

# Faraday wave pattern selection via multi-frequency forcing

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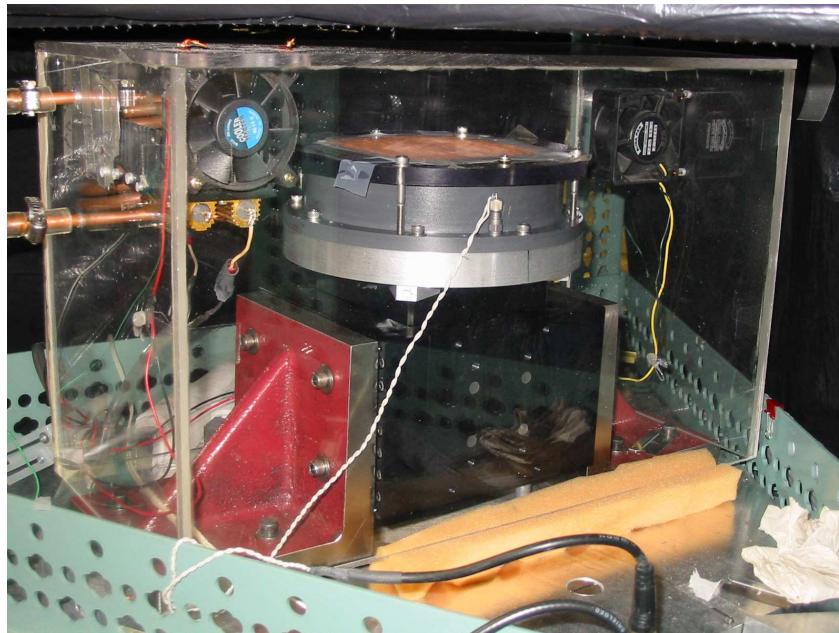
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# The Faraday System

- Periodic vertical acceleration of fluid → surface waves



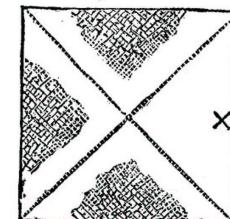
Courtesy of Yu Ding and Paul Umbanhowar

# The Faraday System

- Periodic vertical acceleration of fluid → surface waves
- Faraday's original experiment

65. The general phenomenon now to be considered is easily produced upon a square plate nipped in the middle, either by the fingers or the pincers (2. 6), held horizontally, covered with sufficient water on the upper surface to flow freely from side to side when inclined, and made to vibrate strongly by a bow applied to one edge, ×, fig. 12, in the usual way. Crispations appear on the surface of the water, first at the centres of vibration, and extend more or less towards the nodal lines, as the vibrations are stronger or weaker. The crispation presents the appearance of small conoidal elevations of equal lateral extent, usually arranged

Fig. 12.



– M. Faraday, Phil. Trans. R. Soc. Lond. (1831)

# The Faraday System

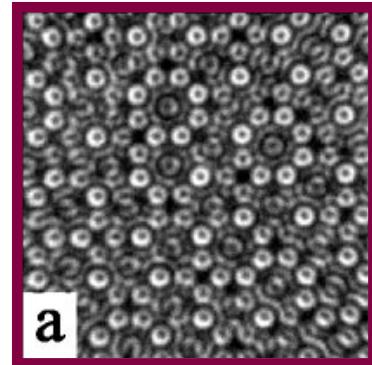
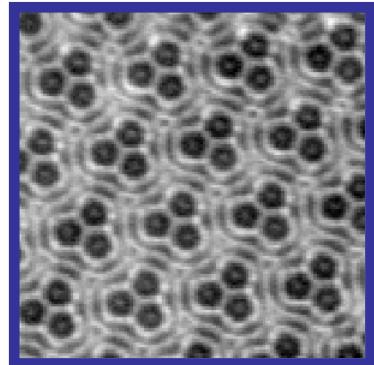
- Periodic vertical acceleration of fluid → surface waves
- Faraday's original experiment
- One-, two-, three-frequency forcing

Müller (PRL 1993)

Edwards & Fauve (PRE 1993, J. Fluid Mech. 1994)

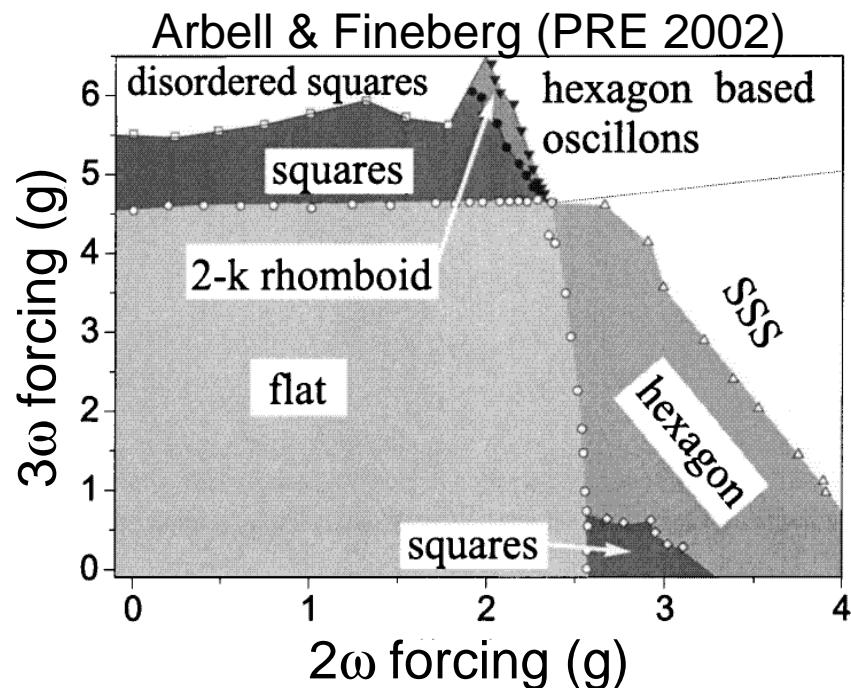
Kudrolli, Pier & Gollub (Physica D, 1998)

Arbell & Fineberg (PRL 1998, PRL 2000, PRE 2002)



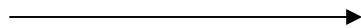
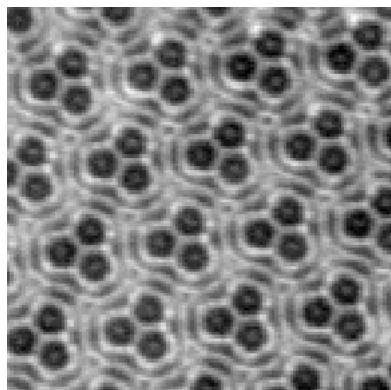
# The Faraday System

- Periodic vertical acceleration of fluid → surface waves
- Faraday's original experiment
- One-, two-, three-frequency forcing
- Exploratory studies



# The End

- Multifrequency forcing enables a prescriptive approach



forcing function

# The End

- Multifrequency forcing enables a prescriptive approach
- Symmetries may give information about coefficients

equivariant  
bifurcation  
theory

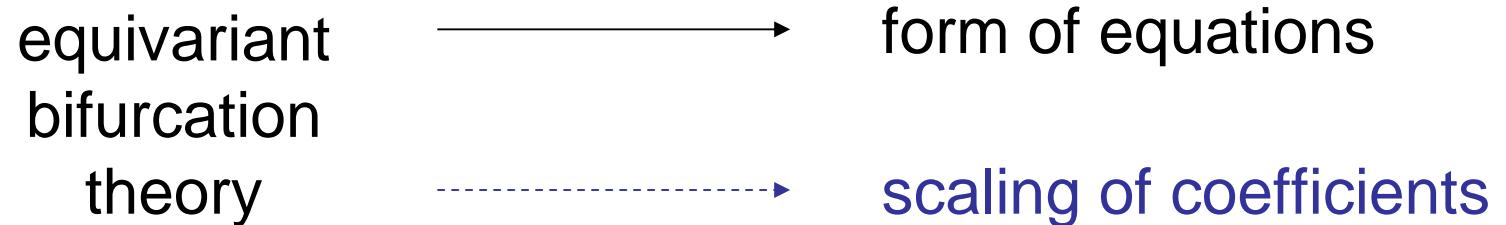


form of equations

# The End

- Multifrequency forcing enables a prescriptive approach
- Symmetries may give information about coefficients

