An Organizing Center for Tracefiring in the Oregonator

Building blocks and instability

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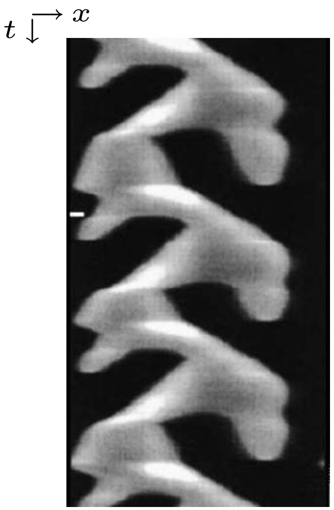
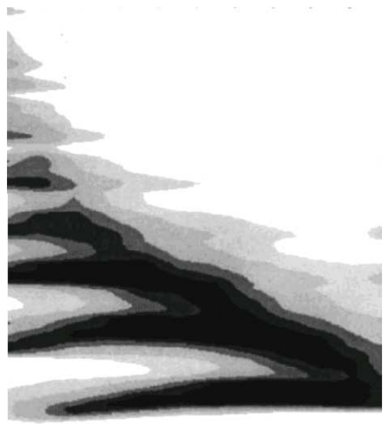
Backfiring and Tracefiring



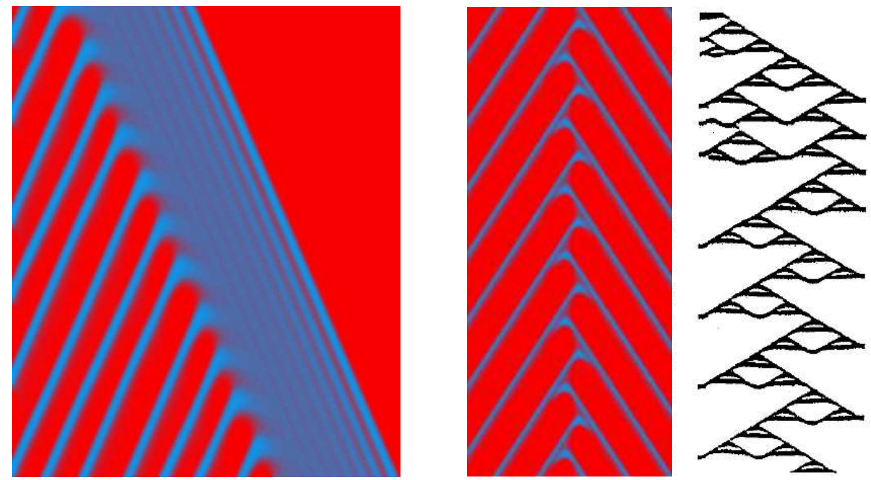
[Annamalai et al '99]

countercurr. flow reactor

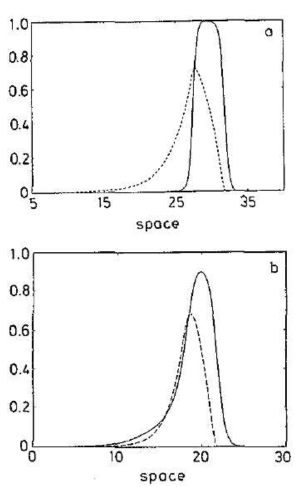


[Otterstedt et al '98]

