A Theory of Rods with Microstructure as a Model for Double Stranded Rods

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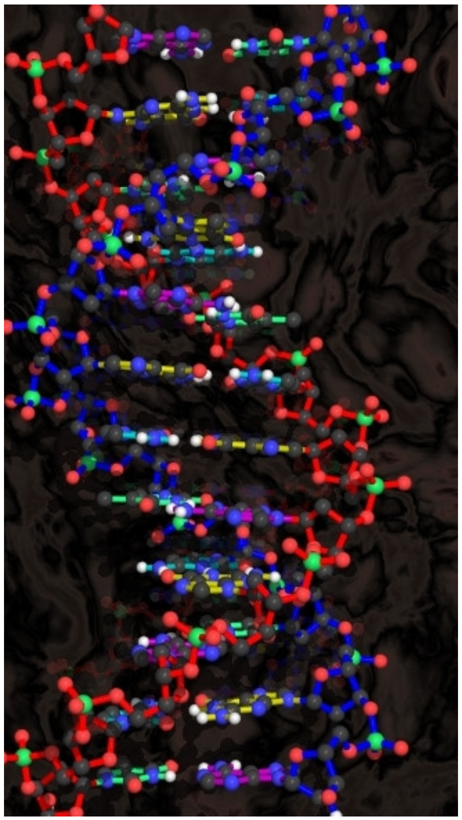
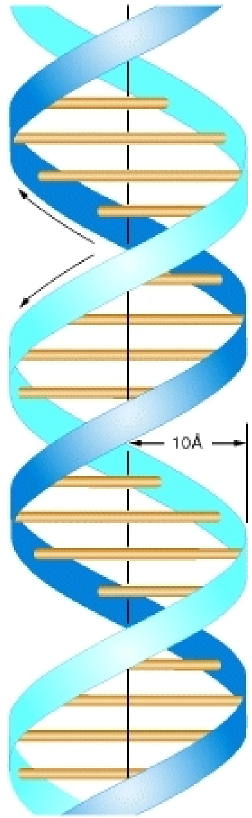
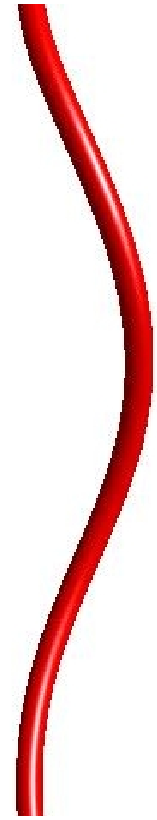
Motivation: DNA Modeling

All Atoms | Watson-Crick | Elastic Rod

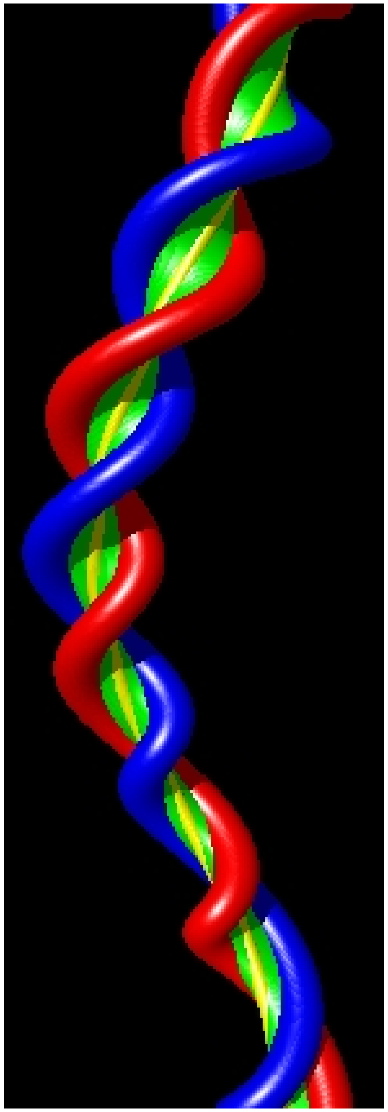






Primary Structure | Secondary Structure | Tertiary Structure

CONFIGURATION



We consider a ribbon $\mathscr{R}$ composed of two special Cosserat rods $\mathscr{R}^+$ and $\mathscr{R}^-$, which we call strands, that are bound together elastically.