(has been said before, but bears repeating)



# Some challenges of the Faraday system

- Huge (potentially infinite dimensional) parameter space

$$\begin{aligned} f(t) &= \sum_{u \in \mathbb{Z}^+} f_u e^{iut} + c.c., \quad f_u \in \mathbb{C} \\ &= \sum_{u \in \mathbb{Z}^+} |f_u| e^{iut + \phi_u} + c.c. \end{aligned}$$

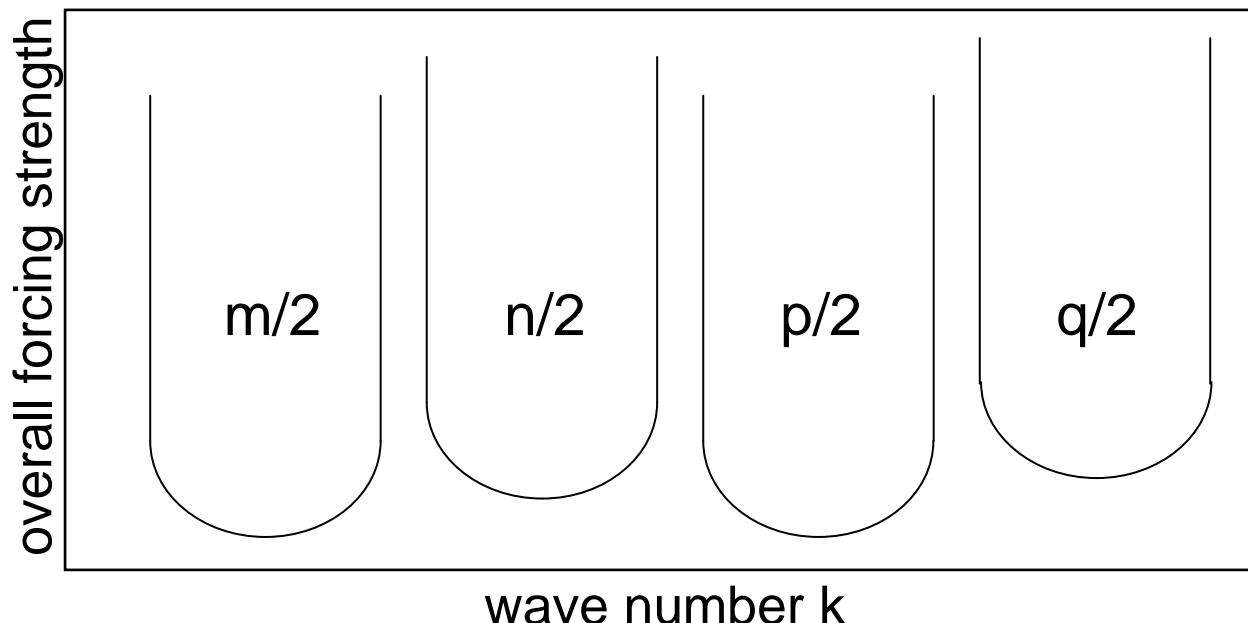
The diagram illustrates the decomposition of a function  $f(t)$  into its constituent parts. It shows three vertical arrows pointing upwards from the labels 'amplitudes', 'frequencies', and 'phases' to the corresponding terms in the equation. The first arrow points to the magnitude term  $|f_u|$ , the second to the frequency term  $iut$ , and the third to the phase term  $\phi_u$ .

# Some challenges of the Faraday system

- Huge (potentially infinite dimensional) parameter space
- Multiple length scales

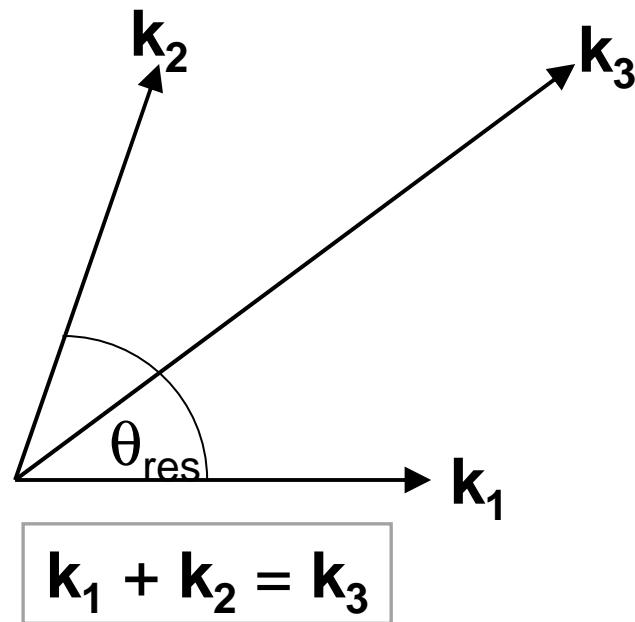
Sample Schematic Neutral Stability Curve:

$$f(t) = f_m e^{imt} + f_n e^{int} + f_p e^{ipt} + f_q e^{iqt} + c.c.$$



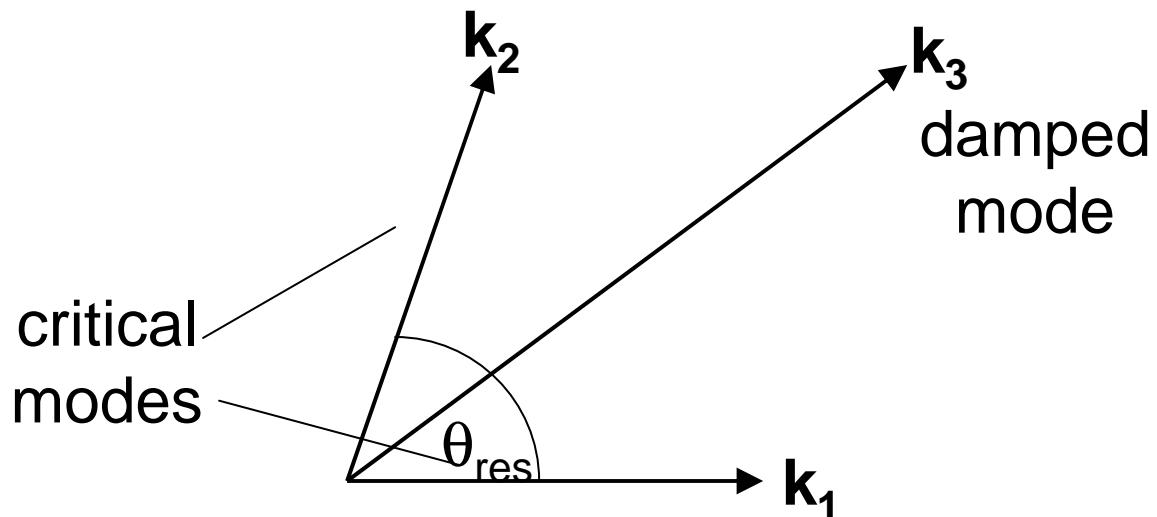
# Resonant triads

- Lowest order nonlinear interactions, affect pattern selection



# Resonant triads

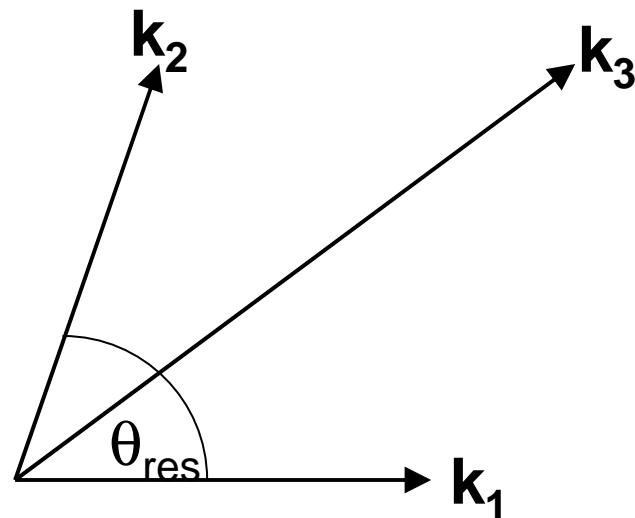
- Lowest order nonlinear interactions, affect pattern selection