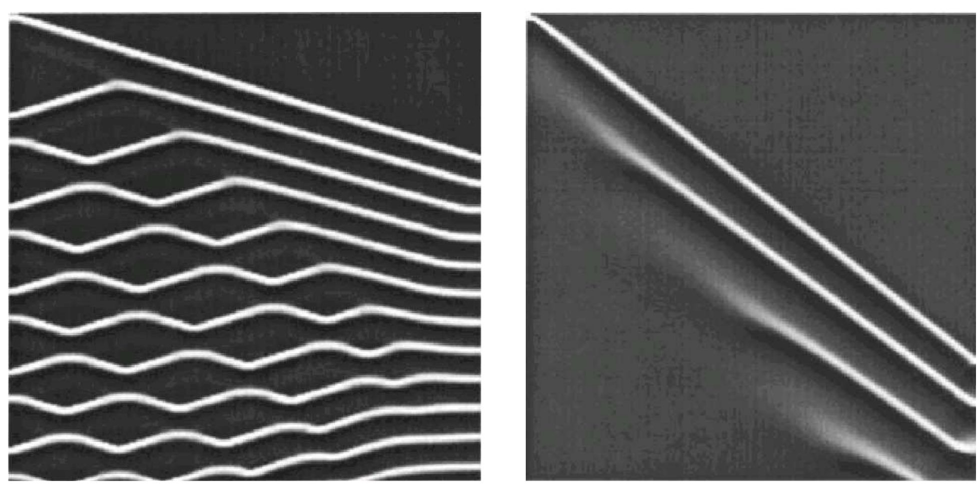
active/passive cobalt



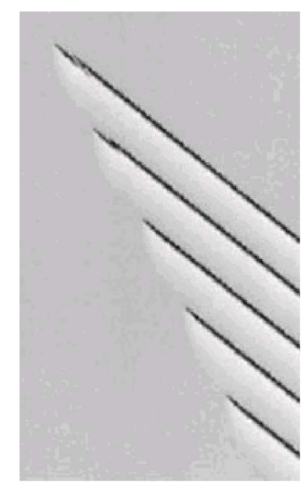
MB model for Co-ox. on Pt110: [Bär et al '94]



MB model

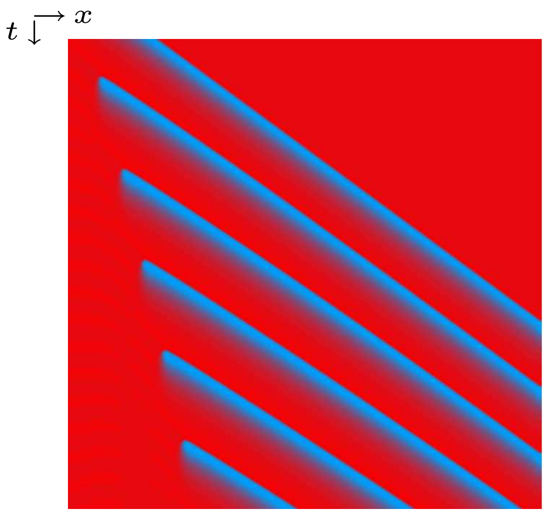


2-state / 3-state Calcium and IP_3-receptor [Sneyd et al '00]



CICR [Snita et al '00]

Tracefiring in the Oregonator model



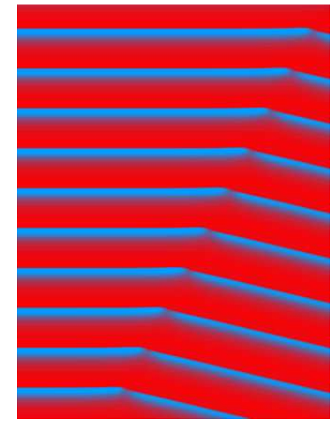
Two characteristic speeds:

- **pulse-chain ~ 3.4 (sink)**

  ω-limit: invading pulse-chain

- **defect ~ 0.6 (source)**

  ω-limit: rest state connected to space-time periodic

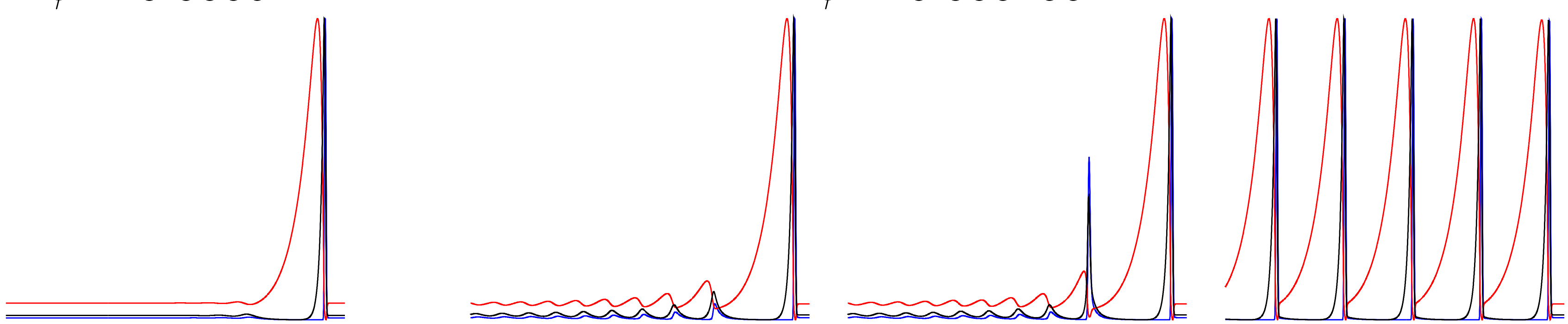


other defect for
$\phi = 0.00016$

Kinetics parameter $\phi \sim$ light intensity $\sim$ excitability

(b.c. in comoving frame: left Neumann, right Dirichlet)

$\phi = 0.0006$ $\phi = 0.000155$



$t = 0$ $t = 50$ $t = 1000$

Spatial Dynamics

Constituents of pulse splitting, (back-) tracefiring

$$U \in \mathbb{R}^N, \quad U_t \quad = \quad DU_{xx} + F(U;\phi)$$

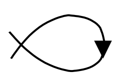
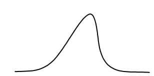
$$\xi = x - ct \quad \to \quad U_t \quad = \quad DU_{\xi\xi} + cU_\xi + F(U;\phi)$$

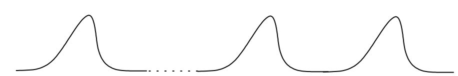
equilibria in these coordinates satisfy travelling wave ODE

$$0 \quad = \quad DU_{\xi\xi} + cU_\xi + F(U;\mu) \quad \Leftrightarrow$$

$$u_\xi \quad = \quad f(u;c,\phi) = f(u;\mu), \quad u \in \mathbb{R}^{2N}$$

bounded ODE solution $\leftrightarrow$ **travelling wave, speed** c

homoclinic  $\leftrightarrow$ **pulse** 

n**–homoclinic** $\leftrightarrow$ n**–pulse** 

heteroclinic: pulse-train to equilibrium $\leftrightarrow$ **invading pulse-chain**

T-point in other models

Organizing Center: Constituents exist nearby,

pulse loses stability nearby

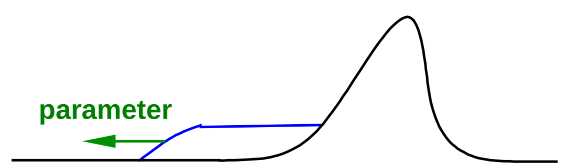
MB [Zimmermann et al ’97], Ca-flow [Sneyd et al ’00] models:

Codim-2 heteroclinic cycle between two equilibria

(T-point [Glendinning, Sparrow ’86])

Interaction of pulse and equilibrium: plateau in pulse’s wake

from other, (absolutely) unstable rest state



Instability: MB [Sandstede, Scheel ’00], Ca [Romero, Jones ’03]

What happens in the Oregonator ?

The light-sensitive Oregonator model

$$\begin{aligned} u_t &= D_u u_{xx} + (u(1-u) - v(u-q))/\epsilon \\ v_t &= D_v v_{xx} + (fw + \phi - v(u+q))/\delta \\ w_t &= u - w \end{aligned}$$

