Presentation

Mean Field Models of Bose-Einstein Condensates

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References

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- NJP **5**, 64 (2003)
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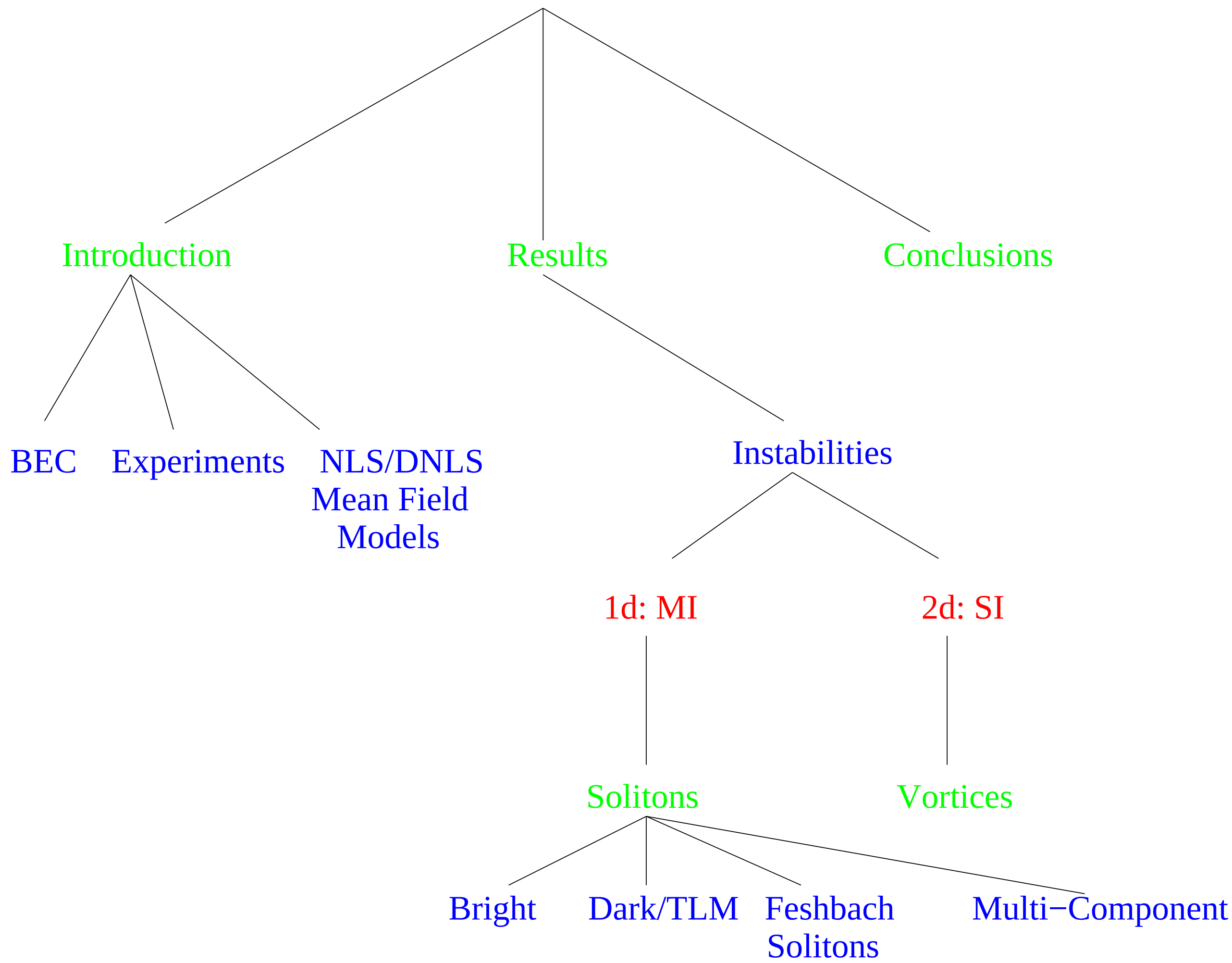
- PRA **68**, 035602 (2003)
- PLA **317**, 513 (2003)
- PRL in press (2003)
- PRL in press (2003)
- EPJD in press (2003)
- cond-mat/0207139

