A Variational Principle for certain dissipative evolution equations

N. Ghoussoub and Leo Tzou

Pacific Institute for the Mathematical Sciences

and The University of British Columbia

Is there a variational Principle for the Heat Equation?

Yes! Brezis-Ekeland (1976)

For the homogeneous heat equation in a smooth bounded domain Ω of $\mathbb{R}^n$.

Minimize the functional

$$I(u) = \int_0^T \left(\frac{1}{2} \int_\Omega (|\nabla u|^2 dx + \frac{1}{2} \int_\Omega |\nabla \Delta^{-1} \dot{u}_t|^2) dx \right) dt + \frac{1}{2} \int_\Omega |u(T)|^2 dx$$

on the set

$$K = \{ u \in C([0,T]; L^2(\Omega)); \int_\Omega |\nabla \Delta^{-1} \dot{u}_t|^2 dx \in L^1(0,T), u(0) = u_0 \}.$$

Euler-Lagrange equation:

$$\begin{cases} (\frac{\partial}{\partial t} - \Delta)(\frac{\partial}{\partial t} + \Delta)u & = & 0 \quad \text{a.e.} \quad \text{on} \quad [0, T] \\ u(0) & = & u_0 \end{cases}$$

However, if one shows that the infimum is actually equal to

$$\inf_K I = \frac{1}{2} \int_\Omega |u_0(x)|^2 dx$$

Then

$$\begin{cases} (\frac{\partial}{\partial t} - \Delta)u & = & 0 \quad \text{a.e.} \quad \text{on} \quad [0, T] \\ u(0) & = & u_0 \end{cases}$$

What is the trick?

$$\begin{cases} \dot{u}(t) + \partial\phi(u(t)) & = & 0 \quad \text{a.e.} \quad \text{on} \quad [0, T] \\ u(0) & = & 0 \end{cases}$$

where $\varphi : H \to \mathbb{R} \cup \{+\infty\}$ is a proper convex and lower semi-continuous functional on a Hilbert space H and where $\partial\varphi$ denotes its subdifferential map.

Let φ^* be the Legendre conjugate of φ on H defined as:

$$\varphi^*(y) = \sup\{\langle y, z \rangle - \varphi(z); \ z \in H\},$$

Minimize $\quad J(v) := \int_0^T \left[\varphi(v(t)) + \varphi^*(-\dot{v}(t))\right] dt + \frac{1}{2}\|v(T)\|_H^2$

on $\quad K = \{v \in C([0, T]; H); \ \varphi^*(-\dfrac{dv}{dt}) \in L^1(0, T), v(0) = u_0\}.$

The proof is based on a simple convex duality principle:

$$\varphi(u(t)) + \varphi^*(-\dot{u}(t)) \geq \langle u(t), -\dot{u}(t) \rangle = -\frac{1}{2}\frac{d}{dt}|u(t)|_H^2 \quad \text{a.e.}$$

with equality if and only if u satisfies

$$-\dot{u}(t) \in \partial\varphi(u(t)) \quad \text{a.e. on } [0, T]$$

But equality is assured only if

$$\text{Min}\{J(v); v \in K\} = \frac{\|u_0\|^2}{2},$$

which is not obvious unless we already know that the equation already has a solution.

To remedy the situation, we change the Brezis-Ekeland principle:

- First, we isolate a concept of self-dual variational problems that seems to be inherent to this type of evolution equations. Let $\psi(u) = \varphi(u + u_0) - \langle u, f \rangle$ and define

$$I(u) = \int_0^T \left[\psi(u(t)) + \psi^*(-\dot{u}(t)) \right] dt + \frac{1}{2} \left(\|u(0)\|_H^2 + \|u(T)\|_H^2 \right)$$

which corresponds to the readily **"self-dual" Lagrangian**:

$$\ell(c_0, c_T) = \frac{1}{2} \|c_0\|_H^2 + \frac{1}{2} \|c_T\|_H^2 \quad \text{and} \quad L(u, v) = \psi(u) + \psi^*(-v).$$

- A boundary-free variational formulation and a Banach space as a constraint set –typically–

$$A_H^2 = \{ u : [0, T] \to H; \, \dot{u} \in L_H^2 \}$$

Now standard methods from the calculus of variations –properly extended to an infinite dimensional framework– can be applied to establish the existence of a unique minimizer.

