LATE STAGE INTERFACE DYNAMICS
AND
RENORMALIZATION GROUP METHODS

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RENORMALIZATION AND SCALING IN PHYSICS

- Renormalization Group (RG)

- Quantum field theory and statistical mechanics, in particularly critical phenomena

- Wilson and Fisher (in early 70's)

RG calculations are generally composed of two steps:

(1) Coarse graining

(2) Rescaling

RG methods have evolved into a broad philosophy rather than a single technique as each new application often involve different method and have been applied to problems in Physics and Applied Mathematics.

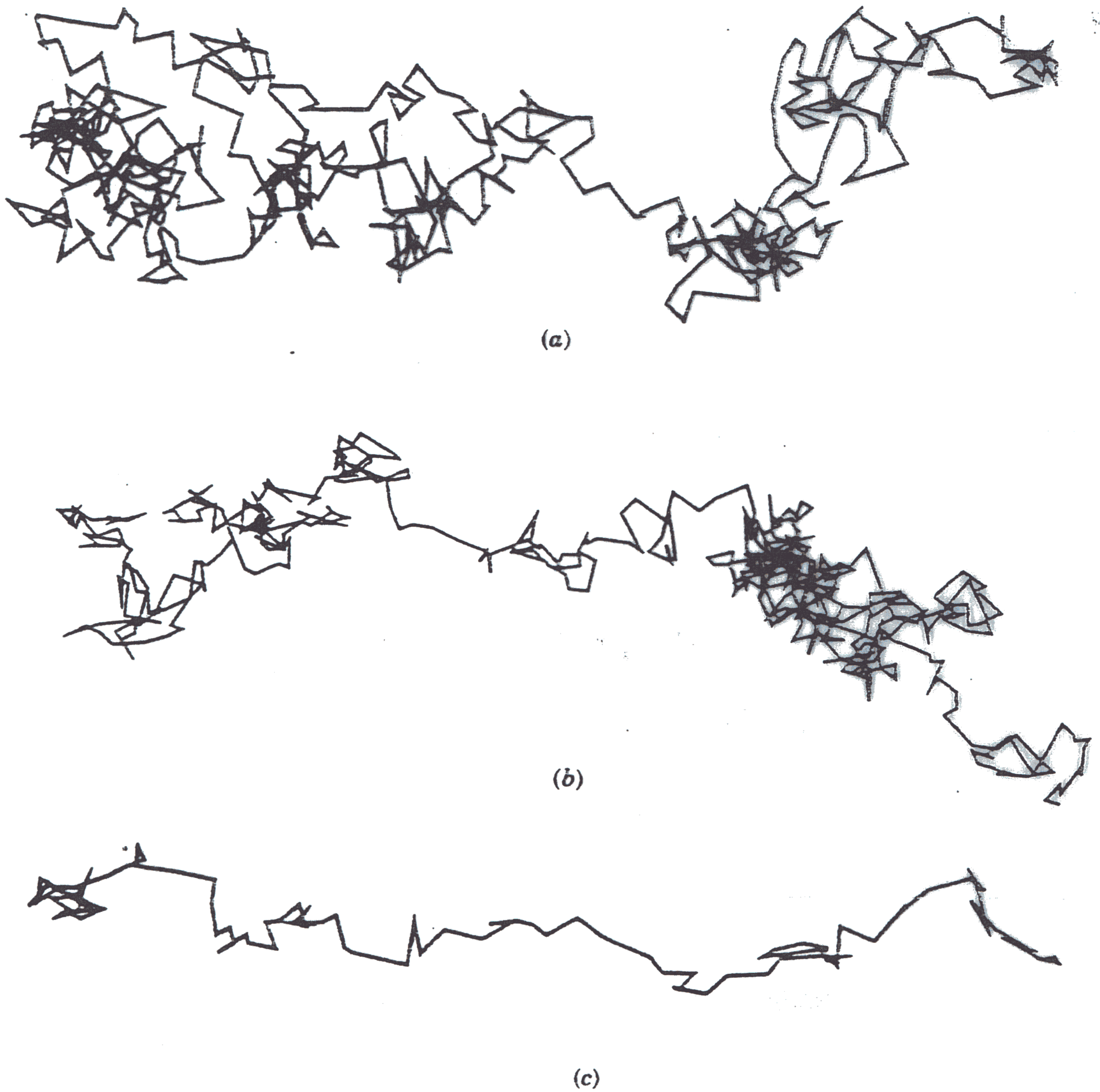


Fig. 1.6.2 Random walk with a drift term. (a) A small segment of the walk is shown and should be compared with the pure random walk, Fig. 1.6.1a. (b) A larger scale view of the same walk shows one-dimensional behavior beginning to emerge. (c) At even larger scales the random component of the walk is much less noticeable and its ultimate one-dimensional nature emerges.

