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Elastic sheets as a model system for studying multiple scale behaviors

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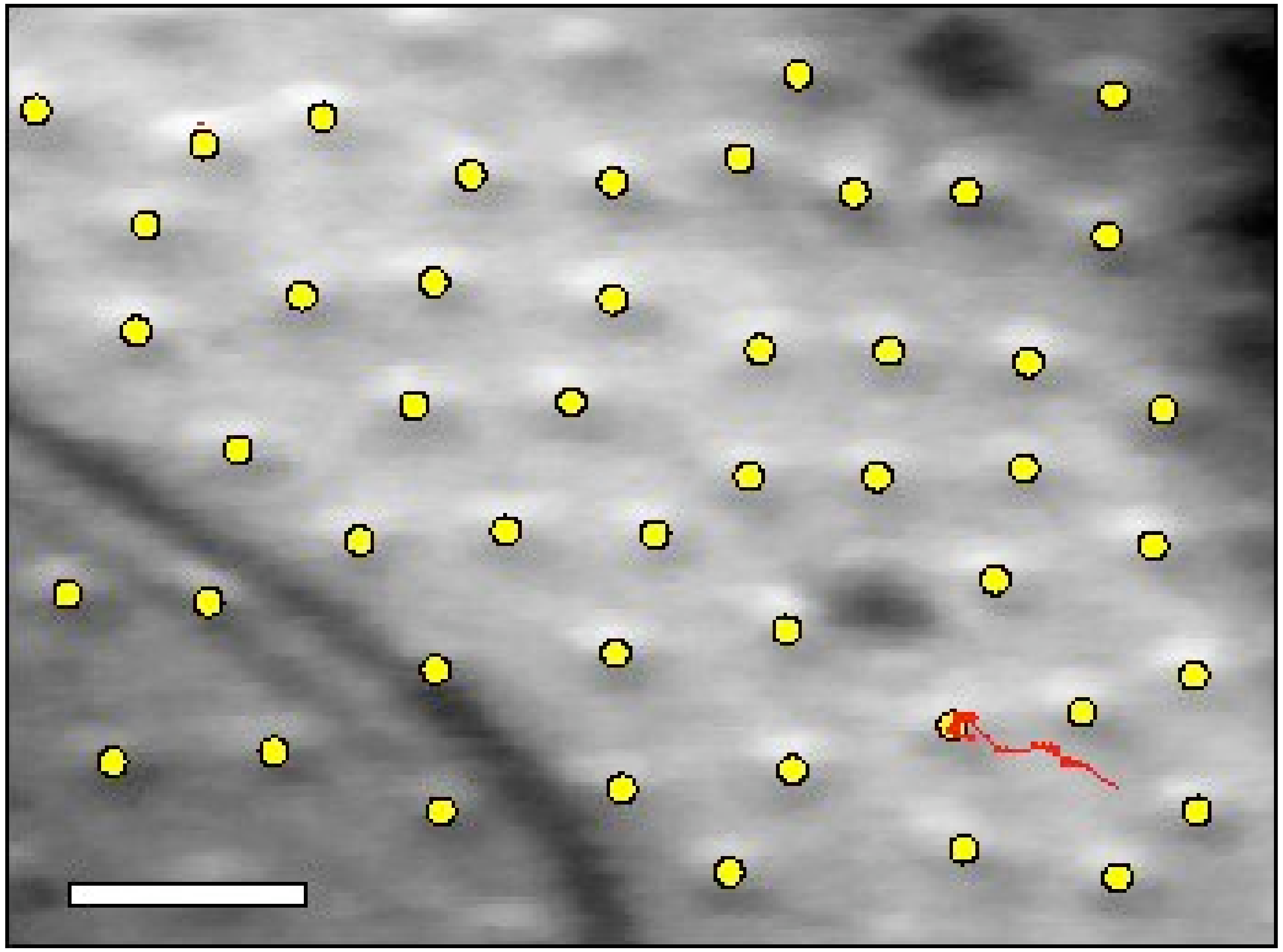
Turbulence in Liquid Helium

Albert Libchaber, et al

Patterns in Vibrated Sand

Oscillons

Experiments at the University of Texas
P. B. Umbanhowar, et al



Vortices in a Type II Superconductor

Crumpling

Confinement $\Rightarrow$ Geometry becomes "rough" on the scale of the forcing.

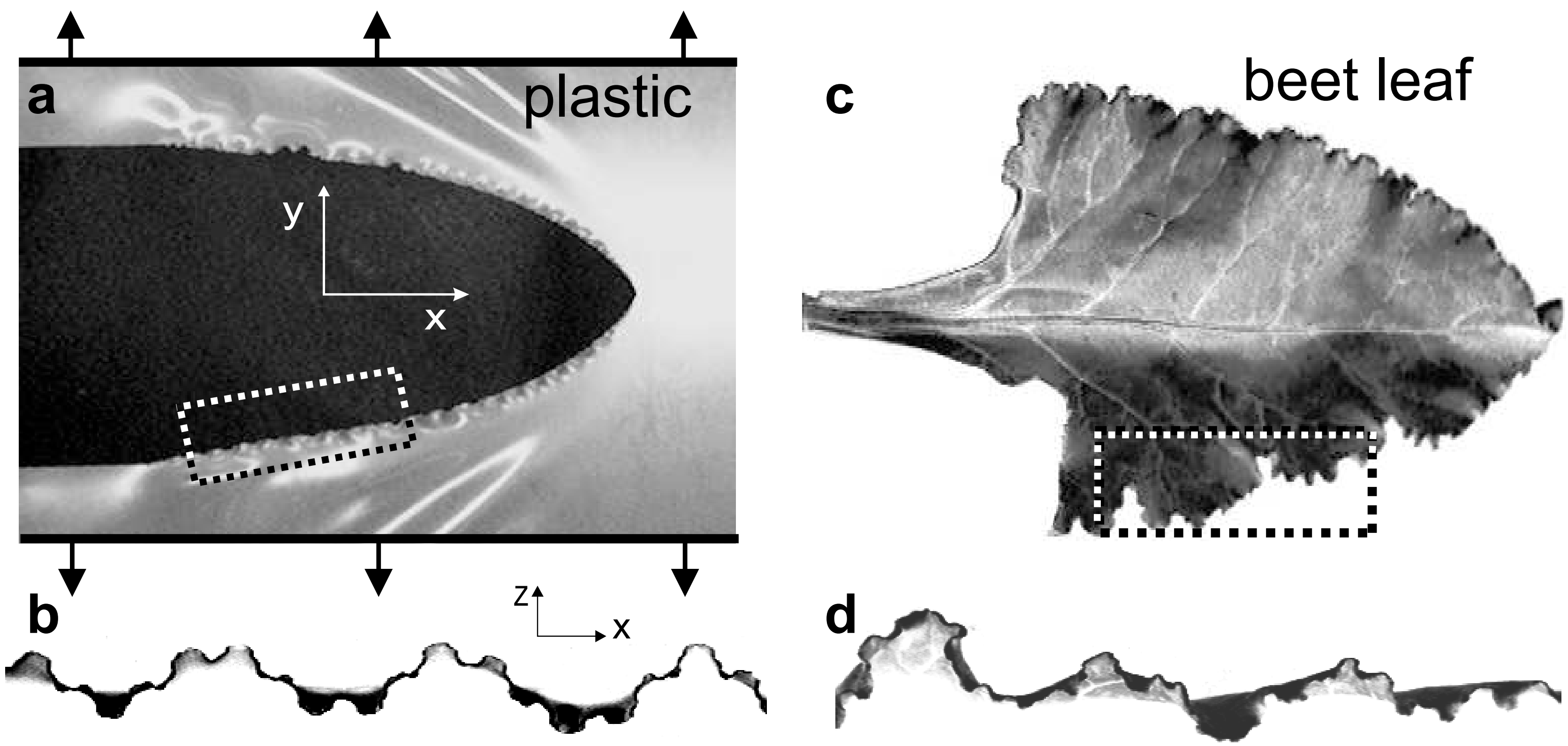
Basic Puzzles:

- Why does a sheet of paper crumple when it is crushed?
- What can we say about the geometry of a crumpled sheet?

Observation: A 2-Sheet crumples when it is crushed in 3-dimensions but a 1-sheet (a rod) does not crumple when it is confined in 2-dimensions.

- Energy and Stress Condensation in Physical systems.
- Non convex variational problems.
- Isometric Immersions and other geometric questions.

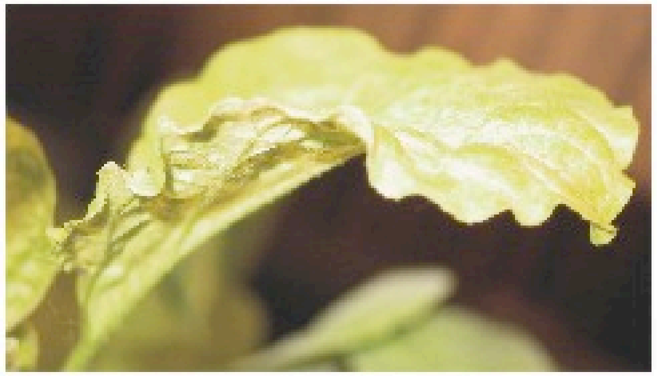
Multiple scale buckling



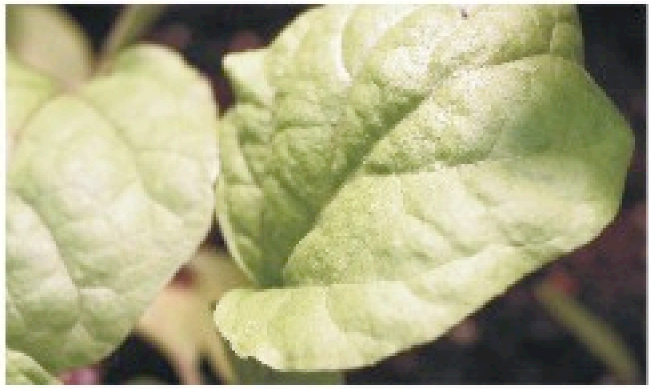
"Buckling cascade in free thin sheets", E. Sharon, *et al.*