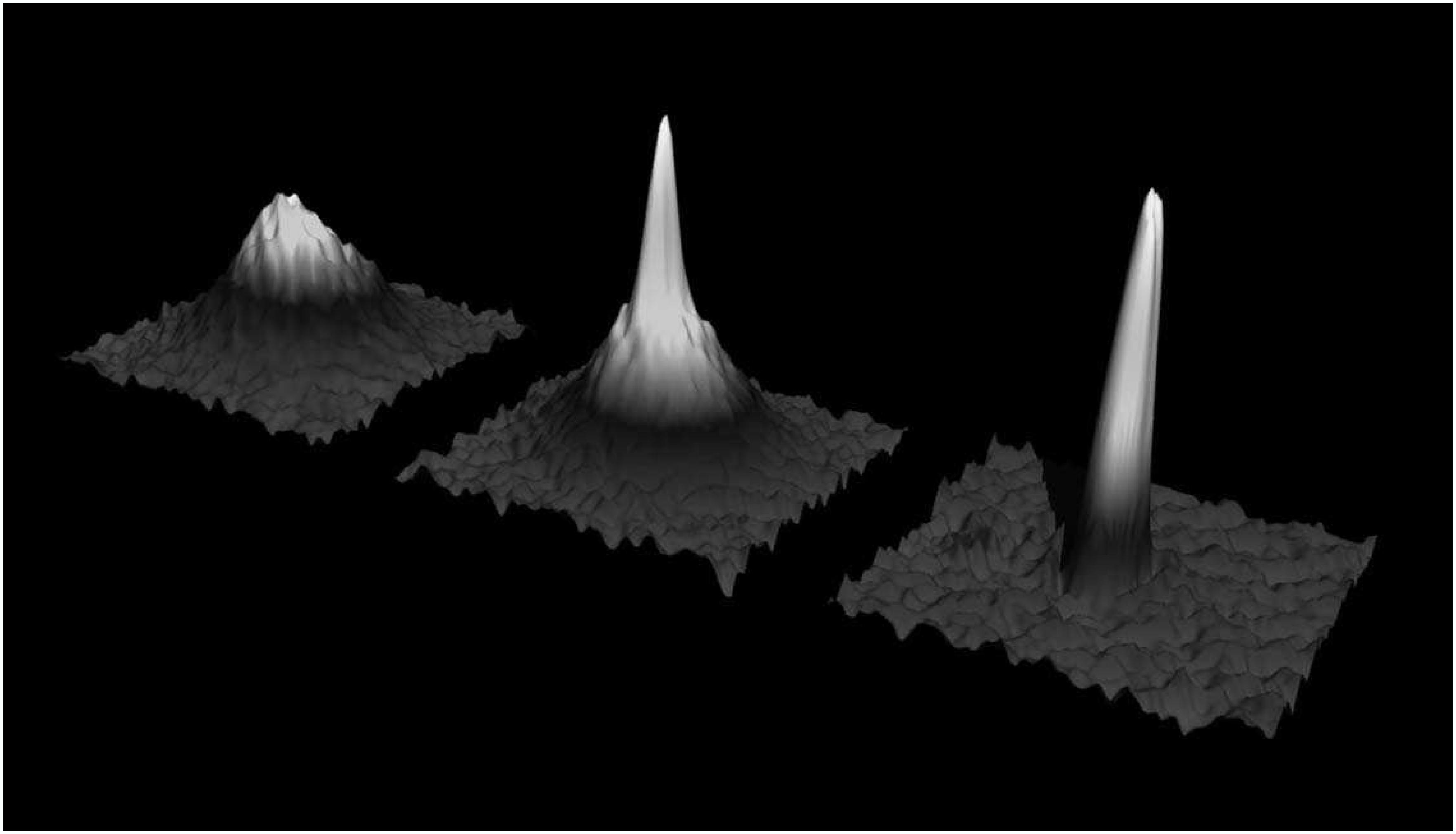
Concept Map / Outline



Introduction

BEC

- 1924: S. Bose and A. Einstein realize that Bose statistics predicts a Maximum Atom Number in the Excited States.
- 1995: E. Cornell, C. Wieman and W. Ketterle realize BEC in a dilute gas of ^{87}Rb and ^{23}Na: 2001 Nobel Prize.
- In between: Mostly ^{4}He (London, Tisza, Landau, Bogoliubov, Gross, Pitaevskii).
- Today:
  - ~ 35 Experimental Groups have achieved BEC (in 100-10^8 atoms of Rb, Li, Na, H).
  - O(10^3) Theoretical and O(10^2) Experimental papers ! Check out: http://amo.phy.gasou.edu/bec.html/bibliography.html



And Why On Earth Should We Care ?