APPLICATION IN APPLIED MATHEMATICS

Creswick, Farach, Poole text. Applications include Cantor sets (and calculating fractal dimensions), random walk (and central limit theorem), self-avoiding random walk, period doubling and chaos.

RENORMALIZATION FOR DIFFERENTIAL EQUATIONS

Decay, finite time blow-up and extinction have asymptotically self-similar characteristics and can be treated by RG methods.

ADVANTAGE: Analog of universality and some distinguishing features of PDEs emerge.

Example : The Porous Medium Equation (Barenblatt). Goldenfeld et al. considered

$$u_t - \frac{1}{2}u_{xx} = \frac{\varepsilon}{2}H(-u_{xx})u_{xx}$$

and found that the decay was non-classical $\left(\neq -\frac{1}{2}\right)$.

2. RENORMALIZATION AND SCALING IN INTERFACE PROBLEMS

In many interface problems we need to know about the large time behavior. Numerical computations and even experiments have difficulty resolving the issues.

KEY QUESTIONS:

- Can we determine the characteristic length, $R(t) \sim t^{\beta}$?
- What parameters in the system are "relevant" ?

- Early work by Jasnow and Vinals indicates $\beta = 1$ in quasi-static one sided (i.e. $R(t) \sim t$).

- Later work by Caginalp indicates $\beta = \frac{1}{2}$ in fully dynamic case (i.e. $R(t) \sim t^{1/2}$).

Merdan and Caginalp consider the two sided model in quasi-static regime below:

$$\Delta T = 0 \qquad \text{in } \Omega \tag{2.1}$$

$$l v_n = -K[\nabla T \bullet \hat{n}]_{-}^{+} \qquad \text{on } \Gamma(t) \tag{2.2}$$

$$T = \frac{-\sigma_0}{[s]_{eq}}(\kappa + \alpha v_n) \qquad \text{on } \Gamma(t). \tag{2.3}$$

We consider two cases:

(i) $\alpha \neq 0$ in (2.3)

(ii) $\alpha = 0$ in (2.3)

<u>idea</u> in the RG calculations

→ Find a basic solution using Green's function representation

→ Implement RG methods to calculate the characteristic length, $R(t)$, as a function of time.

We can rewrite (2.1) and (2.2) in the Oleinik formulation,

$$\Delta T = \frac{l}{2K}\phi_t \tag{2.4}$$

so that the latent heat is treated like a source term.

Use Green's function (assume Ω is infinite or very large):

$$T(x) = \int_{\Omega} G(\vec{x}-\vec{y})\left(\frac{l}{2K}\phi_t(\vec{y},t)\right)d^d y + \int_{\partial\Omega}\left(T(\vec{y})\frac{\partial G}{\partial v}(\vec{x}-\vec{y}) + G(\vec{x}-\vec{y})\frac{\partial T}{\partial v}(\vec{y})\right)d^{d-1}y \tag{2.5}$$

where the Green's function G is

$$G(\vec{x}-\vec{y}) = \begin{cases} \frac{1}{d(2-d)\omega_d}|\vec{x}-\vec{y}|^{2-d} & \text{if } d > 2 \\ \frac{1}{2\pi}\log|\vec{x}-\vec{y}| & \text{if } d = 2 \end{cases}. \tag{2.6}$$

Smoothing ϕ, using h as the distance of point on interface in section, and r the distance in the normal direction, we write

$$\begin{aligned} \phi_t(\vec{x},t) &= \Phi_t(r - v_n t) \\ &= -\left(\hat{k}\bullet\hat{n}\frac{dh}{dt}\right)\Phi_r\left(r - \left(\hat{k}\bullet\hat{n}\frac{dh}{dt}\right)t\right). \end{aligned} \tag{2.7}$$