- **Self-duality** always lead to zero as minimal value, so that *under the right conditions*, there is a unique $\hat{u}$ such that:

$$I(\hat{u}) = \inf_{A_H^\alpha} I(u) = 0. \tag{1}$$

- On the other hand, **Fenchel-Young inequality** gives that:

$$I(u) \geq \|u(0)\|_H^2 \quad \text{for any } u \in A_H^2. \tag{2}$$

It follows that $\hat{u}(0) = 0$, while the limiting case of Young's inequality applied to ψ, implies that the path $\hat{u}(t)$ is a weak solution for the evolution equation

$$-\dot{u}(t) \in \partial\varphi(u(t) + u_0(t)) \quad \text{a.e. on } [0, T]$$

In summary: we are proposing the following variational principle for gradient flows:

Theorem 1. *Let φ be proper convex and lower semi-continuous on a Hilbert space H, with a non-empty subdifferential at 0. For any $u_0 \in \mathrm{Dom}(\partial\varphi)$ and any $f \in H$, the following functional:*

$$\Phi(u) \;=\; \int_0^T \left[\varphi(u(t) + u_0) + \varphi^*(f - \dot{u}(t)) - \langle u(t), f \rangle + \langle \dot{u}(t), u_0 \rangle \right] dt$$

$$+ \frac{1}{2}\left(\|u(0)\|_H^2 + \|u(T)\|_H^2 \right) - T\langle f, u_0 \rangle$$

on A_H^2, has a unique minimum $\hat{u}$ such that $\hat{u}(t) \in \mathrm{Dom}(\varphi) - u_0$ for almost all $t \in [0, T]$, $\Phi_{u_0, f}(\hat{u}) = \inf_{\tilde{K}} \Phi_{u_0, f}(u) = 0$, and the path $u(t) = \tilde{u}(t) + u_0$ is a weak solution for

$$\begin{cases} \dot{u}(t) + \partial\varphi(u(t)) & = & f \quad \text{a.e.} \quad \text{on} \quad [0, T] \\ u(0) & = & u_0 \end{cases}$$

The heat equation: For any $u_0 \in H_0^1(\Omega)$ and any $f \in H^{-1}(\Omega)$ the infimum of the functional

$$\Phi(u) = \frac{1}{2} \int_0^T \int_\Omega |\nabla(u(t,x) + u_0(x))|^2 + |\nabla\Delta^{-1}(f - \dot{u}(t,x))|^2) dx dt$$

$$+ \int_0^T \int_\Omega [u_0(x)\dot{u}(t,x) - f(x)u(x,t)] \, dx dt$$

$$+ \frac{1}{2} \left(\int_\Omega |u(0,x)|^2 |dx + \int_\Omega |u(T,x)|^2 |dx \right)$$

$$- \int_\Omega f(x)u_0(x) \, dx,$$

on the space

$$\left\{ u \in C([0,T], L^2(\Omega)); u(t) \in H_0^1 \cap H^2; \int_0^T \|\dot{u}(t)\|_{L^2(\Omega)}^2 \, dt < +\infty \right\}$$

is equal to zero!

and is attained uniquely at a path $\tilde{u} \in C([0,T]; L^2(\Omega))$ in such a

way that $u(t) = \tilde{u}(t) + u_0$ is a weak solution of the equation:

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) & = & \Delta u + f \ \ \text{on} \ \ \Omega \times [0,T] \\ \\ u(0,x) & = & u_0 \ \ \ \text{on} \ \ \Omega \\ \\ u(t,x) & = & 0 \ \ \text{on} \ \ \partial\Omega. \end{cases}$$

Quasi-linear parabolic equations

For $p \geq 1$, let $\varphi(u) = \frac{1}{p} \int_\Omega |\nabla u|^p$ on $W_0^{1,p}(\Omega)$ and $+\infty$ elsewhere.
Its conjugate is then $\varphi^*(v) = \frac{p-1}{p} \int_\Omega |\nabla \Delta^{-1} v|^{\frac{p}{p-1}} dx$.
For any $u_0 \in W_0^{1,p}(\Omega)$ and any $f \in L^2(\Omega)$, the infimum of

$$\Phi(u) \ \ = \ \ \int_0^T \int_\Omega (\frac{1}{p}|\nabla(u(t,x) + u_0(x))|^p + \frac{p-1}{p}|\nabla\Delta^{-1}(f - \dot{u}(t,x))|^{\frac{p}{p-1}})dxdt$$

$$+ \int_0^T \int_\Omega [u_0(x)\dot{u}(t,x) - f(x)u(x,t)] \, dxdt + \frac{1}{2}\left(\|u(0)\|_2^2 + \|u(T)\|_2^2\right)$$

$$-T \int_\Omega f(x)u_0(x) \, dx$$

on $A^2_{L^2(\Omega)}$ is equal to zero and is attained uniquely at an $W^{1,p}_0(\Omega)$-valued path $\tilde{u}$ such that $\int_0^T \|\dot{u}(t)\|_2^2 dt < +\infty$. The path $u(t) = \tilde{u}(t) + u_0$ is a solution of the equation:

$$\begin{cases} u_t(t,x) & = & \Delta_p u + f \ \ \text{on} \ \ \Omega \times [0,T] \\ u(0,x) & = & u_0 \ \ \text{on} \ \ \Omega \\ u(t,0) & = & 0 \ \ \text{on} \ \ \partial\Omega. \end{cases}$$