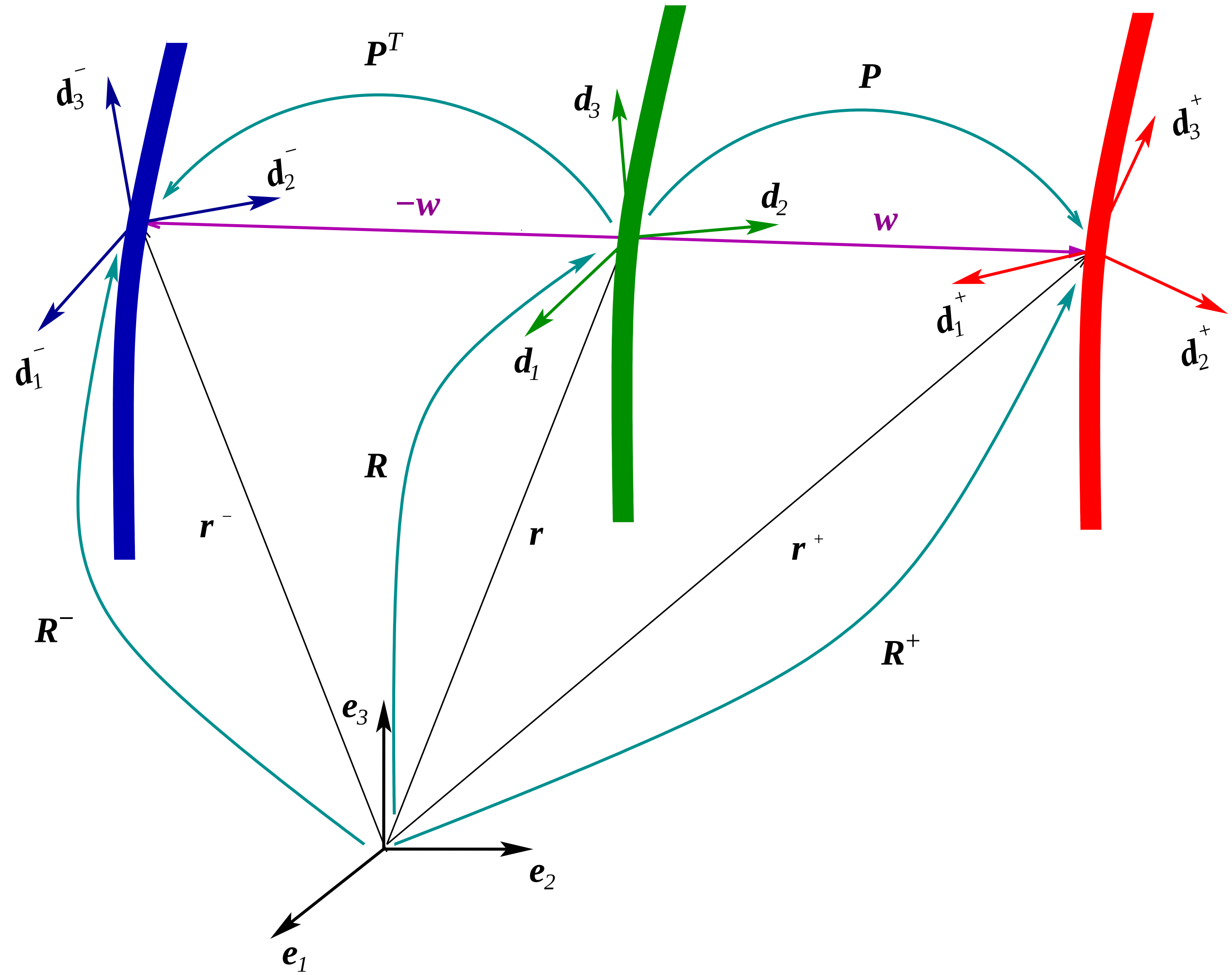
Each strand is completely described by

- a position vector $\boldsymbol{r}^{\pm}(s,t)$,
- a frame of directors $\{\boldsymbol{d}_i^{\pm}(s,t)\}$.

The curves $\mathscr{C}^{\pm}(t) \equiv \{\boldsymbol{r}^{\pm}(s,t),\ s \in [0,L]\}$ represent the lines of centroids, in the current configuration, of $\mathscr{R}^{\pm}$.

The triads $\{\boldsymbol{d}_i^{\pm}(s,t)\}$ give the orientation of the material cross section at s of $\mathscr{R}^{\pm}$.

Configuration: Schematic



KINEMATICS: Strains associated with the centerlines

We define the strain vectors $\boldsymbol{v}$ and $\boldsymbol{v}^{\pm}$ by

$$\begin{aligned}
\boldsymbol{v} &:= \boldsymbol{r}_s, \\
\boldsymbol{v}^{+} &:= \boldsymbol{r}_s^{+}, \\
\boldsymbol{v}^{-} &:= \boldsymbol{r}_s^{-}.
\end{aligned}$$

Then we have the following relations

$$\begin{aligned}
\boldsymbol{v}^{+} &= \boldsymbol{v} + \boldsymbol{w}_s, \\
\boldsymbol{v}^{-} &= \boldsymbol{v} - \boldsymbol{w}_s.
\end{aligned}$$