Substituting into the Green's function equation, we have

$$T(\vec{x}) = \int_{\Omega} d^d y G(\vec{x}-\vec{y})\left(\frac{l}{2K}\right) \bullet$$
$$\left(-\left(\hat{k} \bullet \hat{n}\frac{dh}{dt}\right)\Phi_{r_{\vec{y}}}\left(r - \left(\hat{k} \bullet \hat{n}\frac{dh}{dt}\right)t\right)\right) + \mathrm{BI}. \tag{2.8}$$

The derivatives of Φ vanish just outside of the interfacial region, we can perform integral in the normal direction reducing the integral over Ω to one over Γ, with the result,

$$T(\vec{x}) = \int_{\Gamma(t)} d^{d-1}\sigma_y G(\vec{x}-\vec{y})\left(\frac{l}{2K}\right) \bullet \left(-2\hat{k} \bullet \hat{n}\frac{dh}{dt}\right) + \mathrm{BI} \tag{2.9}$$

For points (x,t) on the interface, one has (with neglecting BI):

$$\frac{-\sigma_0}{[s]_{eq}}\left(\kappa(\vec{x},t) + \alpha v_n(\vec{x},t)\right) = \int_{\Gamma(t)} d^{d-1}\sigma_y G(\vec{x}-\vec{y})\left(\frac{l}{K}\right) v_n(\vec{y},t) \tag{2.10}$$

Letting $D := \frac{K}{C}$ and $d_0 := \frac{\sigma_0/[s]_{eq}}{l/C}$, we rewrite (2.10) as

$$d_0\left(\kappa(\vec{x},t) + \alpha v_n(\vec{x},t)\right) = \frac{1}{D}\int_{\Gamma(t)} G(\vec{x}-\vec{y}) v_n(\vec{y},t) d^{d-1}\sigma_y. \tag{2.11}$$

We need now to convert to dimensionless counterparts in order to compare pure numbers.

THE RG PROCEDURE

Stage 1. For any $b > 0$ and any real λ, make the algebraic substitutions in (2.11)

$$b\vec{x} \text{ for } \vec{x} \quad \text{and} \quad b^{-\lambda}t \text{ for } t. \tag{2.12}$$

Next define new variables

$$\vec{y}' = \vec{y}/b, \qquad \sigma_{y'} = \sigma_y/b. \tag{2.13}$$

These two substitutions yield:

$$\begin{aligned} &d_0\{\kappa(b\vec{x}, b^{-\lambda}t) + \alpha\, v_n(b\vec{x}, b^{-\lambda}t)\} \\ &= \frac{1}{D}\int_\Gamma b^{d-1} d^{d-1}\sigma_{y'} G(b\vec{x} - b\vec{y}')v_n(b\vec{y}', b^{-\lambda}t) \end{aligned} \tag{2.14}$$

Stage 2. Examine scaling of individual terms. We have the purely algebraic transformation for the Green's function (for $d > 2$) :

$$G(b\vec{x} - b\vec{y}') = b^{2-d}G(\vec{x} - \vec{y}'). \tag{2.15}$$

Basic physical assumption of **Single Scale Self Similarity**:

All physical lengths in the problem scale as

$$\xi(b\vec{x}, b^{-\lambda}t) = b\xi(\vec{x}, t) \tag{2.16}$$

while all time mesaurements scale as

$$\mathrm{T}(b\vec{x}, b^{-\lambda}t) = b^{-\lambda}\mathrm{T}(\vec{x}, t) \tag{2.17}$$

so that velocity scales as

$$\begin{aligned} \upsilon_n(b\vec{x}, b^{-\lambda}t) &= \frac{\text{dis tan ce}}{\text{time}}(b\vec{x}, b^{-\lambda}t) \\ &= \frac{b \bullet \text{dis tan ce}}{b^{-\lambda} \bullet \text{time}}(\vec{x}, t) \\ &= b^{1+\lambda}\upsilon_n(\vec{x}, t) \end{aligned} \tag{2.18}$$

and curvature scales as

$$b\kappa(b\vec{x}, b^{-\lambda}t) = \kappa(\vec{x}, t). \tag{2.19}$$

Stage 3. Use relation above to rewrite the equation (2.14) as:

$$\frac{d_0}{b^{3+\lambda}}\left\{\kappa(\vec{x},t)+\frac{\alpha}{b^{-2-\lambda}}\,v_n(\vec{x},t)\right\}$$
$$=\frac{1}{D}\int_\Gamma d^{d-1}\sigma_{y'}\,G(\vec{x}-\vec{y}')v_n(\vec{y}',t)$$