$D_v = 1.12$, $\epsilon = 0.09$, $\delta = \epsilon/8$, $q = 0.001$, $f = 1.5$

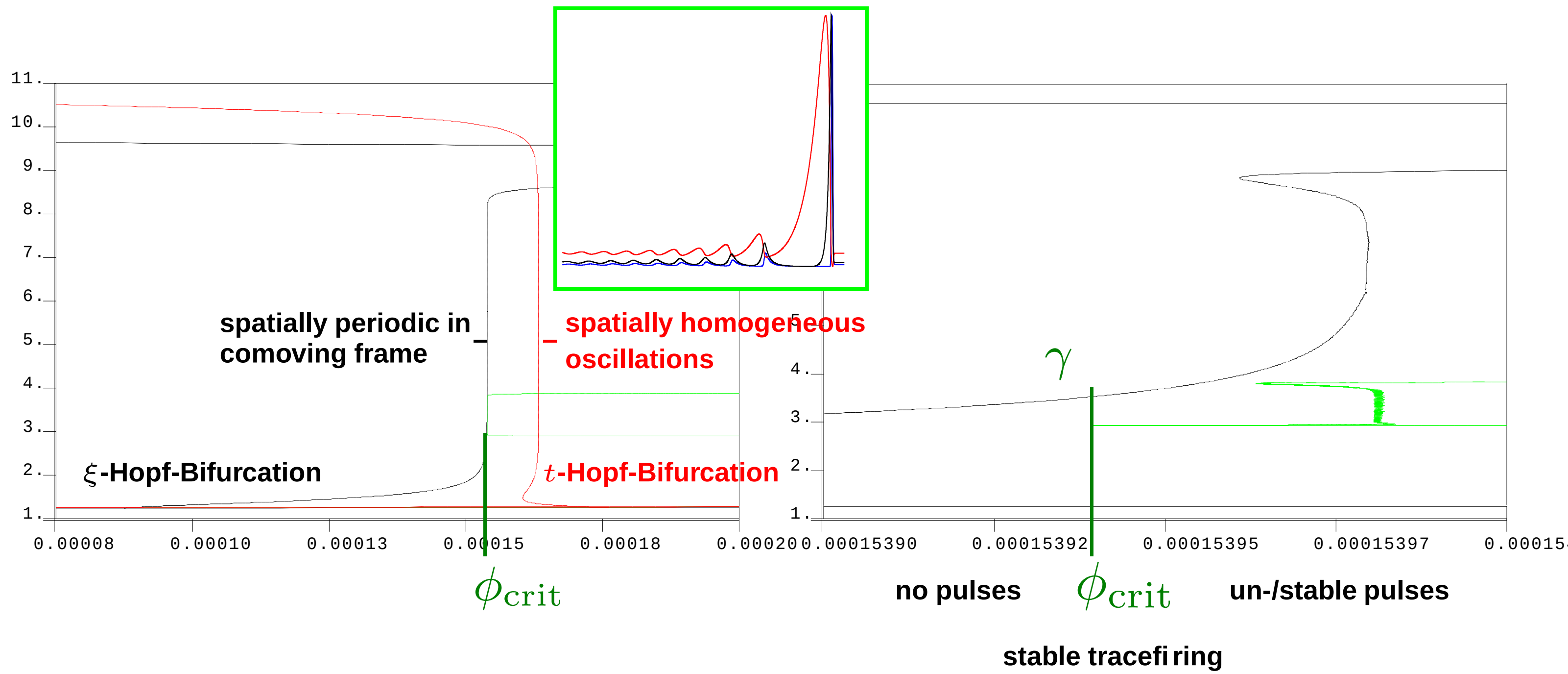
Chemical species in $\mathbb{R}^3$: $u \sim$ bromous acid, $v \sim$ bromide, $w \sim$ catalyst

Travelling wave ODE in $\mathbb{R}^5$ ($c \approx 3.4$):

$$\begin{aligned} u_\xi &= \tilde{u} \\ \tilde{u}_\xi &= -cu_\xi - (u(1-u) - v(u-q))/\epsilon \\ v_\xi &= \tilde{v} \\ \tilde{v}_\xi &= -(cv_\xi + (fw + \phi - v(u+q))/\delta)/D_v \\ w_\xi &= (w-u)/c \end{aligned}$$

Bifurcations and Tracefiring Onset

Hopf-bifurcations of kinetics ODE and travelling wave ODE, and path of homoclinic - instability when *decreasing* ϕ:

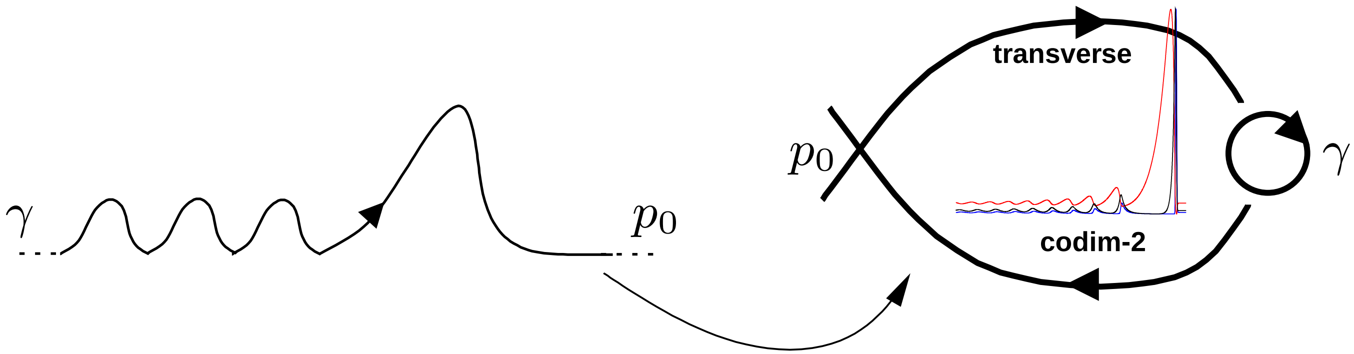


Periodic or Neumann boundary conditions: transient tracefiring

Dirichlet b.c. ahead of primary pulse: permanent tracefiring

Organizing Center for Oregonator's Tracefiring

Idea: codim-2 heteroclinic cycle with *periodic* orbit



Oregonator Morse indices:

$$i(p_0) = \dim(W^{\mathrm{u}}(p_0)) = 3,\ i(\gamma) := \dim(W^{\mathrm{cu}}(\gamma)) = 2$$

Generally for ODE in $\mathbb{R}^n$:

periodic orbit: $i(\gamma) \geq 2$, $\dim(W^{\mathrm{sc}}(\gamma)) = n - i(\gamma) + 1 \geq 2$

codim-2: $i(p_0) = i(\gamma) + 1 \quad \rightarrow \quad i(p_0) \geq 3$

get 2D transversality: $i(p_0) + \dim(W^{\mathrm{sc}}(\gamma)) = n + 2$

for heteroclinic cycle: $\dim(W^{\mathrm{s}}(p_0)) \geq 1 \quad \rightarrow \quad n \geq 4$

Such cycles have recently been found in other models [Sieber '02, Sneyd '03].

Existence of Constituents near Organizing Center

Theorem (R.)

Assume such a heteroclinic cycle with maximal transversality. Then a family of curves of locally unique 1-pulses exist, emanating from heteroclinic cycle in parameter space.

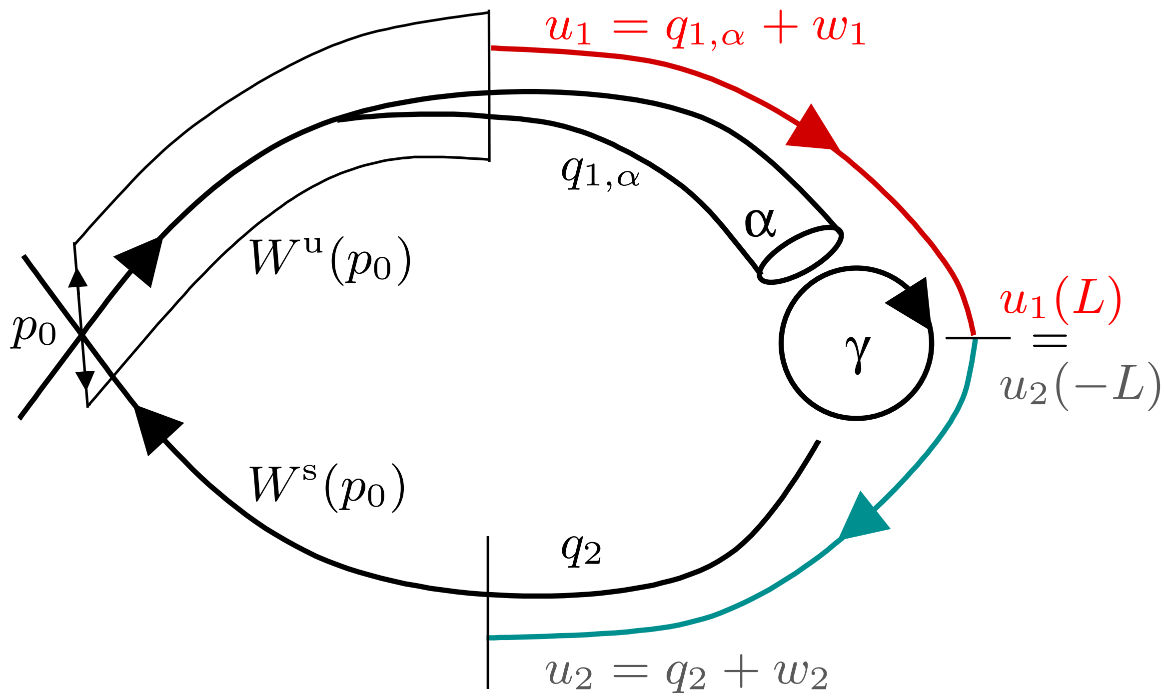
(Also for other codim and more general heteroclinic nets.)