(Zhang/Viñals, Müller, Porter/Silber/Skeldon/Topaz,...)

# Resonant triads

- Lowest order nonlinear interactions, affect pattern selection

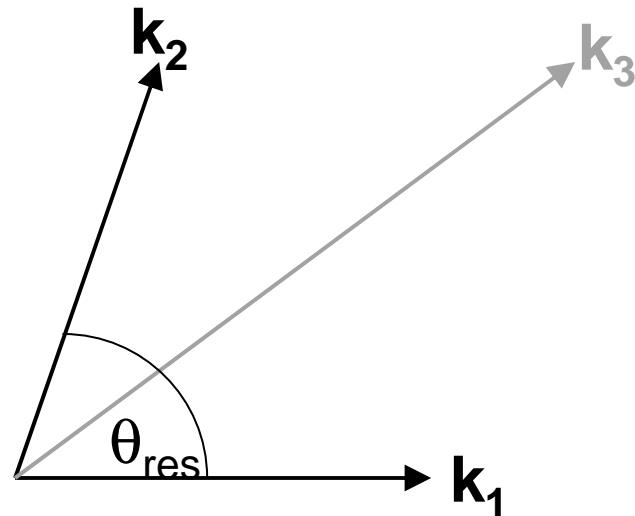


spatial translation, reflection, rotation by  $\pi$

$$\begin{aligned}\dot{A}_1 &= \Lambda_1 A_1 + \alpha_1 \overline{A}_2 A_3 + (a |A_1|^2 + b_0 |A_2|^2 + b_1 |A_3|^2) A_1 \\ \dot{A}_2 &= \Lambda_1 A_2 + \alpha_1 \overline{A}_1 A_3 + (a |A_2|^2 + b_0 |A_1|^2 + b_1 |A_3|^2) A_1 \\ \dot{A}_3 &= \Lambda_2 A_3 + \alpha_2 A_1 A_2 + (b_2 |A_1|^2 + b_2 |A_2|^2 + b_3 |A_3|^2) A_3\end{aligned}$$

# Resonant triads

- Lowest order nonlinear interactions, affect pattern selection

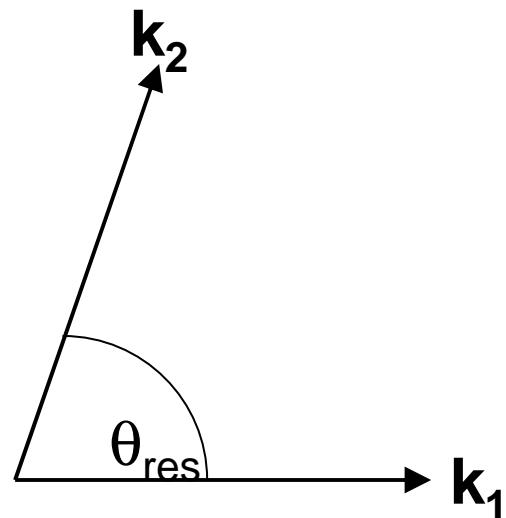


center manifold reduction

$$\begin{aligned}\dot{A}_1 &= \Lambda_1 A_1 + \alpha_1 \bar{A}_2 A_3 + (a |A_1|^2 + b_0 |A_2|^2 + b_1 |A_3|^2) A_1 \\ \dot{A}_2 &= \Lambda_1 A_2 + \alpha_1 \bar{A}_1 A_3 + (a |A_2|^2 + b_0 |A_1|^2 + b_1 |A_3|^2) A_1 \\ A_3 &= -(\alpha_2 / \Lambda_2) A_1 A_2 + \dots\end{aligned}$$

# Resonant triads

- Lowest order nonlinear interactions, affect pattern selection



rhombic equations

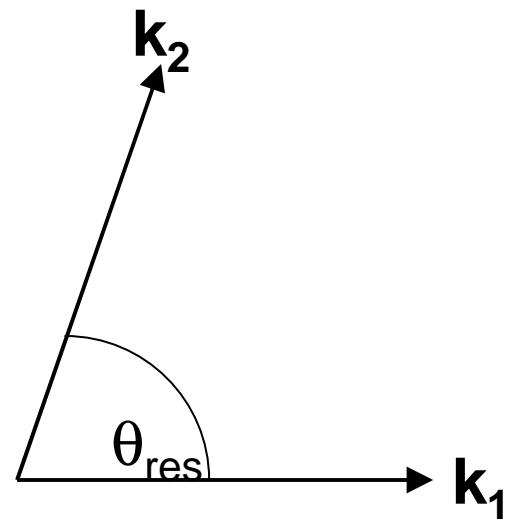
$$\dot{A}_1 = \Lambda_1 A_1 + a|A_1|^2 A_1 + b(\theta_{\text{res}})|A_2|^2 A_1$$

$$\dot{A}_2 = \Lambda_1 A_2 + a|A_2|^2 A_2 + b(\theta_{\text{res}})|A_1|^2 A_2$$

$$b(\theta_{\text{res}}) = b_0 + b_{\text{res}}, \quad b_{\text{res}} = -(\alpha_1 \alpha_2)/\Lambda_2$$

# Resonant triads

- Lowest order nonlinear interactions, affect pattern selection



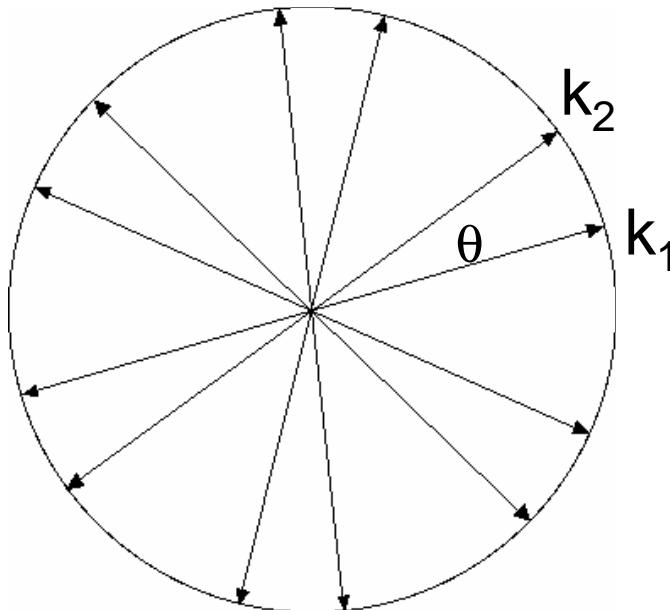
$b_{\text{res}} < 0 \rightarrow$  suppress rhombic state with angle  $\theta_{\text{res}}$

$b_{\text{res}} > 0 \rightarrow$  enhance rhombic state with angle  $\theta_{\text{res}}$

