Problem

Understanding the asymptotic behavior, as $\varepsilon \downarrow 0$, of the variational problems generated by the energies

$$F_\varepsilon(u) = \int_\Omega \frac{(1 - |\nabla u|^2)^2}{\varepsilon} + \varepsilon |\nabla\nabla u|^2$$

where $u : \mathbf{R}^2 \supset \Omega \to \mathbf{R}$

and some boundary conditions must be added.

Motivations

a) Smectic liquid crystals:

P. Aviles, Y. Giga (1987)

b) Thin-films under biaxial compression:

G. Gioia, M. Ortiz (1994)

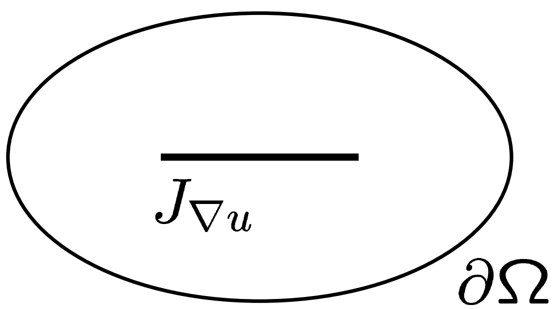
c) Micromagnetics:

A. De Simone, R. Kohn, S. Müller, F. Otto.

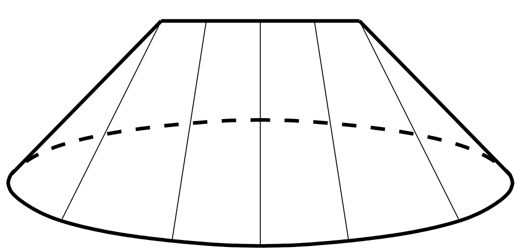
d) Analogies with lower order functionals (successfully studied by many authors).

A solution of the eikonal equation:

Ω is an ellipse
$u(x) = \text{dist}\,(x, \partial\Omega)$
(Hence $u \equiv 0$ on $\partial\Omega$).



graph of u



A conjectured asymptotic energy

Aviles–Giga 87

If $\nabla u = 1$–a.e.

∇u is "smooth" out of a 1–d set J_u

∇u has right and left trace on J_u

then

the asymptotic energy of u should be

$$F(u) = \int_{J_{\nabla u}} |\nabla u^{+} - \nabla u^{-}|^{3} \, d\mathcal{H}^{1}$$

Results

Compactness If $\limsup_\varepsilon F_\varepsilon(u_\varepsilon) \leq K < \infty$, then $\{u_\varepsilon\}$ is strongly precompact in $W^{1,3}$.
(Ambrosio–D.–Mantegazza 99
De Simone–Kohn–Müller–Otto 99).

Asymptotic functional The conjectured asymptotic energy F has a weak formulation $\tilde{F}$ in a larger space. Direct methods give the existence of minimizers.
(Ambrosio–D.–Mantegazza 99)

Lower bound If an asymptotic energy G exists, then $G(u) \geq \tilde{F}(u)$
Jin–Kohn + Aviles–Giga 99
ADM for the weak setting

Upper bound G coincides with F on piecewise affine functions.
D. 00

Entropies

Jin–Kohn 99

If $|\nabla u| = 1$ and $\underline{u \text{ is smooth}}$, then

$$\partial_x[(u_x)^3] - \partial_y[(u_y)^3] = 0$$

There exist many Φ such that

$$\nabla \cdot [\Phi(\nabla u)] = 0\,.$$

Connections to the theory of hyperbolic conservation laws.

Non–smooth solutions

If u is not smooth, then $\nabla \cdot [\Phi(\nabla u)]$ detects the one–dimensional singularities of ∇u.

u piecewise $C^1 \Longrightarrow \nabla \cdot [\Phi(\nabla u)]$ is a 1–dimensional measure supported on the set of discontinuity of ∇u

Jin–Kohn computations:

a) $\|\nabla \cdot [\Phi(\nabla u)]\| \leq C_\Phi F(u)$;

b) If there exists $u_\varepsilon \to u$ with $F_\varepsilon(u_\varepsilon) \leq C$, then $F(u) \leq C$;

c) Some Φ's "calibrate" $F(u)$.

Weak setting

$$\mathcal{E} \ := \ \Big\{ \Phi \text{ entropies for } |\nabla u| = 1 \Big\}.$$

$$\mathcal{S} \ := \ \Big\{ |\nabla u| = 1 \text{ and } \mu_\Phi := \nabla \cdot [\Phi(\nabla u)] \\ \text{is a measure } \forall \Phi \in \mathcal{E} \Big\}$$

$$\mathcal{F}(u) \ := \ \text{"supremum" of } c_\Phi \nabla \cdot [\Phi(\nabla u)]$$

Natural questions

If $u \in \mathcal{S}$, is it true (in some suitable weak sense) that ∇u is "smooth" outside a 1–dimensional set $J_{\nabla u}$ and has right and left traces on $J_{\nabla u}$?

Can we compute $\mathcal{F}(u)$ as

$$\int_{J_{\nabla u}} |\nabla u^+ - \nabla u^-|^3 \, d\mathcal{H}^1 \, ?$$

Internal motivations

a) Natural path in variational formulations:
Classical $\Rightarrow$ Weak $\Rightarrow$ "Almost classical"

b) Mild regularity theorem for the class $\mathcal{S}$ and hence for a minimizer.

c) A first step towards closing the Γ–convergence problem (and hence towards a full answer to the starting problem).