- Many Body Hamiltonian

$$\hat{H} = \int d\mathbf{r} \hat{\Psi}^{\dagger} \left[-\frac{\hbar^2}{2m} \Delta + V_{\text{ext}}(\mathbf{r}) \right] \hat{\Psi} + \frac{1}{2} \int d\mathbf{r} d\mathbf{r}' \hat{\Psi}^{\dagger}(\mathbf{r}) \hat{\Psi}^{\dagger}(\mathbf{r}') V(\mathbf{r}-\mathbf{r}') \hat{\Psi}(\mathbf{r}') \hat{\Psi}(\mathbf{r}) (1)$$

- Bogoliubov Decomposition:

$$\widehat{Psi} = \Phi(\mathbf{r}, t) + \hat{\Psi}'(\mathbf{r}, t) \tag{2}$$

- Φ is now a regular wavefunction (the expectation value of the field operator). Its equation:

$$i\hbar \frac{\partial \Phi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \Phi + V_{\text{ext}}(\mathbf{r}) \Phi + g |\Phi|^2 \Phi \tag{3}$$

- for dilute, cold, binary collision gas.
- But: This is 3D NLS with a Potential: GP !

Anything Simpler ?

Typical Potentials

- Magentic Trap

$$V(\mathbf{r}) = \frac{1}{2}\left(\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2\right) \tag{4}$$

- Optical Lattice

$$V = V_0\left(\sin^2(k_x x) + \sin^2(k_y y) + \sin^2(k_z z)\right) \tag{5}$$

Typical Reductions

- Relevant length scales: Scattering Length (a_s), Harmonic Oscillator Lengths ($a_{x,y,z} = \sqrt{\hbar/(m\omega_{x,y,z})}$).
- For Weak Interactions, Strong Transverse Confinement: ($\delta^2 = (Na_s/a_l)k \ll 1$ and $a_t/a_l \sim \delta\sqrt{k}$), one can use:

$$\Phi(\mathbf{r}, t) = \frac{\delta}{a_t a_s^{1/2}} e^{-i\omega_t t} e^{-\mathbf{r}^2/(2a_t^2)} u(\delta x/a_t, \delta^2\omega_t t/2) \tag{6}$$

- Then, the 3D GP reduces to 1D GP:

$$iu_t + u_{xx} + s|u|^2 u - k(t)x^2 u = 0 \tag{7}$$

- Similar Considerations for the Optical Lattice

$$iu_t + u_{xx} + s|u|^2 u - V_0 \sin^2(kx + \theta)u = 0 \tag{8}$$

- If one direction is tightly confined, then we obtain 2D GP:

$$iu_t + \Delta u + s|u|^2 u - V(x, y)u = 0. \tag{9}$$

- Finally, from NLS with OL to DNLS: if $V(x + L) = V(x)$, use

$$\psi(x, t) = \sum_{n\alpha} c_{n,\alpha}(t) w_{n,\alpha}(x) \tag{10}$$

where the Wannier functions w_n of band α are expressed in terms of the Bloch functions $\phi_{k,\alpha}$ as:

$$w_\alpha(x - nL) = \sqrt{\frac{L}{2\pi}} \int_{-\pi/L}^{\pi/L} \varphi_{k,\alpha}(x) e^{-inkL} dk. \tag{11}$$

– Then the GP equation becomes into a Vector Lattice equation:

$$i\frac{dc_{n,\alpha}}{dt} = \widehat{\omega}_{0,\alpha}c_{n,\alpha} + \widehat{\omega}_{1,\alpha}\left(c_{n-1,\alpha} + c_{n+1,\alpha}\right) + s\sum_{\alpha_1,\alpha_2,\alpha_3} W^{nnnnn}_{\alpha\alpha_1\alpha_2\alpha_3} c^*_{n,\alpha_1} c_{n,\alpha_2} c_{n,\alpha_3} \quad (12)$$