A Daffodil

Transition to wavy leaves in Eggplant due to application of Auxin

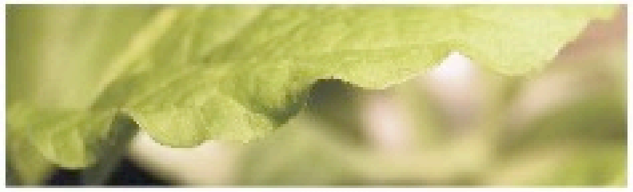


After 2 weeks

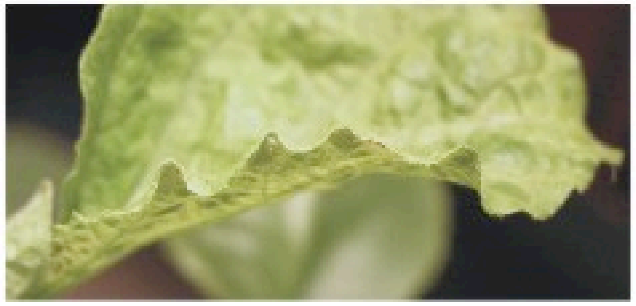


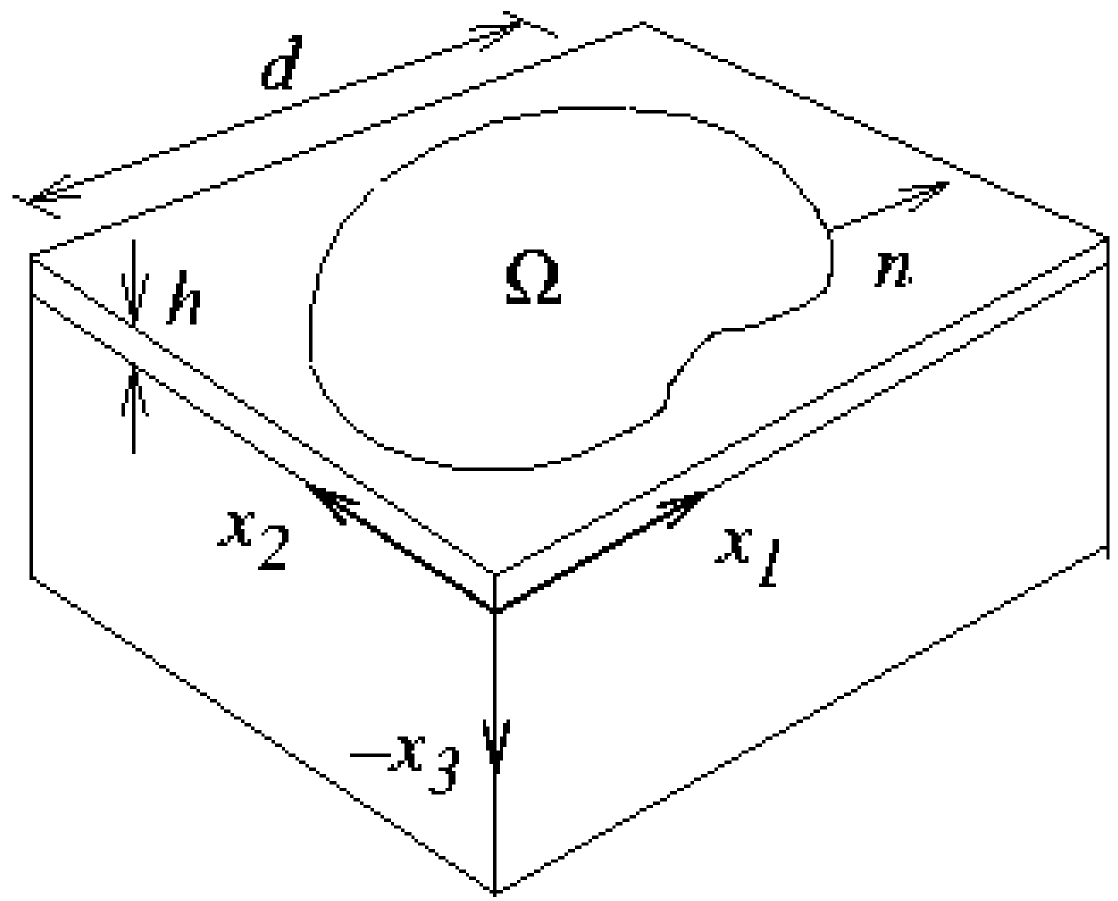
No Auxin

After 1 week