Assume in addition $W_1 := \mathrm{W}^{\mathrm{u}}(p_0) \cap \mathrm{W}^{\mathrm{cs}}(\gamma)$ **contains a curve homotopic to** γ **in** $W_1 \cup \gamma$ **(holds e.g. near spatial Hopf Bifurcation). Then nearby a smooth curve of unique 1-pulses exist, which emanates from the heteroclinic cycle.**

Under Shil'nikov condition a (family of) curve(s) of n**-pulses, and invading pulse-chain exists. (In Oregonator pulses are Shil'nikov-type and** γ **close to spatial Hopf bifurcation).**

Idea of Proof: Adapt Lin's Method

1. Seek solutions near the heteroclinics and 'glued' close to γ
2. Match these away from γ with un/stable manifolds from p_0



Problems here:

- by phase shift: lack of hyperbolicity (essential spectrum)
- periodic distance of forward vs. backward approach to γ

Spectra of Pulses on Large Bounded Domains

and approaching heteroclinic bifurcation

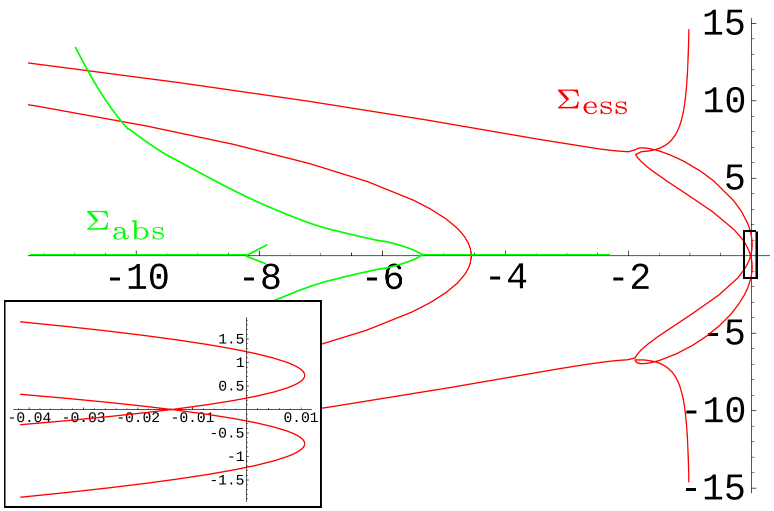
Theorems [Sandstede, Scheel '00] **Typically:**

- **point spectrum from unbounded domain persists**
- **point spectrum for increasing domain length clusters:**
  near essential spectrum Σ_{ess} for *periodic* b.c.
  near absolute spectrum Σ_{abs} for *separated* b.c.
- **If stable pulse approaches heteroclinic orbit with absolutely unstable equilibrium (or periodic orbit) p_2, then its critical point spectrum clusters near absolute spectrum of p_2.**