– which in the Tight Binding, Single Band Limit becomes the DNLS

$$i\frac{dc_{n,\alpha}}{dt} = \widehat{\omega}_{0,\alpha}c_{n,\alpha} + \widehat{\omega}_{1,\alpha}\left(c_{n-1,\alpha} + c_{n+1,\alpha}\right) + sW^{nnnn}_{1111}|c_{n,\alpha}|^2 c_{n,\alpha}, \quad (13)$$

– and more generally the Vector Lattice Model with XPM

$$i\frac{d\tilde{c}_{n,\alpha}}{dt} = \widehat{\omega}_{1,\alpha}(\tilde{c}_{n-1,\alpha} + \tilde{c}_{n+1,\alpha}) + s\sum_{\alpha_1} W_{\alpha\alpha_1}|\tilde{c}_{n,\alpha_1}|^2 \tilde{c}_{n,\alpha}. \quad (14)$$

The GP: Some Short Stories On Teaching An Old Dog New Tricks !

Rediscovering Instabilities !

- Modulational Instability: Soliton Trains
- Snaking Instability: Vortices and Patterns

Rediscovering Solitary Waves !

- 1d Solitons: Bright, Dark, Twisted (and Some New Kids On the Block !)
- 2d Solitons: Ring Dark Solitons, Vortices and Vortex Lattices.

Back to the Future: MI with some Twists

- NLS MI: $V(x) = 0$. Use:

$$u(x,t) = (\phi + \epsilon b)\exp[i((qx - \omega t) + \epsilon\psi(x,t))] \tag{15}$$

- For $b(x,t) = b_0 \exp(i(Qx - \Omega t))$, $\psi(x,t) = \psi_0 \exp(i(Qx - \Omega t))$:

$$(-\Omega + 2qQ)^2 = Q^2(Q^2 - 2s\phi^2) \tag{16}$$

- A Twist: Do it Variationally: Use

$$\psi = [\psi_0 + a(t)e^{i\phi_a(t)}e^{iqx} + b(t)e^{i\phi_b(t)}e^{-iqx}]\exp\left(i(kx - \omega t)\right). \tag{17}$$

inside the Lagrangian !

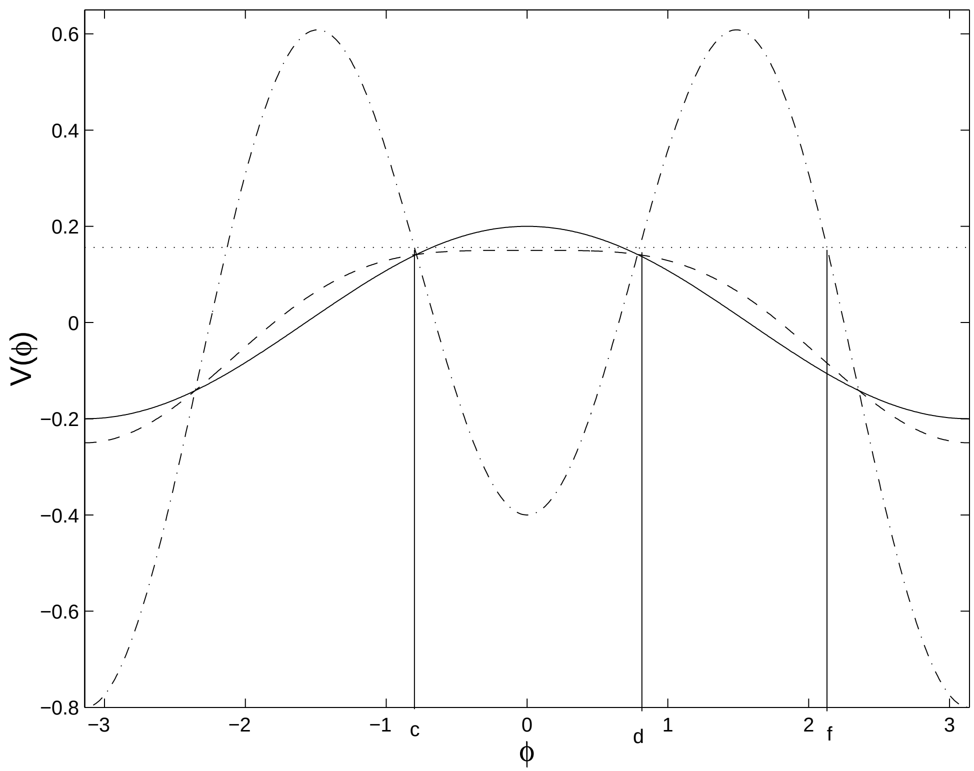
$$L = \int_{-\infty}^{\infty} \left[\frac{i}{2}(\psi^*\psi_t - \psi\psi_t^\star) - |\psi_x|^2 + \frac{U}{2}|\psi|^4\right] dx, \tag{18}$$