KINEMATICS: Strains associated with the directors

Let $\boldsymbol{u}$ and $\boldsymbol{u}^{\pm}$ be the (Darboux) vectors for $\boldsymbol{R}$ and $\boldsymbol{R}^{\pm}$, i.e., such that

$$\boldsymbol{R}_s \boldsymbol{R}^T = \boldsymbol{u}^{\times},$$
$$\boldsymbol{R}_s^+ \boldsymbol{R}^{+T} = (\boldsymbol{u}^+)^{\times},$$
$$\boldsymbol{R}_s^- \boldsymbol{R}^{-T} = (\boldsymbol{u}^-)^{\times}.$$

Given a vector $\boldsymbol{a}$, we denote by $\boldsymbol{a}^{\times}$ the second order skew-symmetric tensor such that, for any vector $\boldsymbol{b}$,

$$\boldsymbol{a}^{\times} \boldsymbol{b} = \boldsymbol{a} \times \boldsymbol{b}.$$

We have the following relations

$$\boldsymbol{u}^+ = \boldsymbol{u} + \alpha \boldsymbol{A} \boldsymbol{\eta}^\triangledown,$$
$$\boldsymbol{u}^- = \boldsymbol{u} - \alpha \boldsymbol{A}^T \boldsymbol{\eta}^\triangledown,$$

where

$\boldsymbol{\eta}$ is the Gibbs rotation vector of $\boldsymbol{P}$,

$\boldsymbol{A} = \boldsymbol{I} + \boldsymbol{\eta}^\times,$

$\alpha = 2/(1 + \boldsymbol{\eta} \cdot \boldsymbol{\eta}),$

$\boldsymbol{\eta}^\triangledown = \boldsymbol{\eta}_s - \boldsymbol{u} \times \boldsymbol{\eta}.$

$$\boldsymbol{P} = \boldsymbol{I} + \alpha[\boldsymbol{\eta}^\times + (\boldsymbol{\eta}^\times)^2].$$

KINEMATICS: Linear velocities

We define the vectors γ and $\gamma^{\pm}$ by

$$\begin{aligned} \boldsymbol{\gamma} &:= \boldsymbol{r}_t, \\ \boldsymbol{\gamma}^{+} &:= \boldsymbol{r}_t^{+}, \\ \boldsymbol{\gamma}^{-} &:= \boldsymbol{r}_t^{-}. \end{aligned}$$

Then we have the following relations

$$\begin{aligned} \boldsymbol{\gamma}^{+} &= \boldsymbol{\gamma} + \boldsymbol{w}_t, \\ \boldsymbol{\gamma}^{-} &= \boldsymbol{\gamma} - \boldsymbol{w}_t. \end{aligned}$$

KINEMATICS: Angular velocities

Let $\boldsymbol{\omega}$ and $\boldsymbol{\omega}^{\pm}$ be the angular velocities for $\boldsymbol{R}$ and $\boldsymbol{R}^{\pm}$, i.e., such that

$$\begin{aligned}
\boldsymbol{R}_t \boldsymbol{R}^T &= \boldsymbol{\omega}^{\times}, \\
\boldsymbol{R}_t^+ {\boldsymbol{R}^+}^T &= (\boldsymbol{\omega}^+)^{\times}, \\
\boldsymbol{R}_t^- {\boldsymbol{R}^-}^T &= (\boldsymbol{\omega}^-)^{\times}.
\end{aligned}$$

We then have the following relations

$$\begin{aligned}
\boldsymbol{\omega}^+ &= \boldsymbol{\omega} + \alpha \boldsymbol{A} \mathring{\boldsymbol{\eta}}, \\
\boldsymbol{\omega}^- &= \boldsymbol{\omega} - \alpha \boldsymbol{A}^T \mathring{\boldsymbol{\eta}},
\end{aligned}$$

where

$$\mathring{\boldsymbol{\eta}} = \boldsymbol{\eta}_t - \boldsymbol{\omega} \times \boldsymbol{\eta}.$$

Two Paradigms

Configuration Variables

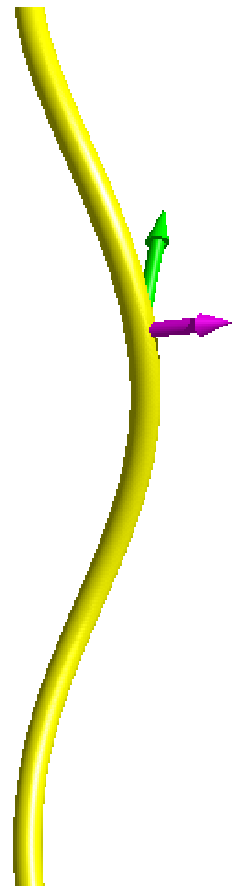
$$\{(\boldsymbol{r}^+, \boldsymbol{R}^+), (\boldsymbol{r}^-, \boldsymbol{R}^-)\}$$

Configuration Space

$$\{\mathbb{R}^3 \times SO(3)\}^2$$

Double-Stranded Rod (Bi-rod)

$$\Longleftrightarrow$$



Rod with Microstructure

Configuration Variables

$$(\boldsymbol{r}, \boldsymbol{R}, \boldsymbol{w}, \boldsymbol{P})$$

Configuration Space

$$\mathbb{R}^3 \times SO(3) \times \mathbb{R}^3 \times SO(3)$$

ACTION DENSITY FOR A ROD WITH MICROSTRUCTURES

We consider a special Cosserat rod $\mathscr{R}$ endowed with two microstructure variables:

- a vector $\boldsymbol{w}$
- a rotation tensor $\boldsymbol{P}$.