# Resonant triads

- Lowest order nonlinear interactions, affect pattern selection

Dionne & Golubitsky (1992)  
Dionne, Silber & Skeldon (1997)  
Silber & Proctor (1998)



stripes, rhombs, hexagons, superlattice

$b_{\text{res}} < 0 \rightarrow$  suppress pattern with angle  $\theta_{\text{res}}$

$b_{\text{res}} > 0 \rightarrow$  enhance pattern with angle  $\theta_{\text{res}}$

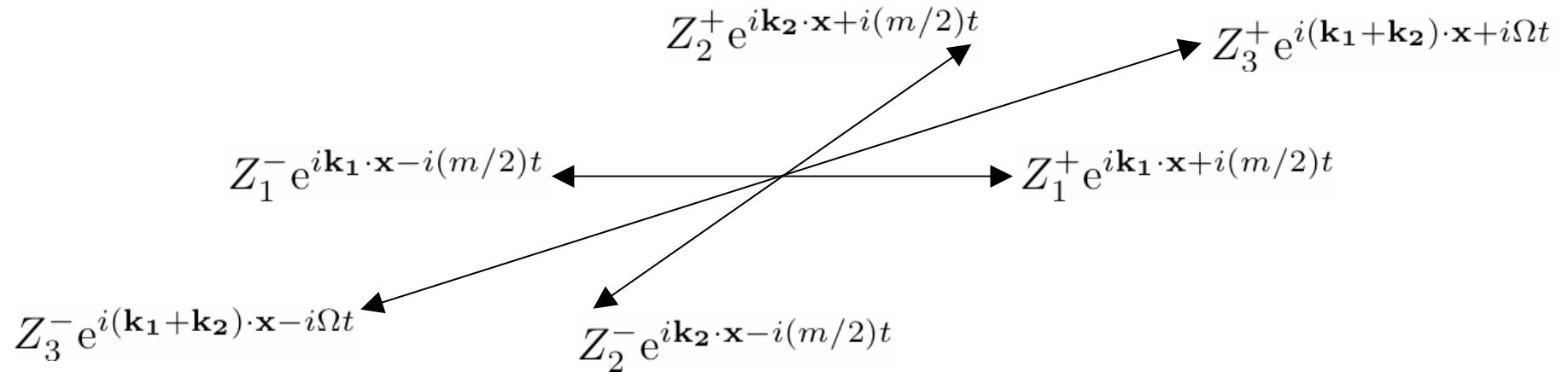
# Some questions to ponder

Which triad interactions are most significant?  
[Which mode for largest  $|b_{\text{res}}|$ ]

How does the triad interaction depend on some parameters?  
[Magnitude and sign of  $b_{\text{res}}(u, |f_u|, \phi_u)$ ]

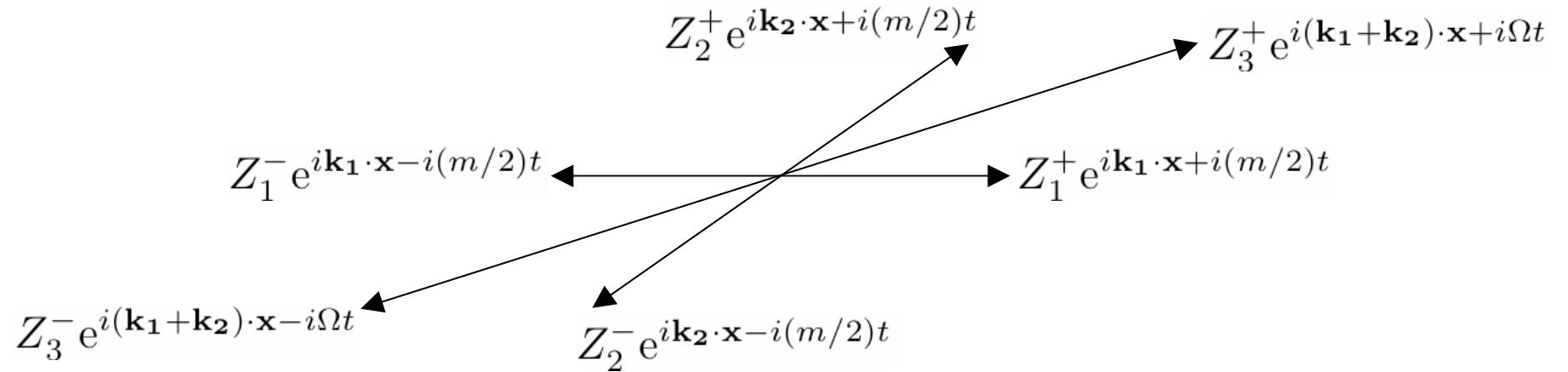
# Symmetry analysis

- Weakly nonlinearity and damping ( $\gamma$ ) and forcing ( $|f_u| \sim \mathcal{O}(\varepsilon)$ )