- Linear Terms $\rightarrow$ Explicit Solution. For all terms:

$$\ddot{\phi} = -\frac{\partial V}{\partial \phi}, \tag{19}$$

with $V(\phi) = A\cos(\phi) - B\cos(2\phi)$, $A = -4U\psi_0^2(U\psi_0^2 - q^2)$ $B = U^2\psi_0^4$

- Taylor Expansion: $V(\phi) = \frac{4B-A}{2}\phi^2 + (A - B)$.



MI for the GP

- For the Parabolic Potential, use Lens Transform:

$$u(x,t) = \ell^{-1} \exp(if(t)x^2)v(\zeta,\tau) \tag{20}$$

with $\zeta = x/\ell$, $\tau_t = 1/\ell^2$, $f_t = -4f^2 - k(t)$ and $\ell(t) = \ell(0)\exp\left(4\int_0^t f(s)ds\right)$.

- Then the GP becomes:

$$iv_\tau + v_{\zeta\zeta} + |v|^2 v - 2i\lambda v = 0, \tag{21}$$

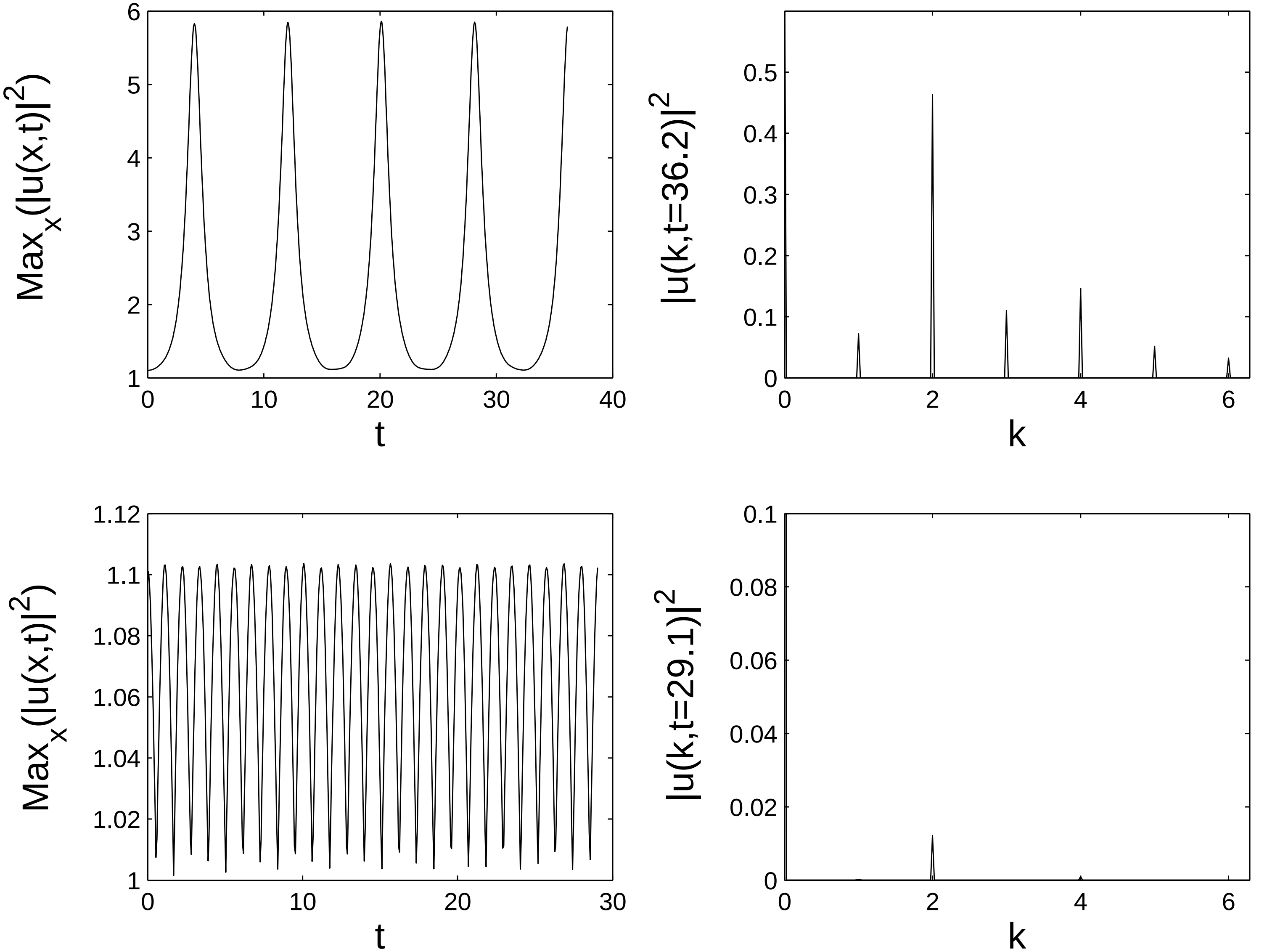
with $f\ell^2 = \lambda$.