BV functions

They would provide a good setting for the limiting variational problem. For two reasons:

Theorem 1 (DG, FF) *If $v : \mathbf{R}^m \to \mathbf{R}^k$ is a BV function, there exists a rectifiable set J_v of dimension $m-1$ s.t.:*

v has right and left trace $\mathcal{H}^{m-1}$ almost everywhere on J;

every $x \in \mathbf{R}^m \setminus J$ is a Lebesgue point for v.

Theorem 2 (Vol'pert chain rule) *v as above. We use the notation:*

$$Dv \;=\; Dv_{diffused} + (v^{+} - v^{-}) \otimes \nu \, \mathcal{H}^{m-1} \llcorner J_v.$$

If $F : \mathbf{R}^k \to \mathbf{R}^h$ is C^1, then

$$D(F(v)) \;=\; DF(v) Dv_{diffused} +$$

$$(F(v^{+}) - F(v^{-})) \otimes \nu \, \mathcal{H}^{m-1} \llcorner J_v.$$

Example 3 $v : \mathbf{R} \to \mathbf{R}$.

$$\begin{aligned} Dv &= Dv_{diffused} + Dv_{atomic} \\ &= Dv_d + \sum_{x \in J_v} [v^+(x) - v^-(x)]\, \delta_x \end{aligned}$$

$$\begin{aligned} D(F(v)) &= DF(v) Dv_d \\ &+ \sum_{x \in J_v} [F(v^+(x)) - F(v^-(x))]\, \delta_x \,. \end{aligned}$$

NO BV

- There exist sequences $\{u_\varepsilon\}$ such that

  $F_\varepsilon(u_\varepsilon) \leq K < \infty$,

  $u_\varepsilon \to u$

  $\nabla u \notin BV$.

  Ambrosio– D.–Mantegazza 99

- Vel. averaging lemmas $\Longrightarrow$

  ∇u almost in $W^{1/3,3/2}$

  Perthame–Jabin '00 – '01

- ∇u not better than $W^{1/3,3}$

  D.–Westdickenberg '02

Remark 4 *Almost optimality of Perthame–Jabin's velocity averaging lemmas.*

Two general problems

Problem 5 *Let* $\mathcal{E} \subset C^\infty(\mathbf{R}^n, \mathbf{R}^n)$ *be given. Let* $v \in L^\infty(\mathbf{R}^k, \mathbf{R}^n)$ *and assume that*

$$\nabla \cdot [\Phi(v)] \text{ is a measure } \forall \Phi \in \mathcal{E}\,.$$

Do the conclusions of De Giorgi–FF structure Theorem hold?

Problem 6 *Assume* $\mathcal{E}$ *is the set of entropies of an appropriate system of PDEs and assume that* v *is a solution of this system.*

Can we compute $\nabla \cdot [\Phi(v)]$ *in a "Vol'pert" way?*

External motivations

a) A regularity theory for entropy solutions to scalar conservation laws, allowing for rough initial data and undercompressive solutions.

b) A regularity theory for compensated– compactness solutions to 2×2 systems of conservation laws.

c) Connections with the theory of renormalized solutions
see Ambrosio 03 (Remark 3.5)
Ambrosio – Bouchut – D. 03 (final section).

Some answers

Theorem 7 (D.–Otto 02) $u \in \mathcal{S}$

Then there is J*, rectifiable, dim .1 s.t.*

(a) ∇u *is VMO outside* J

(b) ∇u *has right and left trace on* J

(c) $\mu_\Phi = \Phi(\nabla u^+) - \Phi(\nabla u^-)\mathcal{H}^1 \llcorner J + \nu$

(Some information on ν*)*

Theorem 8 (D.–O.–Westdick. 02)
$u \in L^\infty(\mathbf{R}^n, \mathbf{R})$ *weak solution of*

$$\nabla \cdot [F(u)] = 0 \tag{1}$$

$\mathcal{E}$ *is the set of entropies for* (1).
Some assumptions on F *needed.*

Then $\exists J$, *rectifiable, cod. 1, s.t.*

(a) u *is VMO outside* J

(b) u *has traces on* J

(c) relations between entropy prod. and traces

Theorem 9 (D.–Rivière 03)

$u \in L^\infty(\mathbf{R}, \mathbf{R})$ *is an* <u>*entropy*</u> *solution of*

$$\partial_t u + \partial_x[f(u)] = 0. \qquad (2)$$

Some assumptions on f.

Then $\exists J$, *rectifiable, dim. 1 s.t.*

(a) every $x \notin J$ *is a Lebesgue point*

(b) u *has traces on* J

(c) **A rule like Vol'pert chain rule applies to the entropy prod.**

Related works

Problem 10 *An asymptotic variational problem which is similar to the starting one and comes from micromagnetics (see the works of Rivière–Serfaty).*

For this problem there are also stronger informations (Ambrosio–Lecumberry–Rivière '02): Interesting (and nontrivial) connections with the viscosity solutions of the eikonal equation.

Exploiting these connections (and using blow–up arguments) one has a structure theorem (Ambrosio–Kirchheim–Lecumberry–Rivière 02)

These methods can be generalized to 1–d conservation laws with f strictly convex, when the entropy production is a *Radon measure*. (Lecumberry–Rivière 02)