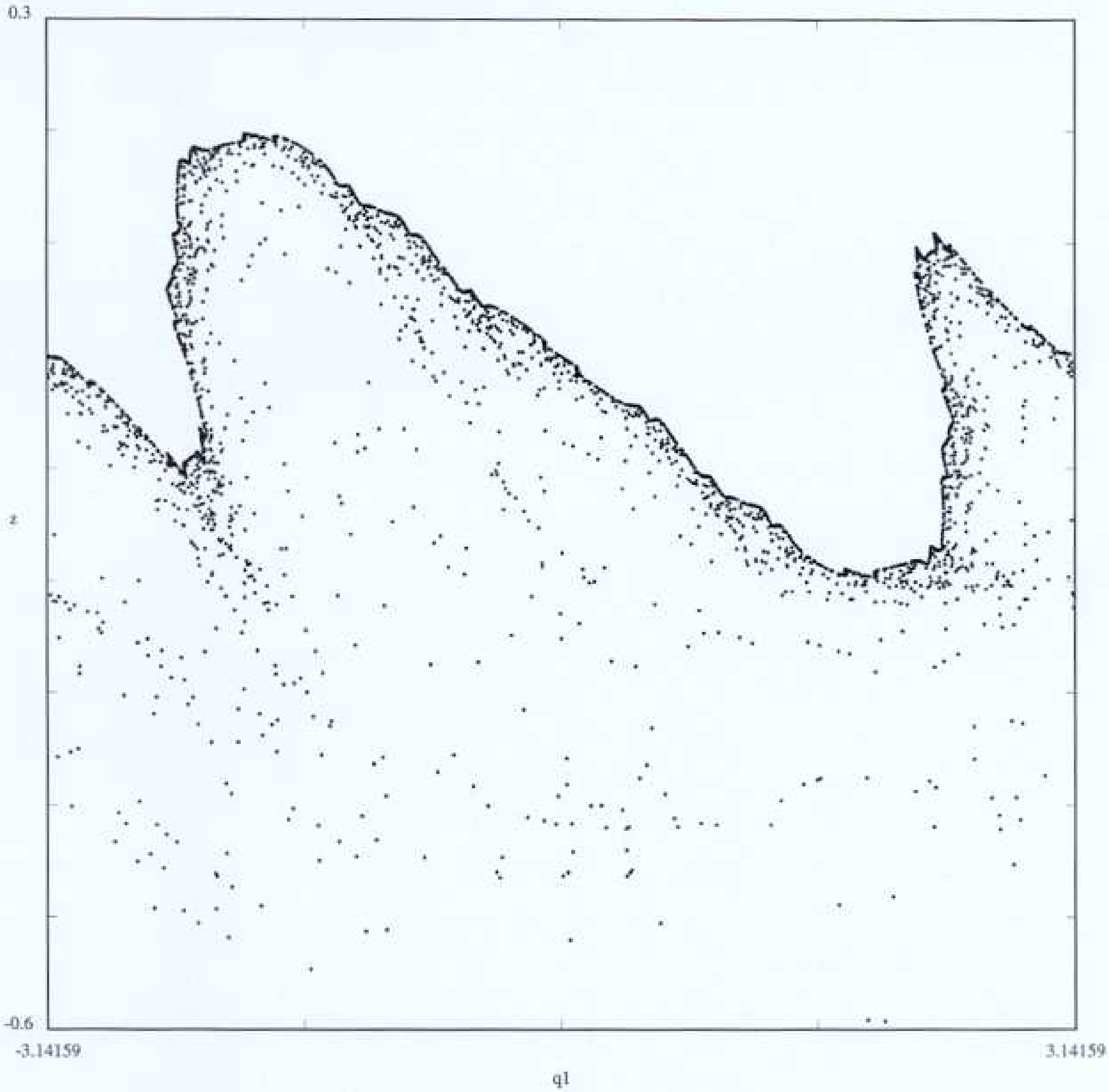


Figure 1: *Orbits for the return map to the cylinder* $q_2 = 0$ *for a critical Hamiltonian at energy zero.*

• **Conclusion**:

Renormalization is a successful non-perturbative alternative to perturbative KAM techniques and (probably) the only hope for a rigorous explanation of the universality of critical phenomena in degenerate Hamiltonian flows.

• **Open (and very hard) questions**:

1) A (computer-assisted ??) proof of existence of the period 12.
2) A (computer-assisted ??) proof of the hyperbolicity of the renormalization operator at this period 12.
3) Do 1) + 2) imply the Greene's criterion for existence of an invariant surface?