- Interesting cases are

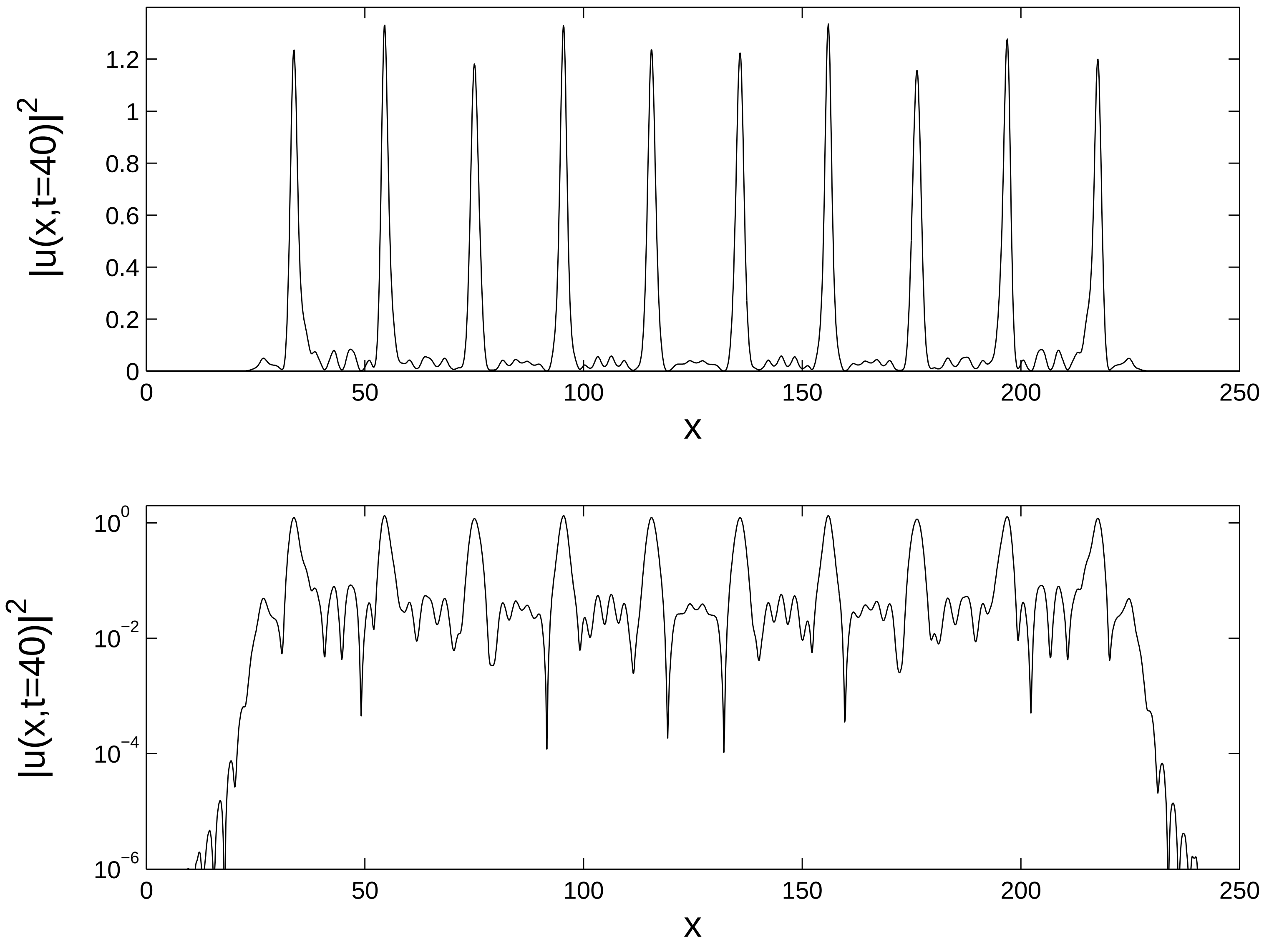
$$k(t) = (t + t^*)^{-2}/16 \tag{22}$$

and