# Symmetry analysis

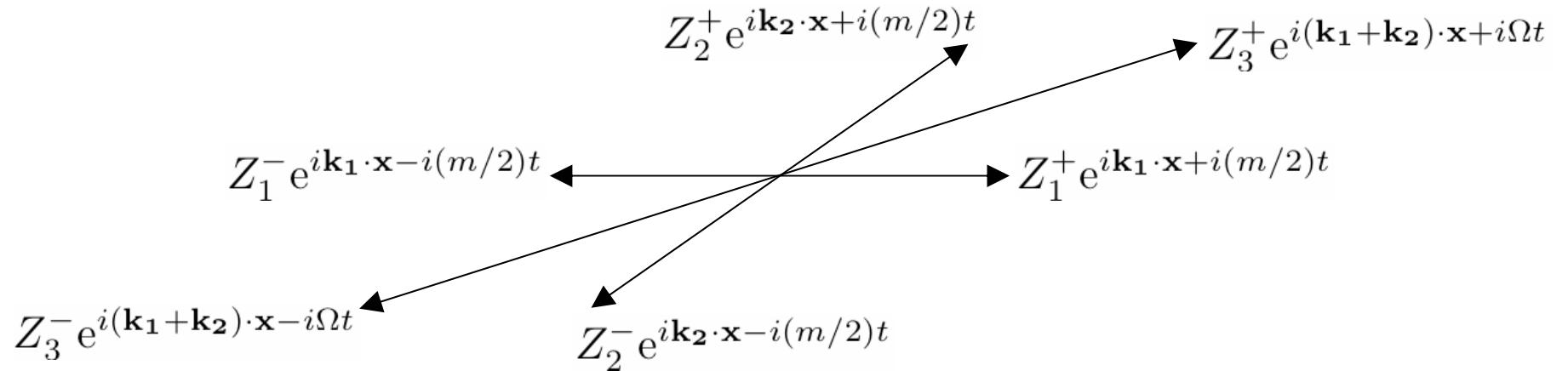
- Spatial symmetries



spatial translation, reflection, rotation by  $\pi$

# Symmetry analysis

- Parameter (broken temporal) symmetries



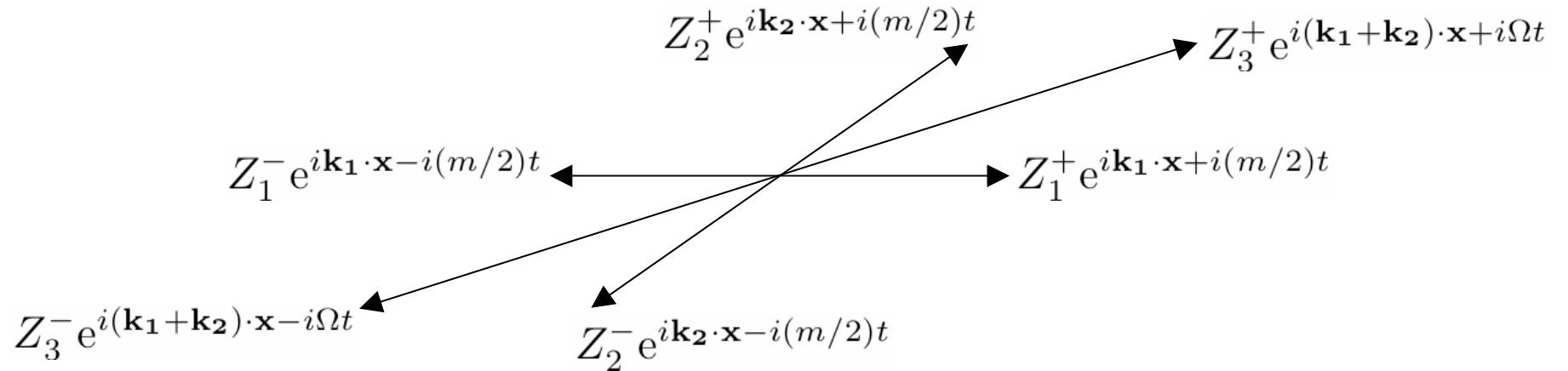
$$f(t) = \sum_{u \in \mathbb{Z}^+} f_u e^{iut} + c.c.$$

time translation symmetry

$$t \rightarrow t + \Delta t : (Z_{1,2}^\pm, Z_3^\pm) \rightarrow (Z_{1,2}^\pm e^{\pm i(m/2)\Delta t}, Z_3^\pm e^{\pm i\Omega\Delta t}), \quad f_u \rightarrow f_u e^{iu\Delta t}$$

# Symmetry analysis

- Parameter (broken temporal) symmetries



$$f(t) = \sum_{u \in \mathbb{Z}^+} f_u e^{iut} + c.c.$$

time reversal symmetry

$$(t, \gamma) \rightarrow - (t, \gamma) : \quad Z_j^\pm \rightarrow Z_j^\mp, \quad f_u \rightarrow \bar{f}_u$$

# Symmetry analysis

- Enforce symmetries  $\rightarrow$  TW amplitude equations

$$\begin{aligned}\dot{Z}_1^+ &= L_1 Z_1^+ + L_2 Z_1^- + Q_1 \bar{Z}_2^+ Z_3^+ + Q_2 \bar{Z}_2^+ Z_3^- + Q_3 \bar{Z}_2^- Z_3^+ + Q_4 \bar{Z}_2^- Z_3^- \\ \dot{Z}_3^+ &= L_3 Z_3^+ + L_4 Z_3^- + Q_5 Z_1^+ Z_2^+ + Q_6 Z_1^+ Z_2^- + Q_6 Z_1^- Z_2^+ + Q_7 Z_1^- Z_2^-\end{aligned}$$

$L_1$	$= -v_r \gamma + \dots$	$v_r > 0$	damping
$L_2$	$= -if_m \lambda_i + \dots$	$\lambda_i \in \mathbb{R}$	parametric forcing
$L_3$	$= -\varrho_r \gamma + \dots$	$\varrho_r > 0$	damping

# Symmetry analysis

- Enforce symmetries → TW amplitude equations

$$\begin{aligned}\dot{Z}_1^+ &= L_1 Z_1^+ + L_2 Z_1^- + Q_1 \bar{Z}_2^+ Z_3^+ + Q_2 \bar{Z}_2^+ Z_3^- + Q_3 \bar{Z}_2^- Z_3^+ + Q_4 \bar{Z}_2^- Z_3^- \\ \dot{Z}_3^+ &= L_3 Z_3^+ + L_4 Z_3^- + Q_5 Z_1^+ Z_2^+ + Q_6 Z_1^+ Z_2^- + Q_6 Z_1^- Z_2^+ + Q_7 Z_1^- Z_2^-\end{aligned}$$

$$\begin{aligned}L_4 &= -i f_{2\Omega} \mu_i + \dots \quad \mu_i \in \mathbb{R} && \text{parametric forcing} \\ Q_\ell &= -i q_\ell F_\ell + \dots \quad q_\ell \in \mathbb{R} && \text{coupling}\end{aligned}$$

involve products of the  $f_u$

# Symmetry analysis

- Enforce symmetries → TW amplitude equations

$$\begin{aligned}\dot{Z}_1^+ &= L_1 Z_1^+ + L_2 Z_1^- + Q_1 \bar{Z}_2^+ Z_3^+ + Q_2 \bar{Z}_2^+ Z_3^- + Q_3 \bar{Z}_2^- Z_3^+ + Q_4 \bar{Z}_2^- Z_3^- \\ \dot{Z}_3^+ &= L_3 Z_3^+ + L_4 Z_3^- + Q_5 Z_1^+ Z_2^+ + Q_6 Z_1^+ Z_2^- + Q_6 Z_1^- Z_2^+ + Q_7 Z_1^- Z_2^-\end{aligned}$$

Time translation invariants:

$$(Q_1, \bar{Q}_5) e^{i(\Omega-m)t}$$

$$(Q_2, Q_7) e^{-i(m+\Omega)t}$$

$$(Q_3, \bar{Q}_4, \bar{Q}_6) e^{i\Omega t}$$

$$L_4 e^{-2i\Omega t}$$

ex. (m,n) forcing,  $\Omega = 3m - n$

$$Q_1 e^{i(2m-n)t} \rightarrow Q_1 \sim \bar{f}_m^2 f_n$$

# Symmetry analysis

- Enforce symmetries → TW amplitude equations

$$\begin{aligned}\dot{Z}_1^+ &= L_1 Z_1^+ + L_2 Z_1^- + Q_1 \bar{Z}_2^+ Z_3^+ + Q_2 \bar{Z}_2^+ Z_3^- + Q_3 \bar{Z}_2^- Z_3^+ + Q_4 \bar{Z}_2^- Z_3^- \\ \dot{Z}_3^+ &= L_3 Z_3^+ + L_4 Z_3^- + Q_5 Z_1^+ Z_2^+ + Q_6 Z_1^+ Z_2^- + Q_6 Z_1^- Z_2^+ + Q_7 Z_1^- Z_2^-\end{aligned}$$

Focus on  $Q_\ell \gtrsim \mathcal{O}(\epsilon)$

Possible only for  $\Omega \in \{m, 2m, n, n - m, m \pm n\}$

At most 5 relevant forcing frequencies for fixed  $\Omega$

# Some key (general) results

- Strongest interaction is for  $\Omega = m$

$$|b_{\text{res}}| \sim O(1) \quad \text{for } \Omega = m$$

$$|b_{\text{res}}| \sim O(\varepsilon) \quad \text{for } \Omega = 2m, n, m \pm n, n - m$$