Let $\mathscr{P}$ be a part of $\mathscr{R}$ which is defined by $s_1 \leq s \leq s_2$ and let $\mathscr{I}$ be the time interval $t_1 \leq t \leq t_2$.

To every set of events $\mathscr{P} \times \mathscr{I}$ we associate the action $\mathcal{A}$ given by

$$\mathcal{A}(\mathscr{P} \times \mathscr{I}) = \int_{t_1}^{t_2} \int_{s_1}^{s_2} \widetilde{\mathfrak{L}}(s, t, \boldsymbol{r}, \boldsymbol{r}_s, \boldsymbol{r}_t, \boldsymbol{R}, \boldsymbol{R}_s, \boldsymbol{R}_t, \boldsymbol{w}, \boldsymbol{w}_s, \boldsymbol{w}_t, \boldsymbol{P}, \boldsymbol{P}_s, \boldsymbol{P}_t) \, ds dt.$$

The density of action $\widetilde{\mathfrak{L}}$ is function of s, t and the macro- and micro-kinematical variables as well as their spatial and time derivatives.

Modified Form of the Action Density

It is more convenient to work with $\boldsymbol{w}^{\triangledown}$ and $\overset{\circ}{\boldsymbol{w}}$ instead of $\boldsymbol{w}_s$ and $\boldsymbol{w}_t$ and with $\boldsymbol{\eta}$, $\alpha\boldsymbol{\eta}^{\triangledown}$ and $\alpha\overset{\circ}{\boldsymbol{\eta}}$ instead of $\boldsymbol{P}$, $\boldsymbol{P}_s$ and $\boldsymbol{P}_t$.

We therefore introduce the variables

$$\boldsymbol{v}^c = \boldsymbol{w}^{\triangledown}, \qquad \boldsymbol{\gamma}^c = \overset{\circ}{\boldsymbol{w}},$$
$$\boldsymbol{u}^c = \alpha\boldsymbol{\eta}^{\triangledown}, \qquad \boldsymbol{\omega}^c = \alpha\overset{\circ}{\boldsymbol{\eta}},$$

where $\alpha = 2/(1+\boldsymbol{\eta}\cdot\boldsymbol{\eta})$, and for any vector $\boldsymbol{a}$ we have

$$\boldsymbol{a}^{\triangledown} = \boldsymbol{a}_s - \boldsymbol{R}_s\boldsymbol{R}\boldsymbol{a}, \quad \overset{\circ}{\boldsymbol{a}} = \boldsymbol{a}_t - \boldsymbol{R}_t\boldsymbol{R}\boldsymbol{a}.$$

Accordingly, we define a new density of action $\mathfrak{L}$ by

$$\begin{aligned}\mathfrak{L}(s,t,\boldsymbol{r},\boldsymbol{r}_s,\boldsymbol{r}_t,\boldsymbol{R},\boldsymbol{R}_s,\boldsymbol{R}_t,\boldsymbol{w},\boldsymbol{v}^c,\boldsymbol{\gamma}^c,\boldsymbol{\eta},\boldsymbol{u}^c,\boldsymbol{\omega}^c) &:= \\ &\widetilde{\mathfrak{L}}(s,t,\boldsymbol{r},\boldsymbol{r}_s,\boldsymbol{r}_t,\boldsymbol{R},\boldsymbol{R}_s,\boldsymbol{R}_t,\boldsymbol{w},\boldsymbol{w}_s,\boldsymbol{w}_t,\boldsymbol{P},\boldsymbol{P}_s,\boldsymbol{P}_t).\end{aligned}$$

Euclidean Invariance

We require $\mathfrak{L}$ be invariant under the group of Euclidean displacements:

- $\mathfrak{L}$ must be independent of t and $\boldsymbol{r}$.
- $\mathfrak{L}$ depends on $\boldsymbol{R}$ only through the reduced variables

$$\begin{array}{lll} & \mathbf{v} = \boldsymbol{R}^T \boldsymbol{r}', & \boldsymbol{\gamma} = \boldsymbol{R}^T \dot{\boldsymbol{r}}, \\ & \mathbf{u}^\times = \boldsymbol{R}^T \boldsymbol{R}', & \boldsymbol{\omega}^\times = \boldsymbol{R}^T \dot{\boldsymbol{R}}, \\ \mathbf{w} = \boldsymbol{R}^T \boldsymbol{w}, & \mathbf{v}^c = \boldsymbol{R}^T \boldsymbol{v}^c, & \boldsymbol{\gamma}^c = \boldsymbol{R}^T \gamma^c, \\ \boldsymbol{\eta} = \boldsymbol{R}^T \boldsymbol{\eta}, & \mathbf{u}^c = \boldsymbol{R}^T \boldsymbol{u}^c, & \boldsymbol{\omega}^c = \boldsymbol{R}^T \boldsymbol{\omega}^c, \end{array}$$