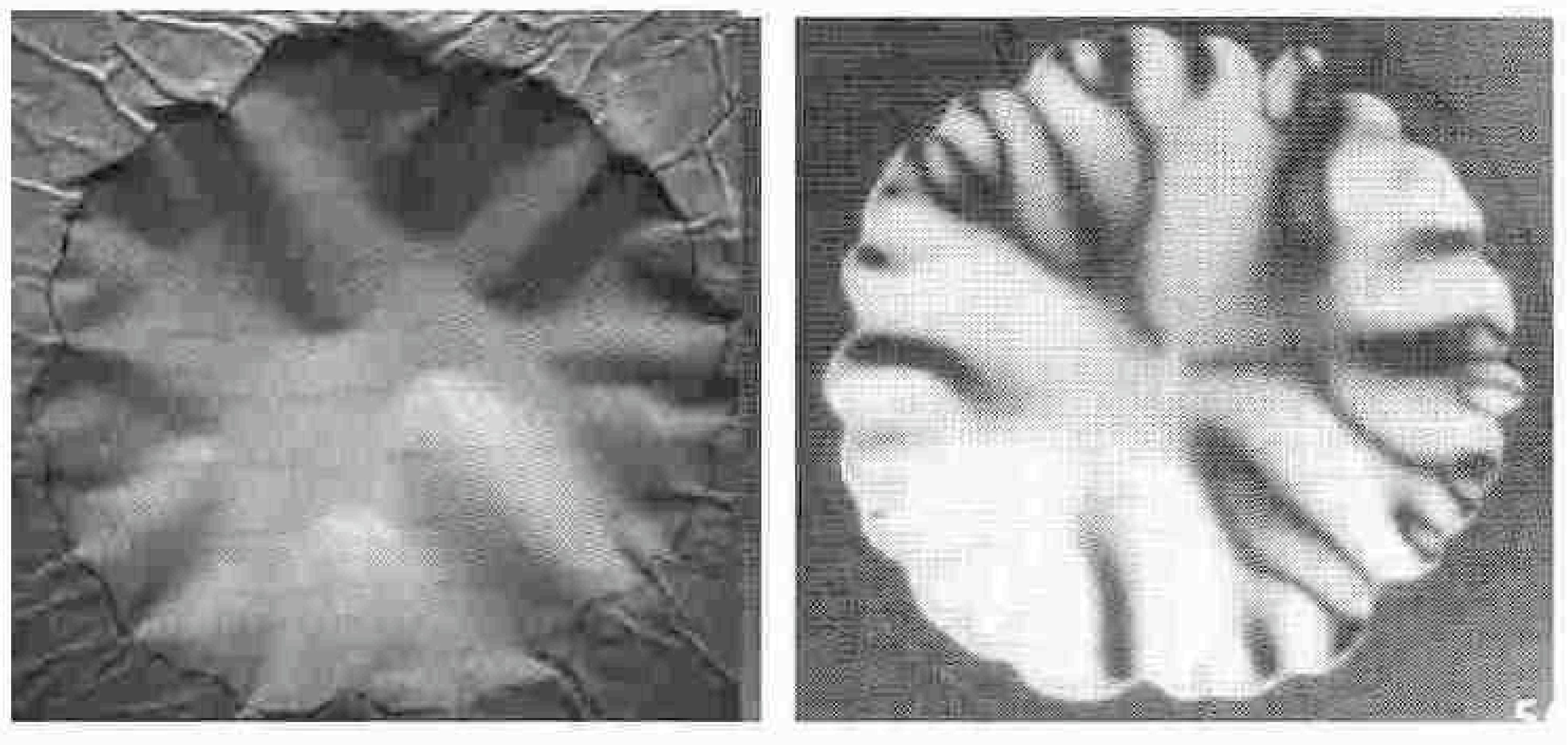
After 12 days



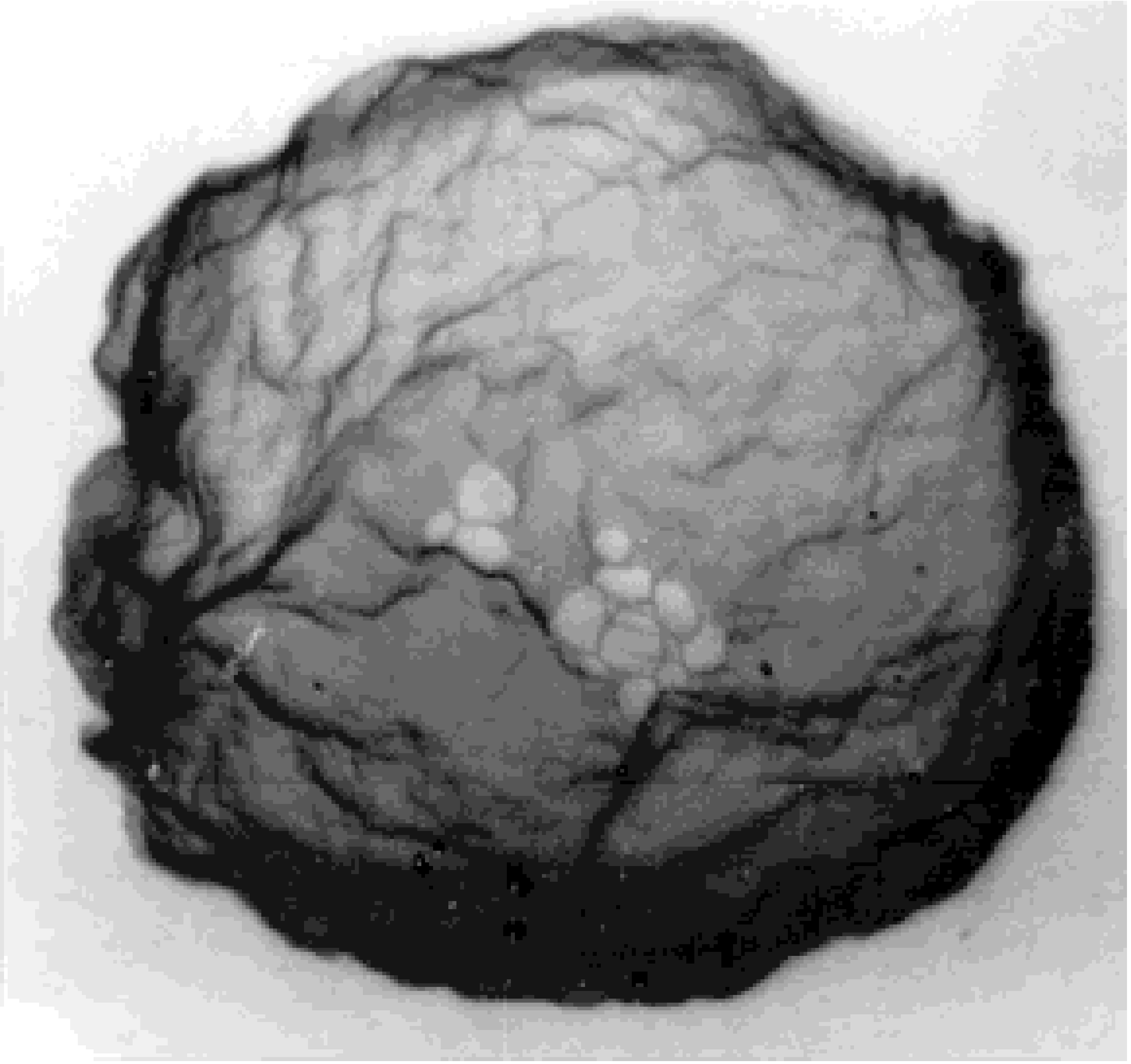


Blistering of Thin films

Courtesy G. Gioia, UIUC



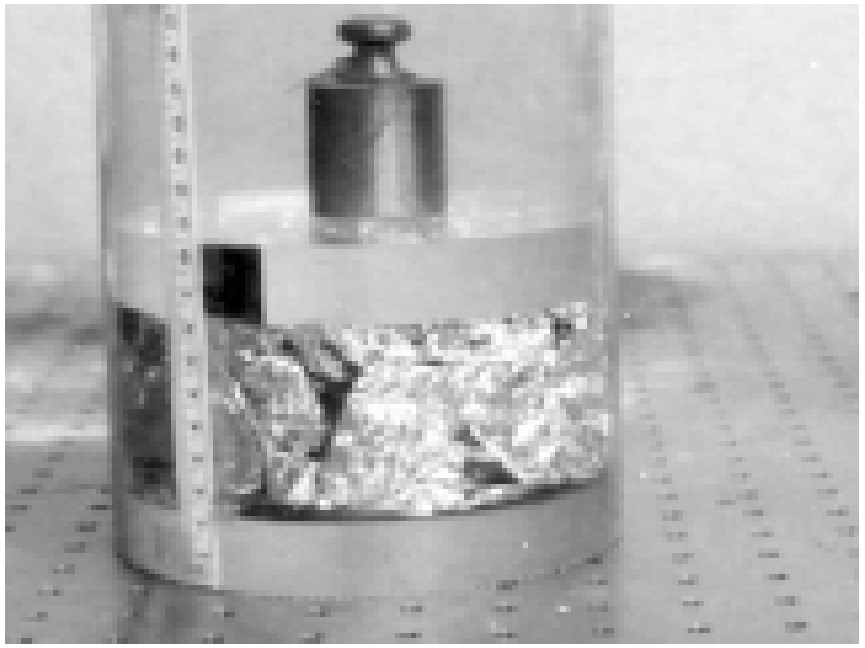
Argon et al., J. Mat. Sci. 24, 1989