Porous media equations

Let $H = H^{-1}(\Omega)$ equipped with the norm induced by the scalar product

$$\langle u, v \rangle = \int_\Omega u(-\Delta)^{-1} v \, dx = \langle u, v \rangle_{H^{-1}(\Omega)}.$$

Consider the functional

$$\varphi(u) = \begin{cases} \frac{1}{m+1} \int_\Omega |u|^{m+1} & \text{on} \quad X = L^{m+1}(\Omega) \\ +\infty & \text{on} \quad H^{-1} \setminus X, \end{cases}$$

and its conjugate

$$\varphi^*(v) = \frac{m}{m+1} \int_\Omega |\Delta^{-1} v|^{\frac{m+1}{m}} \, dx.$$

Let $m > 0$, then for $u_0 \in L^{m+1}(\Omega)$ and $f \in H^{-1}$, the infimum of

$$
\Phi(u) \quad = \quad \int_0^T \left(\frac{1}{m+1} \int_\Omega |u + u_0|^{m+1} + \frac{m}{m+1} \int_\Omega |\Delta^{-1}(f - \dot{u})|^{\frac{m+1}{m}} \right) dxdt
$$
$$
+ \int_0^T \int_\Omega \left[u_0(x)(\Delta^{-1}\dot{u})(t, x) - u(x, t)(\Delta^{-1}f)(x) \right] dxdt
$$
$$
+ \frac{1}{2} \left(\|u(0)\|^2_{H^{-1}} + \|u(T)\|^2_{H^{-1}} \right)
$$
$$
- T \int_\Omega u_0(x)(-\Delta)^{-1} f(x) \, dx
$$

on A^2_H is equal to zero and is attained uniquely at $\tilde{u}$ such that $\int_0^T \|\dot{u}(t)\|^2_H dt < +\infty$. The path $u(t) = \tilde{u}(t) + u_0$ is a solution of:

$$
\begin{cases}
u_t(t, x) & = \quad \Delta u^m + f \ \text{ on } \ \Omega \times [0, T] \\
u(0, x) & = \quad u_0 \quad \text{on} \quad \Omega.
\end{cases}
$$

Self-Dual Variational Problems

$$L : H \times H \to \mathbb{R} \cup \{+\infty\}, \quad \ell : H \times H \to \mathbb{R} \cup \{+\infty\}$$

be two convex and lower semi-continuous functions on $H \times H$. Associate the action functional

$$\Phi_{\ell,L}(u) = \int_0^T L(u(t), \dot{u}(t)) dt + \ell(u(0), u(T))$$

on the Banach space $A_H^\alpha = \{u : [0, T] \to H; \dot{u} \in L_H^\alpha\}$ equipped with the norm $\|u\|_{A_H^\alpha} = \|u(0)\|_H + (\int_0^T \|\dot{u}\|^\alpha dt)^{\frac{1}{\alpha}}$. A_H^α is a reflexive Banach space that can be identified with the product space $H \times L_H^\alpha$, while its dual $(A_H^\alpha)^*$ can be identified with $H \times L_H^\beta$ where $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. The duality is given by:

$$\langle u, (a, p) \rangle_{A_H^\alpha, H \times L_H^\beta} = (u(0), a)_H + \int_0^T \langle \dot{u}(t), p(t) \rangle dt.$$

Associate to the pair (ℓ, L), the "variation function" $\Psi_{\ell,L}$ defined on $(A_H^\alpha)^* = H \times L_H^\beta$ as:

$$\Psi_{\ell,L}(a, y) = \inf\{\int_0^T L(u + y, \dot{u})dt + \ell(u(0) + a, u(T)) \; ; \; u \in A_H^\alpha\}$$

Bolza duality: For all $p \in A_H^\alpha$,

$$\Psi_{\ell,L}^*(p) = \Phi_{m,M}(p)$$

where M, m are the "Bolza-dual" Lagrangians:

$$M(p, s) = L^*(s, p) \;\; \text{and} \;\; m(r, s) = \ell^*(r, -s)$$

where L^* and ℓ^* are the Legendre duals of L and ℓ respectively, and

$$\Phi_{m,M}(u) = \int_0^T M(u, \dot{u})dt + m(u(0), u(T)).$$

Suppose now $q \in \partial \Psi_{\ell,L}(0,0) \in A_H^\alpha$, then

$$\Psi_{\ell,L}(0,0) + \Psi_{\ell,L}^*(q) = 0 = \inf_{A_H^\alpha} \Phi_{\ell,L} + \Phi_{m,M}(q)$$