We therefore obtain the following reduced form of the action density

$$\mathcal{L}(s, \mathbf{v}, \boldsymbol{\gamma}, \mathbf{u}, \boldsymbol{\omega}, \mathbf{w}, \mathbf{v}^c, \boldsymbol{\gamma}^c, \boldsymbol{\eta}, \mathbf{u}^c, \boldsymbol{\omega}^c) := \\ \mathfrak{L}(s, \boldsymbol{R}^T \boldsymbol{r}_s, \boldsymbol{R}^T \boldsymbol{r}_t, \boldsymbol{R}^T \boldsymbol{R}_s, \boldsymbol{R}^T \boldsymbol{R}_t, \boldsymbol{R}^T \boldsymbol{w}, \boldsymbol{R}^T \boldsymbol{v}^c, \boldsymbol{R}^T \gamma^c, \boldsymbol{R}^T \boldsymbol{\eta}, \boldsymbol{R}^T \boldsymbol{u}^c, \boldsymbol{R}^T \boldsymbol{\omega}^c).$$

Further, we assume that $\mathcal{L}$ has the separated form

$$\mathcal{L}(s, \mathbf{v}, \boldsymbol{\gamma}, \mathbf{u}, \boldsymbol{\omega}, \mathbf{w}, \mathbf{v}^c, \boldsymbol{\gamma}^c, \boldsymbol{\eta}, \mathbf{u}^c, \boldsymbol{\omega}^c) = \mathcal{K}(s, \boldsymbol{\gamma}, \boldsymbol{\omega}, \mathbf{w}, \boldsymbol{\gamma}^c, \boldsymbol{\eta}, \boldsymbol{\omega}^c) - \mathcal{W}(s, \mathbf{v}, \mathbf{u}, \mathbf{w}, \mathbf{v}^c, \boldsymbol{\eta}, \mathbf{u}^c).$$

Hamilton's Principle

According to Hamilton's principle, the equations of motion are given by Euler-Poincaré equations of the variational equation

$$\delta\mathcal{A}(\mathscr{P} \times \mathscr{I}) = 0.$$

We obtain

$$\left(\boldsymbol{R}\frac{\partial\mathcal{W}}{\partial\mathbf{v}}\right)_s - \left(\boldsymbol{R}\frac{\partial\mathcal{K}}{\partial\boldsymbol{\gamma}}\right)_t = \mathbf{0},$$

$$\left(\boldsymbol{R}\frac{\partial\mathcal{W}}{\partial\mathbf{u}}\right)_s + \boldsymbol{r}_s \times \boldsymbol{R}\frac{\partial\mathcal{W}}{\partial\mathbf{v}} - \left(\boldsymbol{R}\frac{\partial\mathcal{K}}{\partial\boldsymbol{\omega}}\right)_t - \boldsymbol{r}_t \times \boldsymbol{R}\frac{\partial\mathcal{K}}{\partial\boldsymbol{\gamma}} = \mathbf{0},$$

$$\boldsymbol{R}\left[\frac{\partial\mathcal{W}}{\partial\mathbf{w}} - \left(\frac{\partial\mathcal{W}}{\partial\mathbf{v}^c}\right)_s - \frac{\partial\mathcal{K}}{\partial\mathbf{w}} + \left(\frac{\partial\mathcal{K}}{\partial\boldsymbol{\gamma}^c}\right)_t\right] = \mathbf{0},$$

$$\boldsymbol{R}\left[\frac{\partial\mathcal{W}}{\partial\boldsymbol{\eta}} - \left(\alpha\frac{\partial\mathcal{W}}{\partial\mathbf{u}^c}\right)_s - \frac{\partial\mathcal{K}}{\partial\boldsymbol{\eta}} + \left(\alpha\frac{\partial\mathcal{K}}{\partial\boldsymbol{\omega}^c}\right)_t\right] + \alpha\left(\boldsymbol{\omega}^c \cdot \boldsymbol{R}\frac{\partial\mathcal{K}}{\partial\boldsymbol{\omega}^c} - \boldsymbol{u}^c \cdot \boldsymbol{R}\frac{\partial\mathcal{W}}{\partial\mathbf{u}^c}\right)\boldsymbol{\eta} = \mathbf{0}.$$