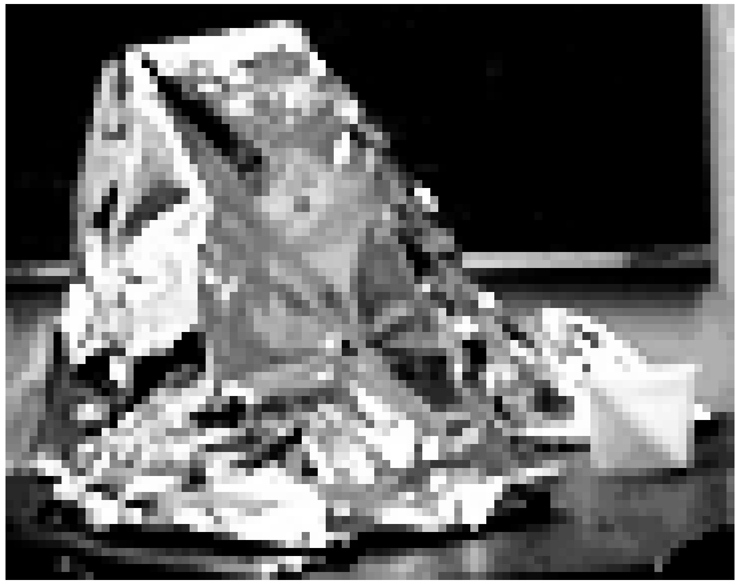
An Osmotically deswollen red blood cell

courtesy Ted Steck

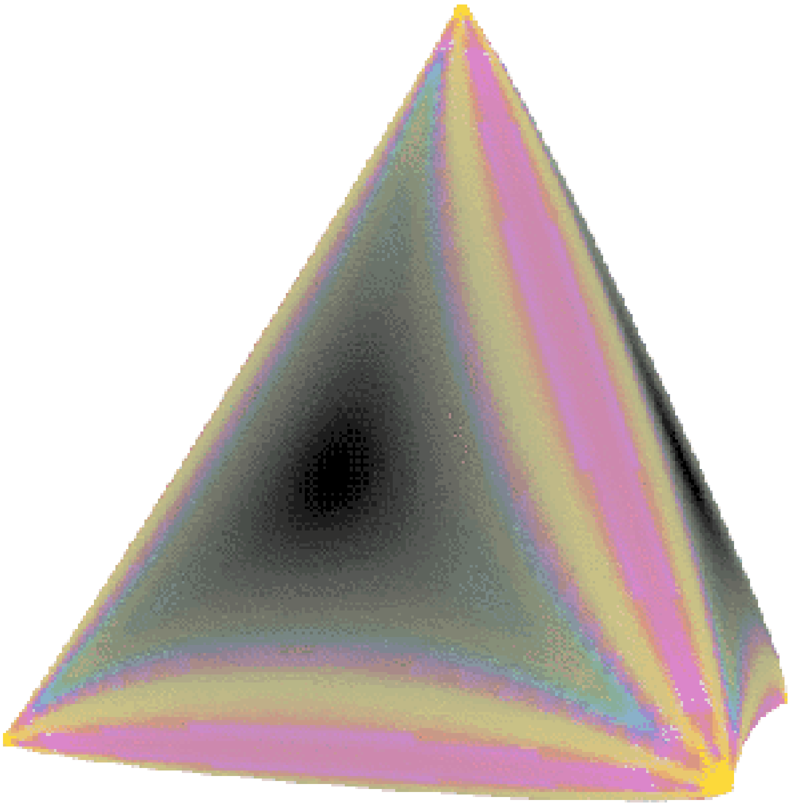
Forced crumpling



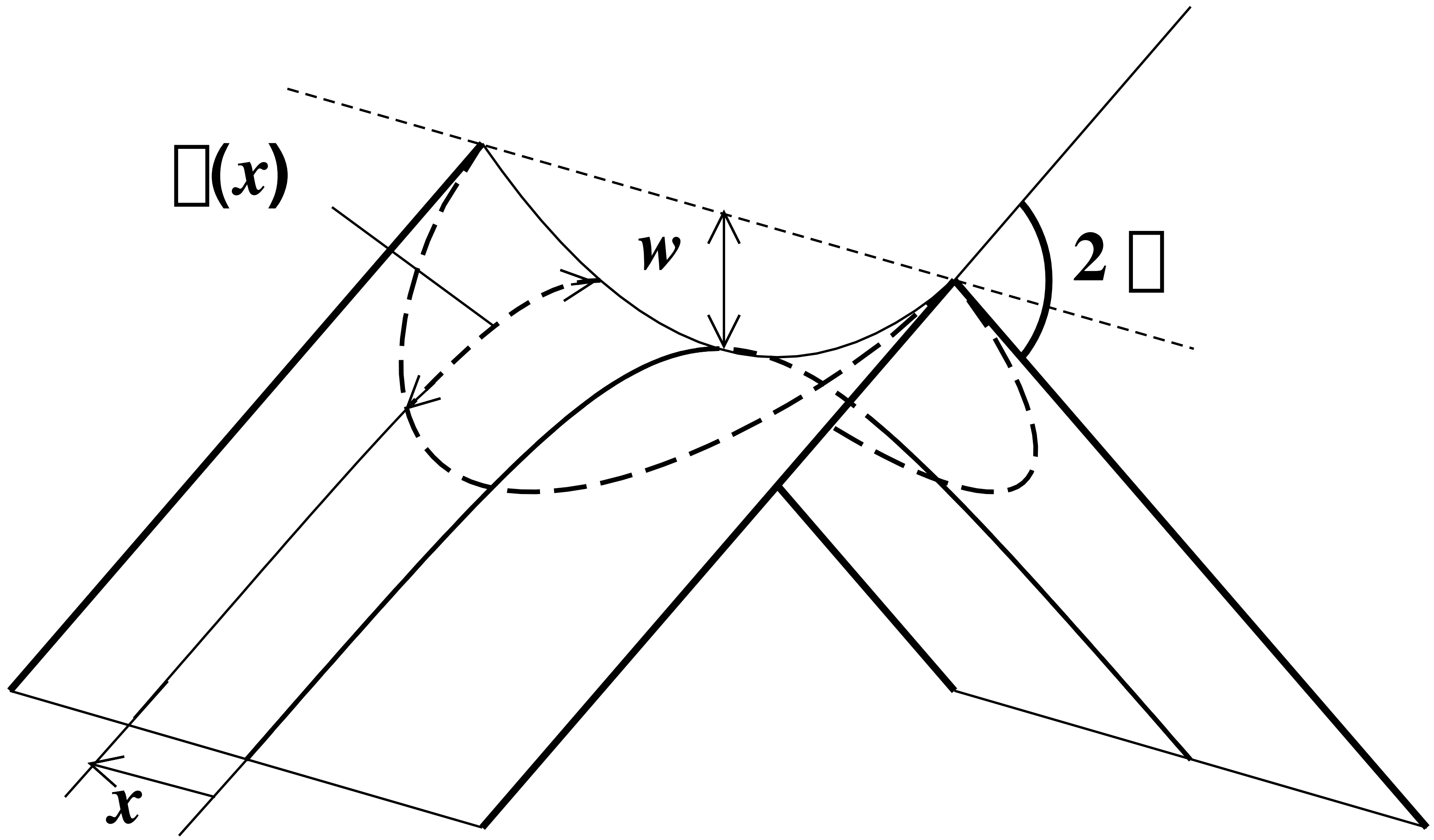
Crumpling experiments
Kittiwit Matan, et al