(2.20)

The key observation is that this new equation (2.20) is identical to the original, (2.11), upon replacing:

$$d_0 \to \frac{d_0}{b^{3+\lambda}}; \qquad \alpha \to \frac{\alpha}{b^{-2-\lambda}}. \qquad (2.21)$$

Hence, one has (with R as the characteristic length):

OLD	**NEW**
d_0	$d_0/b^{3+\lambda}$
α	$\alpha/b^{-2-\lambda}$
$\xi(bx,b^{-\lambda}t)$	$b\xi(x,t)$
$R(b^{-\lambda}t;\alpha,d_0)$	$bR(t;\alpha/b^{-2-\lambda},d_0/b^{3+\lambda})$

By a simple substitution $t_1 = b^{-\lambda}t$, we write

$$R(t_1;\alpha,d_0) = bR(b^{\lambda}t_1;\alpha/b^{-2-\lambda},d_0/b^{3+\lambda}).$$

(2.22)

Stage 4. Choose $b = t_1^{-1/\lambda}$ (and omit subscript 1) yielding,

$$R(t;\alpha,d_0) = t^{-1/\lambda}R(1;\alpha/t^{(2+\lambda)/\lambda}, d_0/t^{-(3+\lambda)/\lambda}). \tag{2.23}$$

Analysis of λ :

▸ If $\lambda < -3$ or $\lambda > 0$, then $d_0 \to \infty$ as $t \to \infty$ (Physically irrelevant)

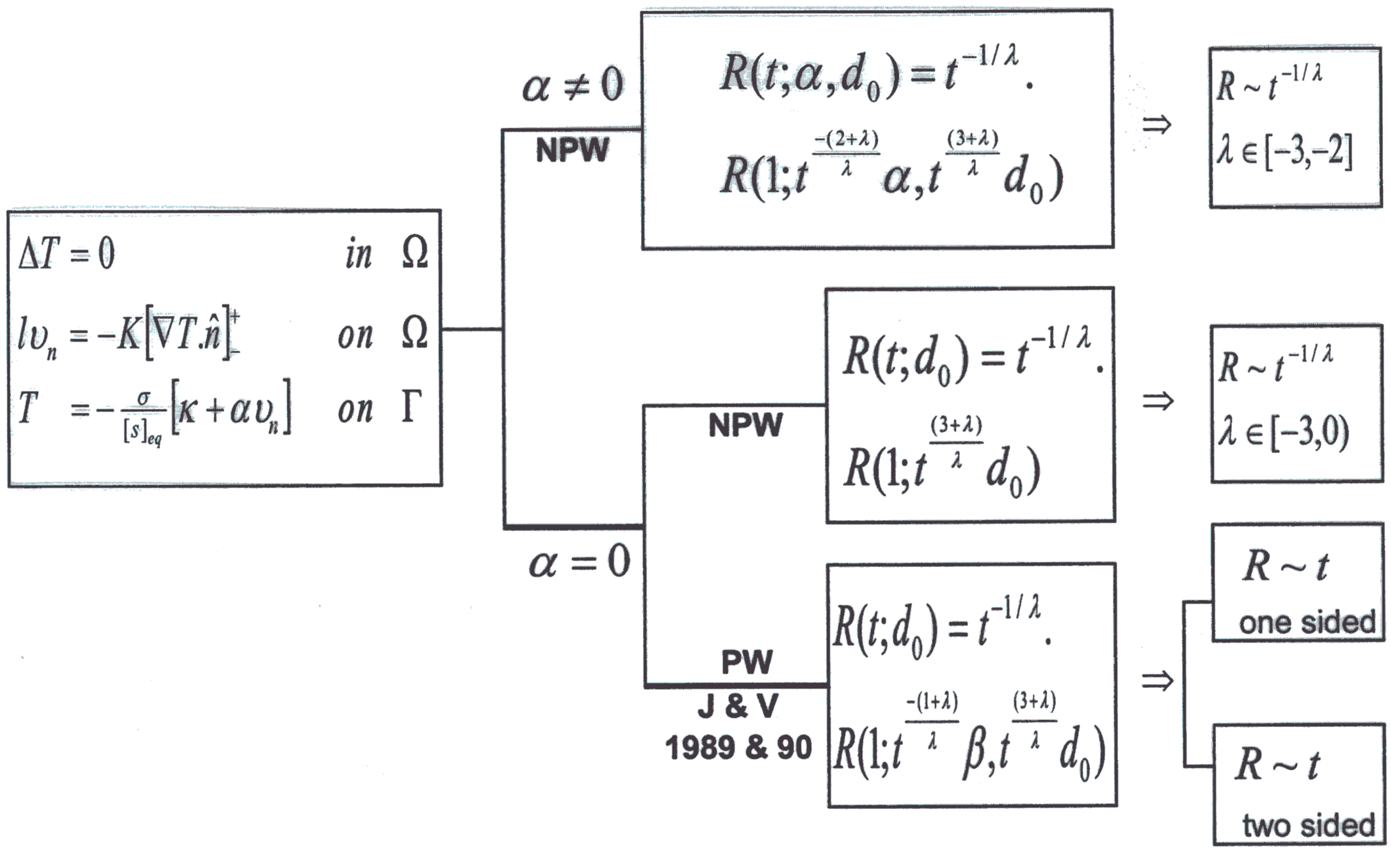
▸ If $-2 < \lambda < 0$, then $d_0 \to 0$, but $\alpha \to \infty$ as $t \to \infty$ (Nonphysical)