Constitutive Equations

$$\boldsymbol{p} = \boldsymbol{R}\frac{\partial \mathcal{K}}{\partial \boldsymbol{\gamma}}, \qquad \boldsymbol{n} = \boldsymbol{R}\frac{\partial \mathcal{W}}{\partial \mathbf{v}},$$

$$\boldsymbol{\pi} = \boldsymbol{R}\frac{\partial \mathcal{K}}{\partial \boldsymbol{\omega}}, \qquad \boldsymbol{m} = \boldsymbol{R}\frac{\partial \mathcal{W}}{\partial \mathbf{u}},$$

$$\boldsymbol{p}^c = \boldsymbol{R}\frac{\partial \mathcal{K}}{\partial \boldsymbol{\gamma}^c}, \qquad \boldsymbol{n}^c = \boldsymbol{R}\frac{\partial \mathcal{W}}{\partial \mathbf{v}^c},$$

$$\boldsymbol{\pi}^c = \boldsymbol{R}\frac{\partial \mathcal{K}}{\partial \boldsymbol{\omega}^c} + \boldsymbol{w}\times\boldsymbol{p} - \boldsymbol{\eta}\times(\boldsymbol{\pi} - \boldsymbol{w}\times\boldsymbol{p}^c), \qquad \boldsymbol{m}^c = \boldsymbol{R}\frac{\partial \mathcal{W}}{\partial \mathbf{u}^c} + \boldsymbol{w}\times\boldsymbol{n} - \boldsymbol{\eta}\times(\boldsymbol{m} - \boldsymbol{w}\times\boldsymbol{n}^c),$$

$$2\boldsymbol{\sigma} = \boldsymbol{R}\frac{\partial \mathcal{K}}{\partial \mathbf{w}} + \boldsymbol{\omega}\times\boldsymbol{p}^c, \qquad 2\boldsymbol{f} = \boldsymbol{R}\frac{\partial \mathcal{W}}{\partial \mathbf{w}} + \boldsymbol{u}\times\boldsymbol{n}^c,$$

$$2\boldsymbol{\tau} = \frac{1}{\alpha}\boldsymbol{R}\frac{\partial \mathcal{K}}{\partial \boldsymbol{\eta}} + \boldsymbol{\omega}\times\boldsymbol{\pi}^c - \boldsymbol{\omega}^c\times(\boldsymbol{\eta}\times\boldsymbol{\pi}^c) + \ldots \qquad 2\boldsymbol{c} = \frac{1}{\alpha}\boldsymbol{R}\frac{\partial \mathcal{W}}{\partial \boldsymbol{\eta}} + \boldsymbol{u}\times\boldsymbol{m}^c - \boldsymbol{u}^c\times(\boldsymbol{\eta}\times\boldsymbol{m}^c) + \ldots$$

Equations of Motion

$$\begin{aligned}
\boldsymbol{n}_s &= \boldsymbol{p}_t, \\
\boldsymbol{m}_s + \boldsymbol{r}_s \times \boldsymbol{n} &= \boldsymbol{\pi}_t + \boldsymbol{r}_t \times \boldsymbol{p}, \\
\boldsymbol{n}^c_s - 2\boldsymbol{f} &= \boldsymbol{p}^c_t - 2\boldsymbol{\sigma}, \\
\boldsymbol{m}^c_s + \boldsymbol{r}_s \times \boldsymbol{n}^c - 2\boldsymbol{c} &= \boldsymbol{\pi}^c_t + \boldsymbol{r}_t \times \boldsymbol{p}^c - 2\boldsymbol{\tau}.
\end{aligned}$$

Specialization to Double-Stranded Rods

When

$$\mathcal{W}(s; \mathbf{v}, \mathbf{u}, \mathbf{w}, \mathbf{v}^c, \boldsymbol{\eta}, \mathbf{u}^c) = \widetilde{\mathcal{W}}(s; \mathbf{v}^+, \mathbf{v}^-, \mathbf{u}^+, \mathbf{u}^-, \mathbf{w}, \boldsymbol{\eta}),$$

$$\mathcal{K}(s; \boldsymbol{\gamma}, \boldsymbol{\omega}, \mathbf{w}, \boldsymbol{\gamma}^c, \boldsymbol{\eta}, \boldsymbol{\omega}^c) = \widetilde{\mathcal{K}}(s; \boldsymbol{\gamma}^+, \boldsymbol{\gamma}^-, \boldsymbol{\omega}^+, \boldsymbol{\omega}^-, \mathbf{w}, \boldsymbol{\eta}),$$

we obtain

$$\boldsymbol{p} = \boldsymbol{p}^+ + \boldsymbol{p}^-, \qquad \boldsymbol{n} = \boldsymbol{n}^+ + \boldsymbol{n}^-,$$

$$\boldsymbol{\pi} = \boldsymbol{\pi}^+ + \boldsymbol{\pi}^- + \boldsymbol{w} \times (\boldsymbol{p}^+ - \boldsymbol{p}^-), \qquad \boldsymbol{m} = \boldsymbol{m}^+ + \boldsymbol{m}^- + \boldsymbol{w} \times (\boldsymbol{n}^+ - \boldsymbol{n}^-),$$