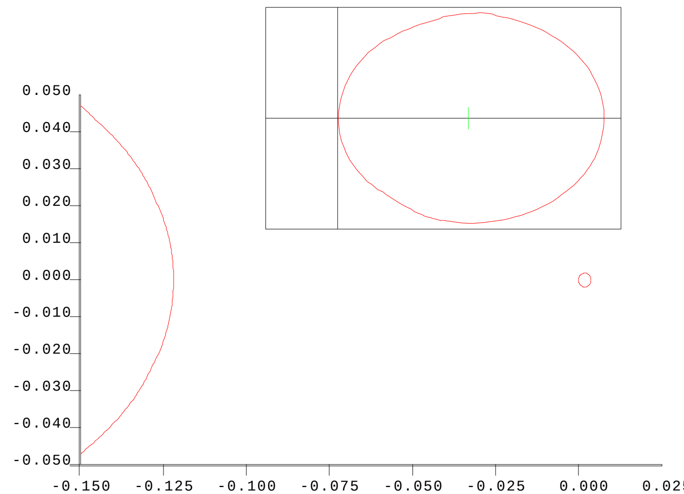
Σ_{abs}**: Solutions $\mathrm{e}^{\lambda t+\nu_1\xi}$, $\mathrm{e}^{\lambda t+\nu_2\xi}$ with $\mathrm{Re}(\nu_1)=\mathrm{Re}(\nu_2)$ and index condition. Distinguishes absolute and convective instabilities.**

Spectra in the Oregonator

Spectra of p_0 (base state)



Spectra of γ for ϕ near onset



Observations

- $\Sigma_{\mathrm{ess}}(p_0)$ **is already unstable: convective instability of pulse on unbounded domain or Neumann b.c. –** ***not*** **onset of tracefi ring**
- $\Sigma_{\mathrm{abs}}(p_0)$ **remains stable: no absolute instability of background**
- $\Sigma_{\mathrm{ess}}(\gamma)$**, and in particular** $\Sigma_{\mathrm{abs}}(\gamma)$ **unstable**

Framework for Oregonator's Tracefiring Instability

Conclusions in light of organizing center idea

Assuming stable isolated point spectrum, the pulse becomes unstable when approaching the organizing center with absolutely unstable γ. In fact, winding around γ increases Morse-index. Similar for n-pulses.

Geometric indication of real-type absolute instability:

Theorem (R.)

All assumptions of theorem 1. Path of homoclinic spirals (logarithmically) into point of heteroclinic cycle, if and only if $0 \in \Sigma_{\mathrm{abs}}$ and Σ_{abs} unstable.

Idea of proof

Use Floquet form of variation about γ and to compute leading order expansion of homoclinic path. This oscillates, if and only if $0 \in \Sigma_{\mathrm{abs}}$ and Σ_{abs} unstable.

Discussion

- several models exhibit tracefiring (or backfiring)
- Oregonator's transient tracefiring permanent with mixed b.c.
- idea for organizing center of Oregonator's tracefiring instability:

theory and observations are coherent, and analogous to T-point:

- constituents exist in unfolding
- instability caused by abs. unstable 'wiggles' in pulse's wake
- spiraling path (saddle-nodes) iff real-type absolute instability

Perspectives

- apply analytic method to bifurcations in other heteroclinic networks with periodic orbits
- periodic nature of tracefiring (also for T-points)

Spectra and Stability

For a solution U_* to $U_t = DU_{\xi\xi} + cU_\xi + F(U;\phi)$, we say $\lambda \in \mathbb{C}$ is in the spectrum $\Sigma(U_*)$ of U_*, if

$$\mathcal{T}(\lambda) = (\lambda - D\partial_{\xi\xi} - c\partial_\xi - \partial_U F(U_*))$$

is not boundedly invertible.

We say $\lambda \in \Sigma(U_*)$ is in the point spectrum, if $\mathcal{T}(\lambda)$ is Fredholm with index 0; otherwise λ lies in the essential spectrum $\Sigma_{\mathrm{ess}}(U_*)$.

Travelling waves spectra can be characterized in terms of exponential dichotomies of

$$v_\xi = \left(\partial_u f(u_*(\xi);\mu) + \lambda \begin{pmatrix} 0 & 0 \\ -\mathrm{Id} & 0 \end{pmatrix} \right) v$$

Here spectral stability ($\mathrm{Re}(\Sigma(U_*)) < 0$) implies nonlinear stability.