A mountain of crumpled
Mylar, Tom Witten

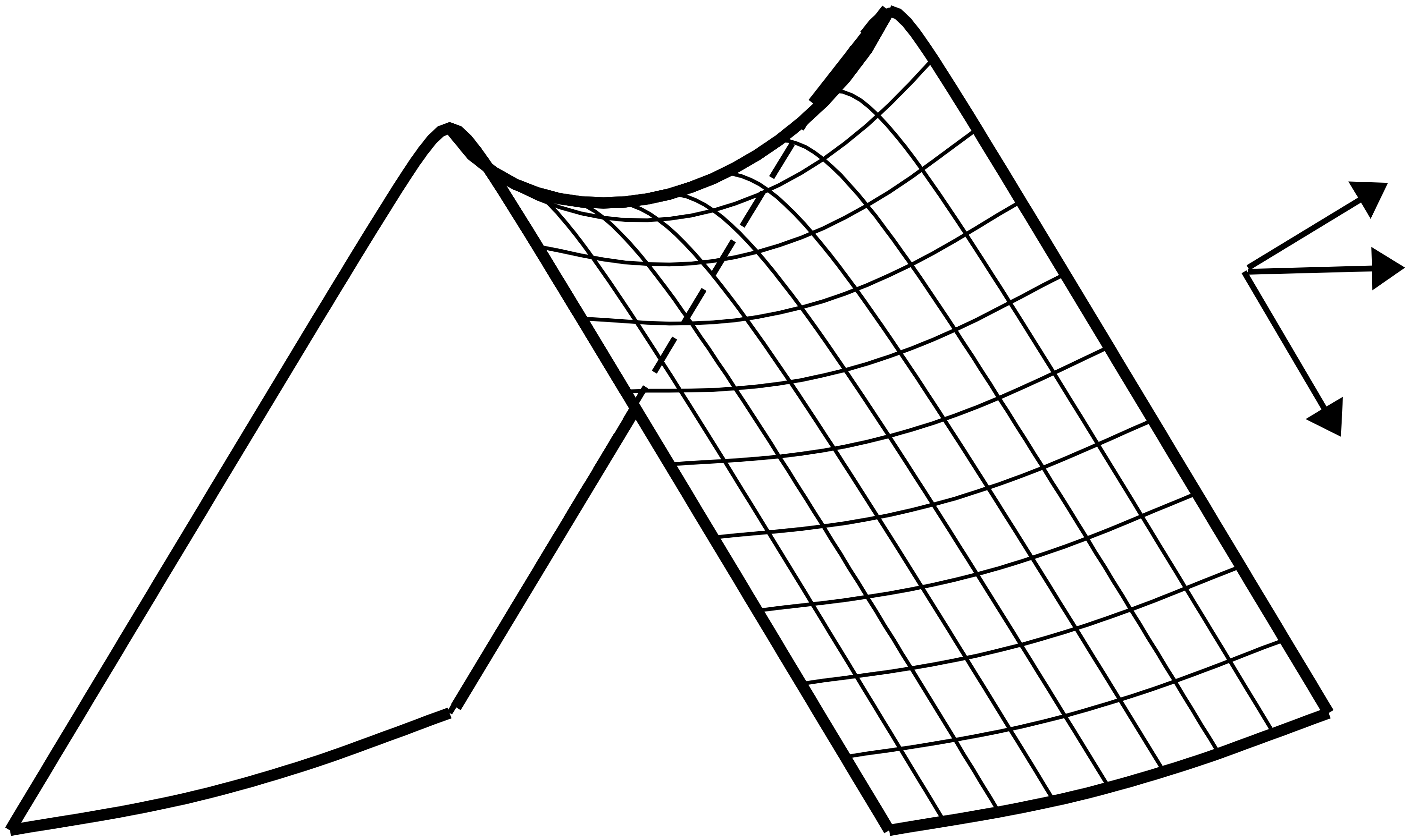


Ridges and vertices in a crumpled sheet



A Minimal ridge

Conformation of the sheet in space



Energetics of Deformed Elastic Sheets

The conformation of the (centroid) of the sheet is given by a map

$$\mathbf{r} : S \to \Omega^d.$$

The strain in the sheet is given by

$$u_{\alpha\beta} = \frac{1}{2}\left(\partial_\alpha \mathbf{r} \cdot \partial_\beta \mathbf{r} - g_{\alpha\beta}\right) = \frac{1}{2}\left[(D\mathbf{r})^T D\mathbf{r} - g\right]$$

The extrinsic curvature of the sheet in the normal direction $\mathbf{n}^{(j)}$ is given by

$$K^{(j)}_{\alpha\beta} = \partial_\alpha \partial_\beta \mathbf{r} \cdot \mathbf{n}^{(j)} = P_j\left[D^2 \mathbf{r}\right],$$

for $j = 1, 2, \ldots, d - m$. The energy is

$$\mathcal{E}^\epsilon = \int_S (E_b + E_s) dx^m = \int_S dx^m \left\{\epsilon^2 \| P_{\mathbf{n}}\left[D^2\mathbf{r}\right]\|^2 + \|(D\mathbf{r})^T D\mathbf{r} - g\|^2\right\}$$

Boundary conditions

- Confinement, as for the crumpling problem.
- Periodic
- Blistering boundary conditions.
- Free boundary conditions.

The *boundary conditions* play an important role in determining the type of solutions we get!

The singular structures are *nonlocal*!

Nonconvex variational problems

Nonconvexity provides a natural mechanism for the formation of small scale structures in the *equilibrium state*.

Nonconvex variational problems, however, have the following "unphysical" features, that real systems cannot exhibit –

1. Structures can form on arbitrarily fine scales.
2. The minimization problem may not have a solution at all!
3. There are a large number of macroscopically distinct low energy states, that arise from different minimizing sequences.

Regularized nonconvex variational problems

$$\mathcal{E}^\epsilon[u] = \int_{\mathcal{S}} W(x, u, Du)dx + \epsilon^2 \int_{\mathcal{S}} F(x, u, Du, D^2u)dx.$$

$W \geq 0$ is *nonconvex* in Du and $F \geq 0$ is *convex* in D^2u.

$\epsilon = 0$ – "bare" nonconvex problem and structures on arbitrarily fine scales.

$\epsilon > 0$ – small scale cutoff through *singular perturbation*.

There is a *minimizer* – u^ϵ for $\mathcal{E}^\epsilon$.

Selection mechanism – small subset of all the low energy states of $\mathcal{E}$ are the "true" minimizers.

We care about the limit $\epsilon \to 0$

Morphology of the solutions Do the minimizers u^ϵ display oscillations or concentration effects? Describe the oscillatory/singular regions in the solutions.

Scaling laws How do $\mathcal{E}^\epsilon(u^\epsilon)$, and the length scales associated with the oscillations/singular regions in the minimizer u^ϵ depend on ϵ?