$$\lambda \in [-3,-2] \Rightarrow \boxed{\begin{array}{c} \text{Nontrivial fixed points} \\ \hline R(t) \sim t^{-1/\lambda} \end{array}}.$$

■ **THE CASE $\alpha = 0$:**

$$\lambda \in [-3,0) \Rightarrow \boxed{\begin{array}{c} \text{Nontrivial fixed points} \\ \hline R(t) \sim t^{-1/\lambda} \end{array}}.$$

▸ $\lambda = -1$ is selected from this continuous spectrum by the plane wave imposed by Jasnow and Vinals.



NPW : There is not a plane wave imposed

PW : There is a plane wave imposed.

$$
\begin{aligned}
&CT_t = K\Delta T && \text{in } \Omega \\
&l\upsilon_n = -K\left[\nabla T.\hat{n}\right]_-^+ && \text{on } \Omega \\
&T = -\frac{\sigma}{[s]_{eq}}\left[\kappa + \alpha \upsilon_n\right] && \text{on } \Gamma
\end{aligned}
$$

- $\alpha \neq 0$ **Caginalp 99 & 01**
  - **NPW**: $R(t;D,\alpha,d_0) = t^{-1/\lambda}.$ $R(1;t^{\frac{2+\lambda}{\lambda}}D,t^{\frac{-(2+\lambda)}{\lambda}}\alpha,t^{\frac{1}{\lambda}}d_0)$ $\Rightarrow$ $\boxed{R \sim t^{1/2}}$
  - **PW**: $R(t;D,\alpha,d_0) = t^{-1/\lambda}.$ $R(1;t^{\frac{2+\lambda}{\lambda}}D,t^{\frac{-(2+\lambda)}{\lambda}}\alpha,t^{\frac{1}{\lambda}}d_0)$ $\Rightarrow$ $\boxed{R \sim t^{1/2}}$
- $\alpha = 0$
  - **NPW**: $R(t;D,d_0) = t^{-1/\lambda}.$ $R(1;t^{\frac{2+\lambda}{\lambda}}D,t^{\frac{1}{\lambda}}d_0)$ $\Rightarrow$ $\boxed{R \sim t^{1/2}}$
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CONCLUSIONS

- Characteristic length evolves as $R(t) \sim t^{\beta}$

where (i) $\beta \in [1/3, 1/2]$ when $\alpha \neq 0$,

(ii) $\beta \in [1/3, \infty)$ when $\alpha = 0$

- In almost all these casess, capillarity length, d_0, is essentially irrelevant for large time-sharp contrast with its stabilizing role in short times.

CHALLENGES AHEAD

▸ Understanding the transition from the exponent of the characteristic length for the quasi-static model to that for the fully dynamic set of equations (through RG)

▸ Understanding the important connection between the dynamic and static renormalization methods

▸ Understanding the transition between short time (linear stability) and the long term asymptotics (through RG). Crossover behavior