# Some key (general) results

- Strongest interaction is for  $\Omega = m$
- Forcing damped mode can drastically strengthen interaction

ex.  $(m, n, p = 2n - 2m)$  forcing,  $\Omega = n - m$

$$b_{\text{res}} = \alpha_2 |f_n|^2 \underbrace{\frac{|\varrho_r \gamma| + \mu_i |f_p| \sin \Phi}{|\varrho_r \gamma|^2 - |\mu_i f_p|^2}}_{|\varrho_r \gamma| > |\mu_i f_p|}$$

since  $\Omega = n - m$  is damped

# Some key (general) results

- Strongest interaction is for  $\Omega = m$
- Forcing damped mode can drastically strengthen interaction
- Phases  $\phi_u$  may tune interaction strength

ex.  $(m, n, p = 2n - 2m)$  forcing,  $\Omega = n - m$

$$b_{\text{res}} = \alpha_2 |f_n|^2 \frac{|\varrho_r \gamma| + \mu_i |f_p| \sin \Phi}{|\varrho_r \gamma|^2 - |\mu_i f_p|^2}$$

$$\Phi = \phi_p + 2\phi_m - 2\phi_n$$

# Some key (general) results

- Strongest interaction is for  $\Omega = m$
- Forcing damped mode can drastically strengthen interaction
- Phases  $\phi_u$  may tune interaction strength
- Only  $\Omega = n - m$  is always enhancing (Hamiltonian argument)

ex. ( $m, n, p = 2n - 2m$ ) forcing,  $\Omega = n - m$

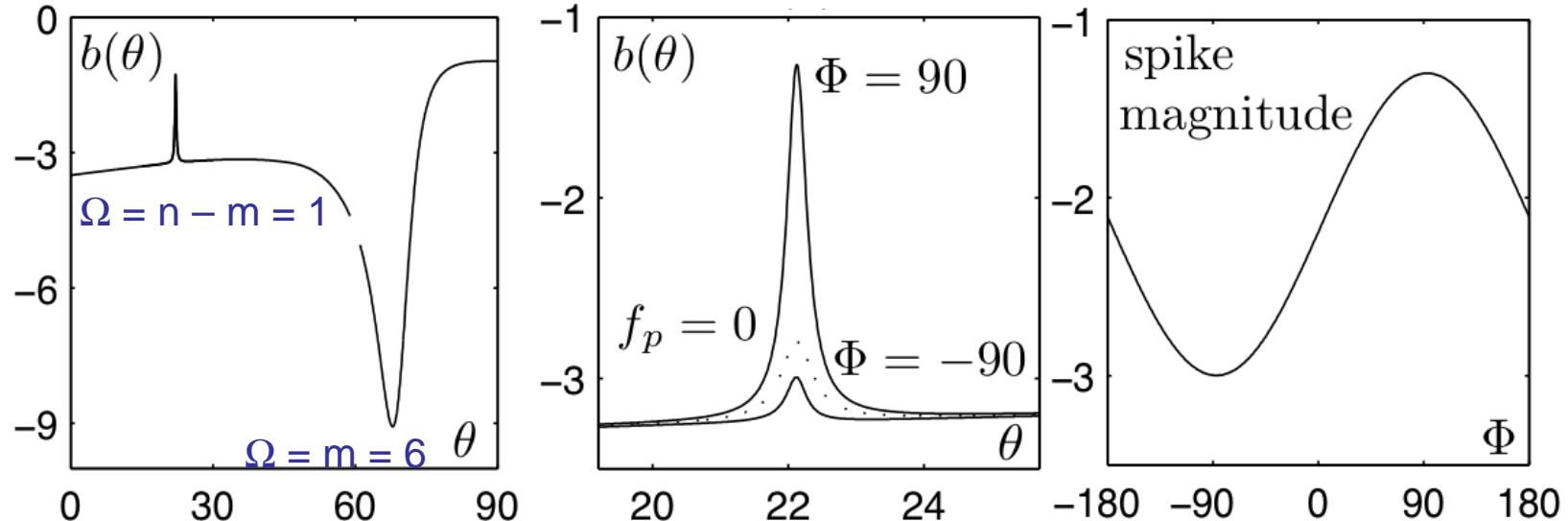
$$b_{\text{res}} = \underbrace{\alpha_2 |f_n|^2}_{> 0} \underbrace{\frac{|\varrho_r \gamma| + \mu_i |f_p| \sin \Phi}{|\varrho_r \gamma|^2 - |\mu_i f_p|^2}}_{P_p(\Phi) > 0}$$

$b_{\text{res}} > 0$  for this case!  
(can get signs for some other cases)

# Some key (general) results

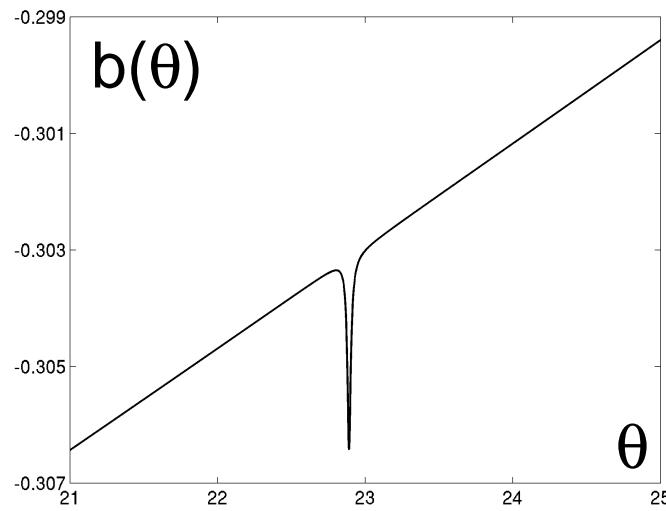
- Strongest interaction is for  $\Omega = m$
- Forcing damped mode can drastically strengthen interaction
- Phases  $\phi_u$  may tune interaction strength
- Only  $\Omega = n - m$  is always enhancing ( $b_{\text{res}} > 0$ )

ex. (6,7,2) forcing, Zhang-Viñals equations (J. Fluid Mech 1997)



# Range of validity of symmetry results

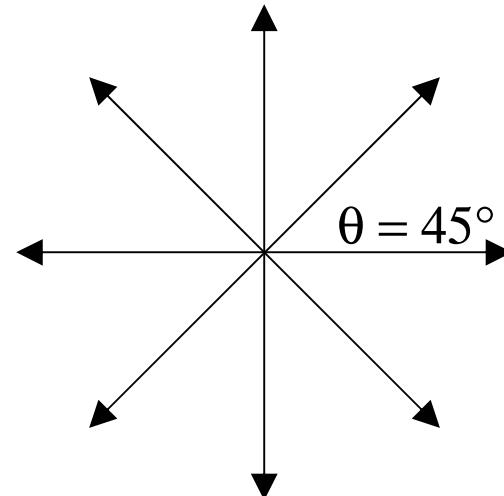
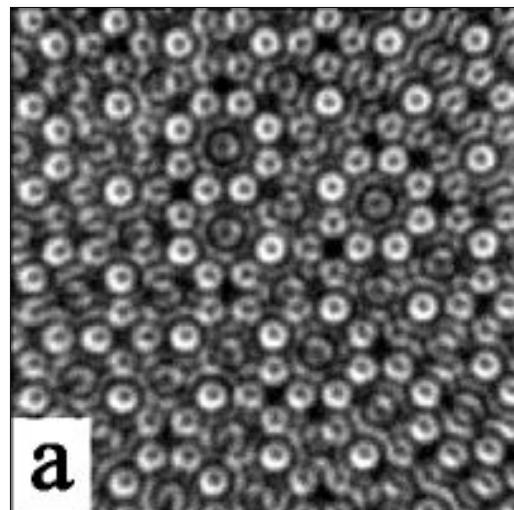
- Scaling laws verified via calculations on Zhang-Vinals eq.
- Ex. (8,7,2) forcing,  $\Omega = 1$
- $|b_{\text{res}}| \sim \gamma$  up to  $\gamma \sim O(10^{-1})$
- Half-width  $\sim \gamma$  up to  $\gamma \sim O(10^{-1})$
- Dependence on  $\Phi$  up to  $\gamma \sim O(10^{-2}) - O(10^{-1})$



# Application I: Experimental quasipattern

Arbell & Fineberg, PRE, 2002

(3,2,4) forcing { Interestingly, this state was exceedingly stable and existed within a single domain over a wide range of parameters (note the sharp peaks in Fig. 38). This is in sharp contrast to the distorted eightfold state shown in Fig. 37, which existed in both a narrow range of parameters and, as evident in its diffuse spectrum, had a tendency to break up into domains. It } (3,2) forcing



# Application I: Experimental quasipattern

From symmetry results:

$(m,n) = (3,2)$  forcing,  $\Omega = 1$

$$b_{\text{res}}(\gamma, f_u) = -\alpha_1 |f_n|^2 P_n(3\phi_n - 2\phi_m) < 0$$

# Application I: Experimental quasipattern

From symmetry results:

$(m,n) = (3,2)$  forcing,  $\Omega = 1$

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$(m,n) = (3,2,4)$  forcing,  $\Omega = 1$

$$b_{\text{res}}(\gamma, f_u) = \underbrace{-\alpha_1 |f_n|^2 P_n(3\phi_n - 2\phi_m)}_{< 0} + \underbrace{\alpha_2 |f_p|^2 P_n(2\phi_m + \phi_n - 2\phi_p)}_{> 0}$$

# Application I: Experimental quasipattern

From symmetry results:

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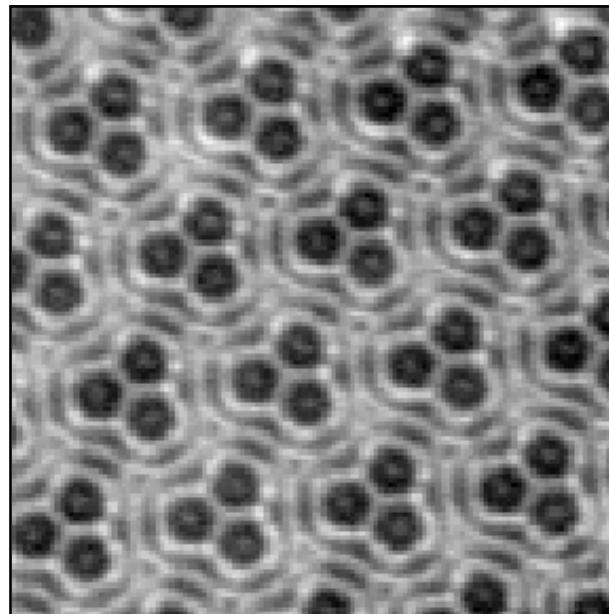
From hydrodynamic dispersion relation:

$$\theta_{\text{res}} \approx 42.8^\circ$$

(expect could spill over to  $45^\circ$  for experimental  $\gamma$ )

# Application II: Superlattice selection

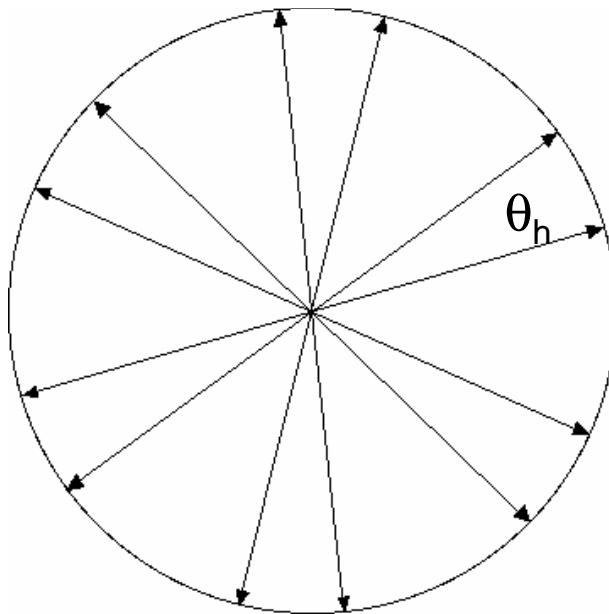
- Idea: use symmetry results to construct a forcing function which favors a chosen pattern



Kudrolli et al., Physica D (1998)

# Application II: Superlattice selection

- Step 1: Use geometry to select  $\theta_{\text{res}}$



Stability of superlattice depends on  
 $b(\theta_h)/a$ ,  $b(60^\circ - \theta_h)/a$ ,  $b(60^\circ + \theta_h)/a$

**Strategy: Make  $b_{\text{res}} > 0$  (via difference frequency mode)**

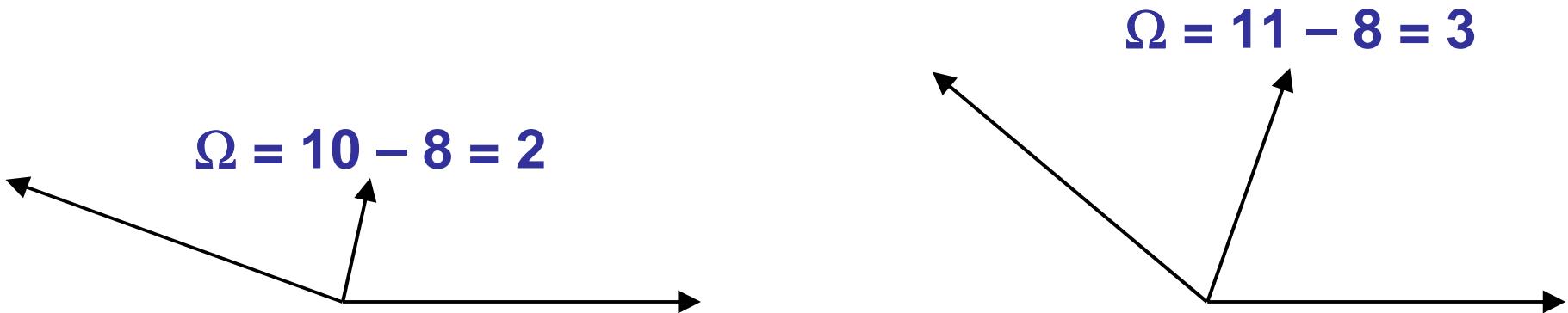
# Application II: Superlattice selection

- Step 2: Use inviscid dispersion relation and trigonometry

Example: Choose  $\theta_h \approx 20.2^\circ \rightarrow 60^\circ - \theta_h \approx 39.8^\circ$

Need  $\{m,n,p\}$  such that  $k(n - m)$  and  $k(p - m)$  determine resonant angles of  $20.2^\circ$  and  $39.8^\circ$  with  $k(m/2)$   
[actually,  $180^\circ - 20.2^\circ$  and  $180^\circ - 39.8^\circ$  are ok]