Variational Convergence Does $\lim_{\epsilon \to 0} u^\epsilon$ exist, in some appropriate sense? If so, is it possible to define a suitable limiting energy functional which describes the limiting behavior of the minimizers?

The first two questions are of direct physical interest. The third question is closely related to the notion of Γ – convergence.

Identify the elementary excitations (singularities) of the system and the energy associated with these structures.

Reduced model – Integrating out an (irrelevant) small scale.

The limit $\epsilon \to 0$

Assumption –

$$\mathcal{E}^\epsilon[u^\epsilon] = \int_{\mathcal{S}} W(Du^\epsilon)dx + \epsilon^2 \int_{\mathcal{S}} F(D^2u^\epsilon)dx \to 0.$$

Let K be the zero set for W. Then, Du^ϵ is nearly in K.

Growth condition: $W(Du) \to \infty$ as $Du \to \infty$. Therefore, $Du^\epsilon \rightharpoonup Du$.

Non-singular limit $Du \in K$ and $\int F(D^2u) < \infty$.

Concentration $Du^\epsilon \to Du \in K$ but $\int F(D^2u) = \infty$ and "concentrates" on a singular set.

Oscillations $Du \notin K$.

Interplay between the rank one constraint and the geometry of K

Intuition – behavior gets increasingly singular as we go down the list, and *this leads to an increase in the asymptotic energy* in the $\epsilon \to 0$ limit.

Criteria for deciding the $\epsilon \to 0$ behavior of the thin sheet.

1. If the imposed boundary conditions allow a strain free configuration $\tilde{u}$, with $\int F dx < \infty$, the $\epsilon \to 0$ behavior of the minimizers will be non singular.

2. If the boundary conditions allow a strain free configuration $\tilde{u}$, but not one with a finite $\int F dx$, the minimizers will concentrate energy on (a set which is "similar" to) the set on which $D^2\tilde{u}$ is singular.

3. If there *no strain free*,"piecewise smooth" configuration compatible with the boundary conditions, the minimizers will display *microstructure* (small scale oscillations) as $\epsilon \to 0$.

Philosophy/Ideology

These arguments purely suggestive, and not mathematically rigorous.

Geometry/Topology – "Smooth" solutions to $W(Du) = 0$. The "scale" of smoothness is given by F. Usually, this is an Underdetermined PDE.

Where applicable, these criteria reduce a "hard" analytic problem to a simpler geometric problem.

Knowing the structure of the minimizers greatly aids in a rigorous analysis of the problem.

Part of this picture can be proved. It is probably not true in every circumstance.

Question – Deduce easily verified conditions on W and F that are sufficient in order for this approach to work.

Isometric Immersions

Theorem 1. *(Nash) : Every manifold M^m can be isometrically embedded in an arbitrarily small volume in $\mathbb{R}^d$ for sufficiently large $d \sim 3m^3/2$.*

Improvements due to M. Gromov – Convex Integration.

Results by Hartman + Nirenberg (1959), and by Pakzad + Kirchheim('02,'03).

Immersion theorems (V., Witten, Kramer and Geroch)

Theorem 2. *If $d \geq 2m$, there exists a smooth isometry $\phi : D_m \to B_r^d$ for all $r > 0$.*

Theorem 3. *If $d < 2m$ and there exists a smooth isometry $\phi : D_m \to B_r^d$, it follows that $r \geq 1/2$.*

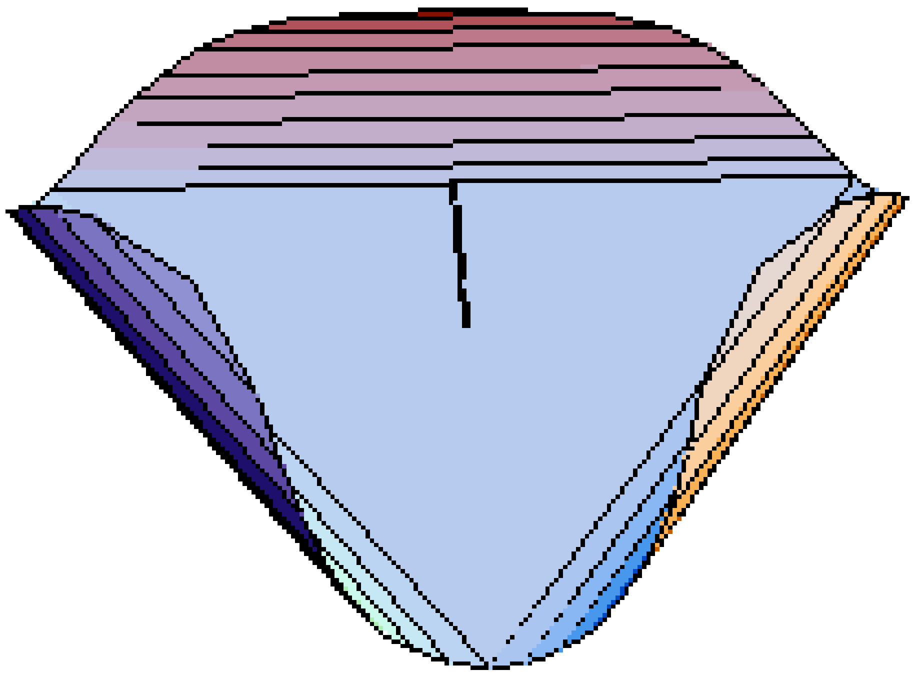
These result + our conjecture ⇒ Predictions for the morphology of minimizers for m-sheets confined in d-space, substantially confirmed by numerical simulations. (DiDonna, Witten, V.)

Idea of the proof

Identify the source of the rigidity.

Gauss curvature = 0 implies there is at least one local flat direction.

Show that the local flat directions are coherent, and give line segments (generators) that are entirely flat in the embedding.



But this is not the whole story ...

Theorem 4. *(Nash-Kuiper-Gromov) : For any C^1 manifold (M^m, g), there is a C^1 isometric immersion $\phi : M^m \to \mathbb{R}^{m+1}$. Further, the C^1 isometric immersions of M^m are C^0 dense in the space of all short immersions $\psi : M^m \to \mathbb{R}^{m+1}$.*

$\phi : M^m \to \mathbb{R}^{m+1}$ is *isometric* if $D\phi^T \cdot D\phi = g$ – No stretching.

$\psi : M^m \to \mathbb{R}^{m+1}$ is *short* if $\mathbf{v}^T \cdot D\phi^T \cdot D\phi \cdot \mathbf{v} < g(\mathbf{v}, \mathbf{v}) \quad \forall \mathbf{v} \in TM$ – Only compression.