$$\boldsymbol{p}^c = \boldsymbol{p}^+ - \boldsymbol{p}^-, \qquad \boldsymbol{n}^c = \boldsymbol{n}^+ - \boldsymbol{n}^-,$$

$$\boldsymbol{\pi}^c = \boldsymbol{\pi}^+ - \boldsymbol{\pi}^- + \boldsymbol{w} \times (\boldsymbol{p}^+ + \boldsymbol{p}^-), \qquad \boldsymbol{m}^c = \boldsymbol{m}^+ - \boldsymbol{m}^- + \boldsymbol{w} \times (\boldsymbol{n}^+ + \boldsymbol{n}^-),$$

$$2\boldsymbol{\sigma} = \boldsymbol{R}\frac{\partial \widetilde{\mathcal{K}}}{\partial \mathbf{w}}, \qquad 2\boldsymbol{f} = \boldsymbol{R}\frac{\partial \widetilde{\mathcal{W}}}{\partial \mathbf{w}},$$

$$2\boldsymbol{\tau} = \frac{1}{\alpha}\boldsymbol{R}\frac{\partial \widetilde{\mathcal{K}}}{\partial \boldsymbol{\eta}} - \boldsymbol{\eta} \times (\boldsymbol{w} \times \boldsymbol{\sigma}), \qquad 2\boldsymbol{c} = \frac{1}{\alpha}\boldsymbol{R}\frac{\partial \widetilde{\mathcal{W}}}{\partial \boldsymbol{\eta}} - \boldsymbol{\eta} \times (\boldsymbol{w} \times \boldsymbol{f}),$$

where

$$\boldsymbol{p}^{+} = \boldsymbol{P}\boldsymbol{R}\frac{\partial\widetilde{\mathcal{K}}}{\partial\boldsymbol{\gamma}^{+}}, \qquad \boldsymbol{p}^{-} = \boldsymbol{P}^{T}\boldsymbol{R}\frac{\partial\widetilde{\mathcal{K}}}{\partial\boldsymbol{\gamma}^{-}},$$

$$\boldsymbol{\pi}^{+} = \boldsymbol{P}\boldsymbol{R}\frac{\partial\widetilde{\mathcal{K}}}{\partial\boldsymbol{\omega}^{+}}, \qquad \boldsymbol{\pi}^{-} = \boldsymbol{P}^{T}\boldsymbol{R}\frac{\partial\widetilde{\mathcal{K}}}{\partial\boldsymbol{\omega}^{-}},$$

$$\boldsymbol{n}^{+} = \boldsymbol{P}\boldsymbol{R}\frac{\partial\widetilde{\mathcal{W}}}{\partial\mathbf{v}^{+}}, \qquad \boldsymbol{n}^{-} = \boldsymbol{P}^{T}\boldsymbol{R}\frac{\partial\widetilde{\mathcal{W}}}{\partial\mathbf{v}^{-}},$$

$$\boldsymbol{m}^{+} = \boldsymbol{P}\boldsymbol{R}\frac{\partial\widetilde{\mathcal{W}}}{\partial\mathbf{u}^{+}}, \qquad \boldsymbol{m}^{-} = \boldsymbol{P}^{T}\boldsymbol{R}\frac{\partial\widetilde{\mathcal{W}}}{\partial\mathbf{u}^{-}},$$

Linearly-elastic Strands

$$\begin{pmatrix} \boldsymbol{p} \\ \boldsymbol{p}^c \end{pmatrix} = \rho \begin{pmatrix} \boldsymbol{r}_t \\ \boldsymbol{w}_t \end{pmatrix},$$

$$\begin{pmatrix} \boldsymbol{\pi} \\ \boldsymbol{\pi}^c \end{pmatrix} = \begin{pmatrix} \mathbb{J} & \mathbb{J}^c \\ \mathbb{J}^c & \mathbb{J} \end{pmatrix} \begin{pmatrix} \boldsymbol{\omega} + \boldsymbol{\eta} \times \boldsymbol{\omega}^c \\ \boldsymbol{\omega}^c \end{pmatrix} + \begin{pmatrix} \boldsymbol{w} \times \boldsymbol{p}^c \\ \boldsymbol{w} \times \boldsymbol{p} \end{pmatrix},$$

$$\begin{pmatrix} \boldsymbol{n} \\ \boldsymbol{n}^c \end{pmatrix} = \begin{pmatrix} \mathbb{L} & \mathbb{L}^c \\ \mathbb{L}^c & \mathbb{L} \end{pmatrix} \left\{ \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{v}^c + \boldsymbol{w} \times \boldsymbol{u} \end{pmatrix} - \begin{pmatrix} \boldsymbol{E} & \boldsymbol{E}^c \\ \boldsymbol{E}^c & \boldsymbol{E} \end{pmatrix} \begin{pmatrix} \widetilde{\boldsymbol{v}} \\ \widetilde{\boldsymbol{v}}^c + \widetilde{\boldsymbol{w}} \times \widetilde{\boldsymbol{u}} \end{pmatrix} \right\},$$