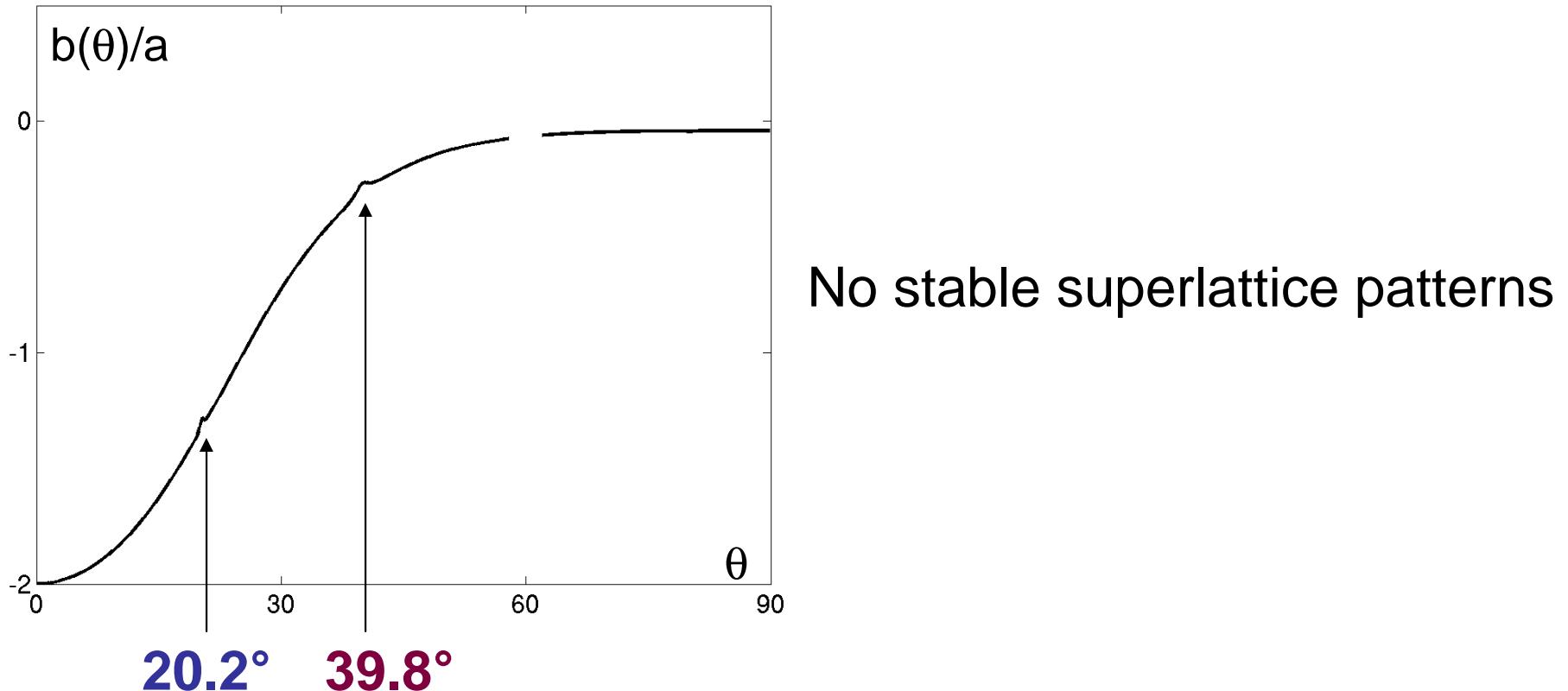
$$(m, n, p) = (8, 10, 11)$$



# Application II: Superlattice selection

- Step 2: Use inviscid dispersion relation and trigonometry

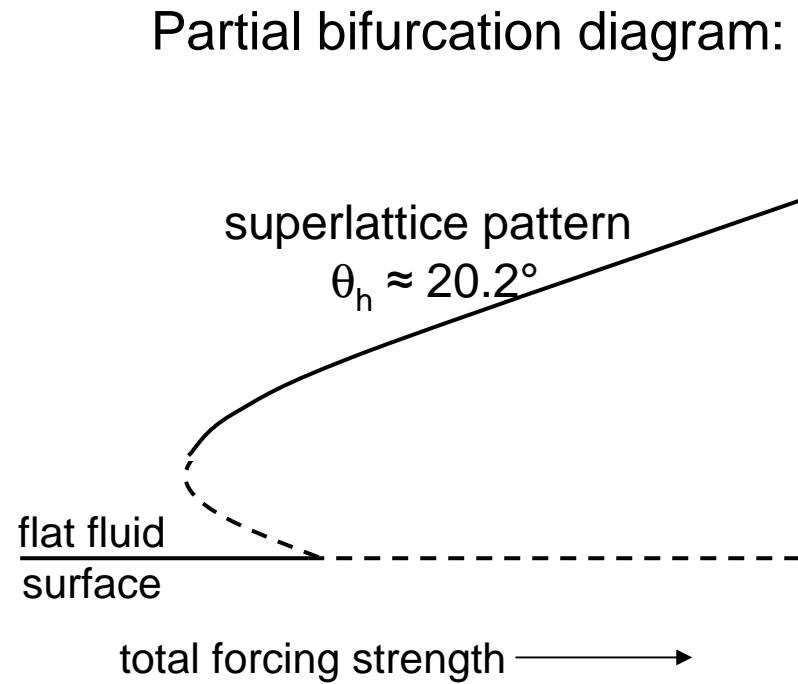
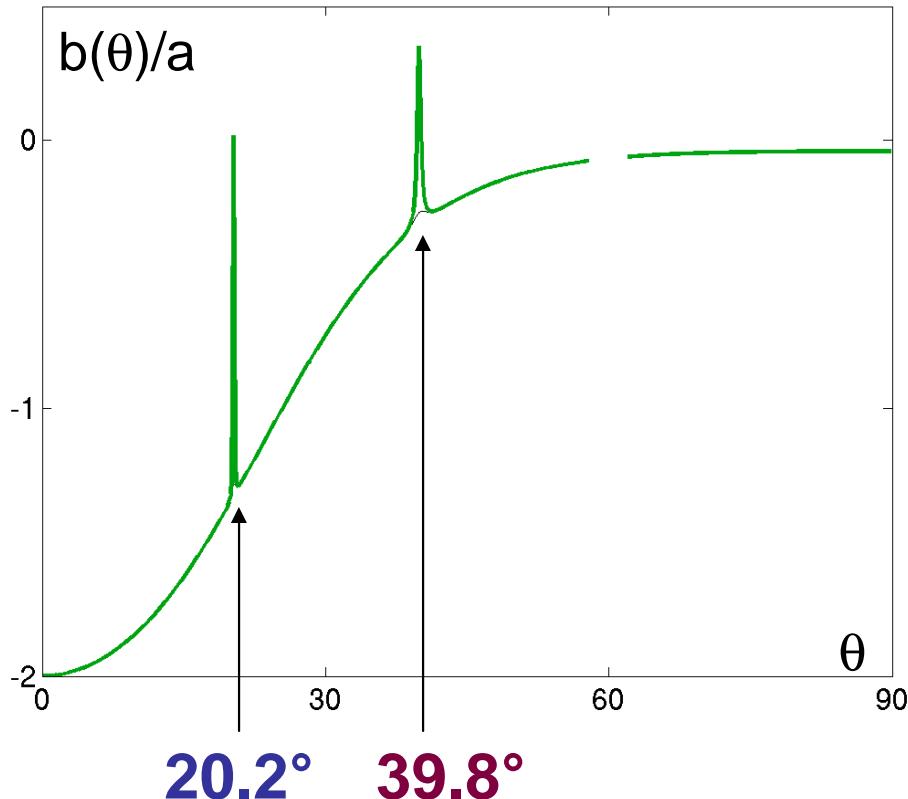
$$(m, n, p) = (8, \mathbf{10}, \mathbf{11}), (\phi_m, \phi_n, \phi_p) = (0^\circ, 0^\circ, 0^\circ)$$



# Application II: Superlattice selection

- Step 3: Force damped mode, use phases to enhance effect

$$(m, n, p, q, r) = (8, \mathbf{10}, \mathbf{11}, \mathbf{4}, \mathbf{6}), (\phi_m, \phi_n, \phi_p, \phi_q, \phi_r) = (0^\circ, 0^\circ, 0^\circ, 10^\circ, -12^\circ)$$



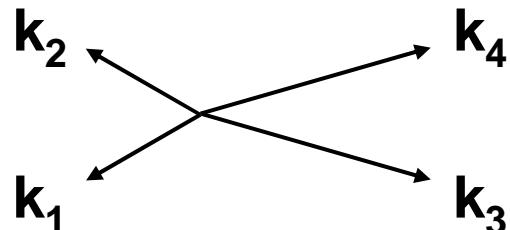
# Conclusions

- Arbitrary periodic forcing function
- Parameter symmetries
- Classification of resonant triads
- Scaling, phase dependence
- Enhancing vs. suppressing (sometimes)
- Results used in explanatory and prescriptive manner

The future:

Other systems, e.g. vibrated convection (Rogers, Schatz et al., PRL, 2000)

Four-wave interactions



$$\mathbf{k}_1 - \mathbf{k}_2 = \mathbf{k}_3 - \mathbf{k}_4$$