ϕ is $\epsilon(x)C^0$ close to ψ if $|\phi(x) - \psi(x)| < \epsilon(x) \quad \forall x \in M^m$.

Eg: Take a map of the earth's surface so that the distance between any two points on the map is less than the corresponding distance on the eart's surface. For any given $\epsilon > 0$, there is a *length preserving* map of the earth that is within ϵ of the original map. Further, the new map is smooth in the sense it has a continuous tangent plane.

Convex integration

The proof is through Convex integration, and the resulting immersions have oscillations on all scales.

Proposition 1. *(S.C.V.) For "reasonable" boundary conditions the minimizing immersion for the elastic energy satisfies* $\mathcal{E}^\epsilon[u^\epsilon] \to 0$.

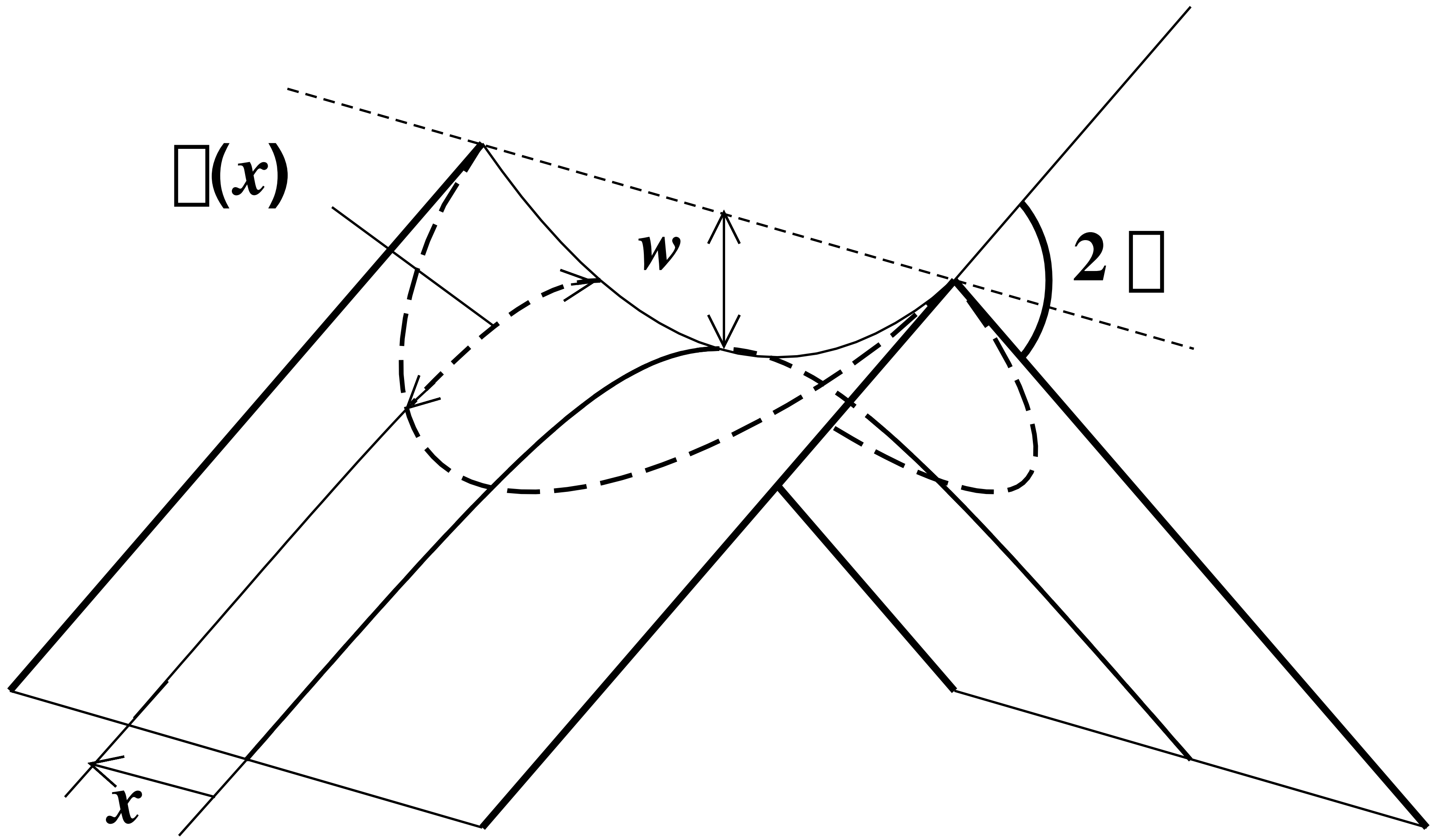
Reasonable – Admit a *short* isometric immersion, and allow for a little "wiggle" room in C^0.

Proof. Let ϕ be a C^1 isometric immersion that is close to the allowed short immersion.

Smooth by convolution on a scale η.

$$\exists \alpha > 0 : \qquad \mathcal{E}^\epsilon[u^\epsilon] \leq C_1 \eta^2 + \frac{\epsilon^2}{\eta^\alpha}.$$

Optimize the choice of η. □



A Minimal ridge

The minimal ridge

Theorem 5. *S.C.V. The minimizing maps* $\mathbf{r}^\epsilon$ *of the Elastic energy functional* $\mathcal{E}^\epsilon$ *subject to the "minimal ridge" boundary condition, have an energy given by*

$$c\alpha^{7/3}\epsilon^{5/3}L^{1/3} \leq \mathcal{E}^\epsilon[u^\epsilon] \leq C\alpha^{7/3}\epsilon^{5/3}L^{1/3},$$

where L *is the length of the ridge.*

The energy scales as $L^{1/3}$!

Idea of the proof

Upper bound by construction, following scaling analysis by Lobkovsky and Witten. Lower bounds need functional analysis inequalities.

$$\mathcal{E}_b\mathcal{E}_s^5 \geq C\epsilon^4 L^2\alpha^{14}$$

Equipartion – $\mathcal{E}_b = 5\mathcal{E}_s$.

	Concentration	Oscillations
Uniform/local	vertices in a crumpled sheet Self-similarity, Gamma-convergence	Free sheets ? Homogenization, matched asymptotics, convex integration?
nonuniform/ nonlocal	Ridges in a crumpled sheet Gamma convergence with "hidden fields"?	Thin film blisters convex integration? multiresolution methods?

Transitions between the various behaviors?