$$\begin{pmatrix} \boldsymbol{m} \\ \boldsymbol{m}^c \end{pmatrix} = \begin{pmatrix} \mathbb{K} & \mathbb{K}^c \\ \mathbb{K}^c & \mathbb{K} \end{pmatrix} \left\{ \begin{pmatrix} \boldsymbol{u} + \boldsymbol{\eta} \times \boldsymbol{u}^c \\ \boldsymbol{u}^c \end{pmatrix} - \begin{pmatrix} \boldsymbol{E} & \boldsymbol{E}^c \\ \boldsymbol{E}^c & \boldsymbol{E} \end{pmatrix} \begin{pmatrix} \widetilde{\boldsymbol{u}} + \widetilde{\boldsymbol{\eta}} \times \widetilde{\boldsymbol{u}}^c \\ \widetilde{\boldsymbol{u}}^c \end{pmatrix} \right\} + \begin{pmatrix} \boldsymbol{w} \times \boldsymbol{n}^c \\ \boldsymbol{w} \times \boldsymbol{n} \end{pmatrix},$$

where

$$\rho = \rho^+ + \rho^-$$

$$\mathbb{M} = \mathbb{M}^+ + \mathbb{M}^-, \qquad \mathbb{M}^c = \mathbb{M}^+ - \mathbb{M}^-,$$

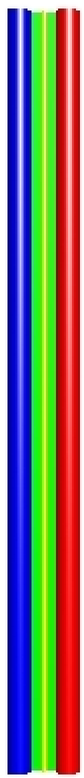
$$\mathbb{M}^+ = \boldsymbol{P}\boldsymbol{R}\mathsf{M}^+\boldsymbol{R}^T\boldsymbol{P}^T, \qquad \mathbb{M}^- = \boldsymbol{P}^T\boldsymbol{R}\mathsf{M}^-\boldsymbol{R}^T\boldsymbol{P},$$

$$\boldsymbol{E} = (\boldsymbol{P}\boldsymbol{R}\widetilde{\boldsymbol{R}}^T\widetilde{\boldsymbol{P}}^T + \boldsymbol{P}^T\boldsymbol{R}\widetilde{\boldsymbol{R}}^T\widetilde{\boldsymbol{P}})/2, \qquad \boldsymbol{E}^c = (\boldsymbol{P}\boldsymbol{R}\widetilde{\boldsymbol{R}}^T\widetilde{\boldsymbol{P}}^T - \boldsymbol{P}^T\boldsymbol{R}\widetilde{\boldsymbol{R}}^T\widetilde{\boldsymbol{P}})/2.$$

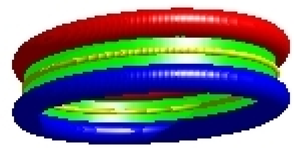
Example: A Double Ring

Reference configuration (unstressed)

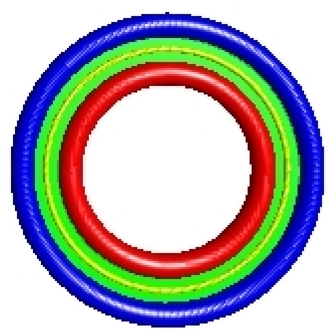
$$\hat{\boldsymbol{r}} = s\boldsymbol{e}_3$$
$$\hat{\boldsymbol{w}} = \hat{h}\boldsymbol{e}_2$$
$$s \in [0, L]$$
$$L = 2\pi R$$



Low strain energy bi-ring



$$\begin{cases} \boldsymbol{r} = R(\cos\frac{s}{R}\boldsymbol{e}_1 + \sin\frac{s}{R}\boldsymbol{e}_3) \\ \boldsymbol{w} = \hat{h}\boldsymbol{e}_2 \\ \boldsymbol{n} = \boldsymbol{0}, \quad \boldsymbol{n}^c = \boldsymbol{0} \\ \boldsymbol{m} = \frac{2K}{R}\boldsymbol{e}_2, \quad \boldsymbol{m}^c = \boldsymbol{0} \end{cases}$$



High strain energy bi-ring

$$\begin{cases} \boldsymbol{r} = R(\cos\frac{s}{R}\boldsymbol{e}_2 + \sin\frac{s}{R}\boldsymbol{e}_3) \\ \boldsymbol{w} = h(\cos\frac{s}{R}\boldsymbol{e}_2 + \sin\frac{s}{R}\boldsymbol{e}_3) \\ h = \frac{G\hat{h}}{A/R^2 + G} \\ \boldsymbol{n} = \boldsymbol{0}, \quad \boldsymbol{n}^c = \frac{2Ah}{R}\boldsymbol{r}' \\ \boldsymbol{m} = \frac{2(K + Ah^2)}{R}\boldsymbol{e}_1, \quad \boldsymbol{m}^c = \boldsymbol{0} \end{cases}$$