Self-duality: Say that the pair (L, ℓ) is self-dual if for all $(r, p, s) \in H^3$, we have

$$m(r,s) = \ell(-r,-s) \quad \text{and } M(s,p) = L(-s,-p),$$

or equivalently

$$\ell^*(r,s) = \ell(-r,s) \quad \text{and } L^*(p,s) = L(-s,-p)$$

In this case, $\Phi_{m,M}(u) = \Phi_{\ell,L}(-u)$ for any u,

$$-\inf_{A_H^\alpha} \Phi_{\ell,L} = \Phi_{m,M}(q) = \Phi_{\ell,L}(-q) \geq \inf_{A_H^\alpha} \Phi_{\ell,L}$$

We are done if the latter is non-negative!

It is the case because the following general:

"Weak duality" formula:

$$\inf_{u \in A_H^\alpha} \Phi_{\ell,L}(u) \geq - \inf_{u \in A_H^\alpha} \Phi_{m,M}(u),$$

combines again with self-duality $\Phi_{m,M}(u) = \Phi_{\ell,L}(-u)$ and the fact that the constraint set is a vector space, to give:

$$\inf_{u \in A_H^\alpha} \Phi_{\ell,L}(u) \geq - \inf_{u \in A_H^\alpha} \Phi_{m,M}(u) = - \inf_{A_H^\alpha} \Phi_{\ell,L}(u)$$

which means that $\inf_{u \in A_H^\alpha} \Phi_{\ell,L}(u)$ is necessarily non-negative.

Theorem 2. *Suppose L and l self-dual and $\Psi_{\ell,L} : H \times L^2_H \to \mathbb{R}$ subdifferentiable at $(0,0)$, then there exists $\hat{u} \in A^2_H$ such that:*

$$\Phi_{\ell,L}(\hat{u}) = \inf_{A^\alpha_H} \Phi_{\ell,L}(u) = 0.$$

For gradient flows: Let $\varphi : H \to \mathbb{R}$ be a convex and lsc. For any $u_0 \in \mathrm{Dom}(\varphi)$, $f \in H$, consider on A^α_H:

$$\Phi_{u_0,f}(u) = \int_0^T \left[\psi(u(t)) + \psi^*(-\dot{u}(t))\right] dt + \frac{1}{2}(\|u(0)\|^2_H + \|u(T)\|^2_H)$$

where $\psi(u) = \varphi(u + u_0) - \langle u, f \rangle$. Here

$$\ell(c_0, c_T) = \frac{1}{2}\|c_0\|^2_H + \frac{1}{2}\|c_T\|^2_H \quad \text{and} \quad L(u,v) = \psi(u) + \psi^*(-v).$$

are clearly self-dual. However for the sub-differentiability of $\Psi_{\ell,L}$ at $(0,0)$, we need that for some $\gamma > 1$ and $C > 0$,

$$\varphi(u) \le C(1 + \|u\|^\gamma_H) \quad \text{for } u \in H.$$

which is never satisfied!!!

One way to remedy this is to regularize φ by using inf-convolution. That is, we define as before $\psi(u) = \varphi(u + u_0) - \langle u, f \rangle$ and for each $\lambda > 0$, let

$$\psi_\lambda(x) = \inf\{\psi(y) + \frac{1}{2\lambda}\|x - y\|_H^2; \ y \in H\},$$

in such a way that for some $C > 0$,

$$\psi_\lambda(x) \le \frac{C}{\lambda}(1 + \|x\|_H^2),$$

while its conjugate is given by

$$\psi_\lambda^*(y) = \psi^*(y) + \frac{\lambda}{2}\|y\|_H^2.$$

The functionals ψ_λ now satisfy the hypothesis and therefore the corresponding evolution equations

$$\begin{cases} \dot{u}_\lambda(t) + \partial\psi_\lambda(u_\lambda(t)) &= 0 \quad \text{a.e.} \quad \text{on} \quad [0,T] \\ u_\lambda(0) &= 0 \end{cases}$$

have weak solutions $u_\lambda(t)$ in A_H^2 that minimize

$$\Psi_\lambda(u) = \int_0^T \left[\psi_\lambda(u(t)) + \psi_\lambda^*(-\dot{u}(t)) \right] dt + \frac{1}{2} (\|u(0)\|_H^2 + \|u(T)\|_H^2).$$

Now we need to argue that $(u_\lambda)_\lambda$ converges as $\lambda \to 0$ to a solution of the original problem. This analysis is reminescent of the approach via the resolvent theory of Hille-Yosida, but is much easier here since the variational approach does not require the uniform convergence of $(u_\lambda)_\lambda$ and their time-derivatives.