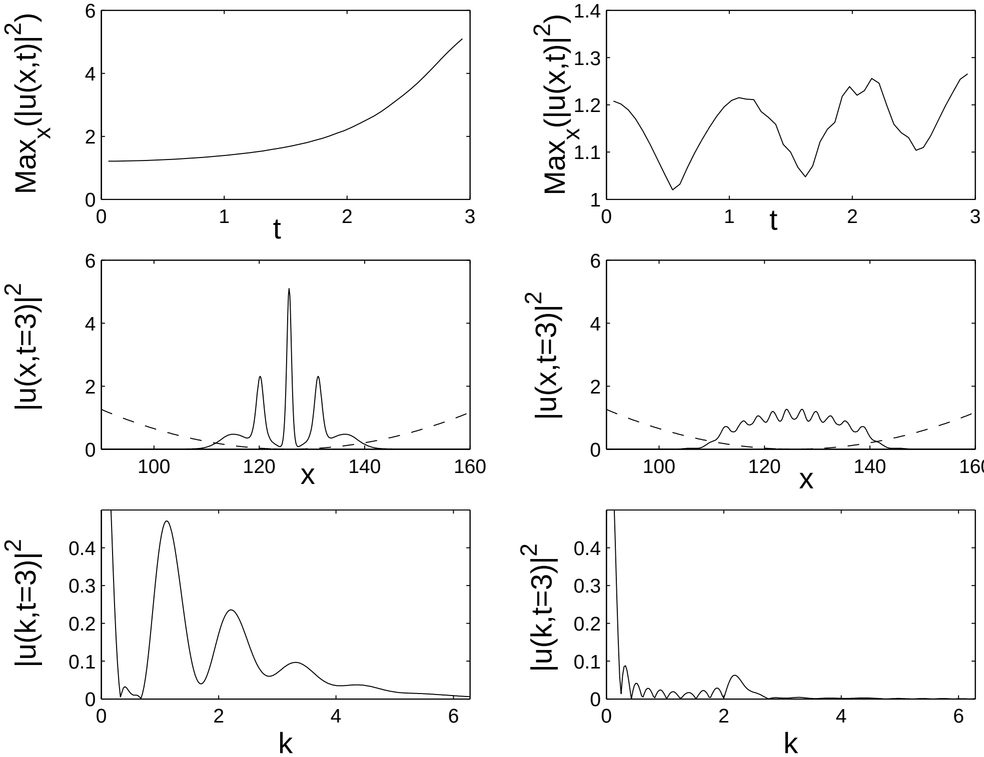
$$k(t) = C. \tag{23}$$



Max_x(|u(x,t)|^2)
t
|u(k,t=36.2)|^2
k
Max_x(|u(x,t)|^2)
t
|u(k,t=29.1)|^2
k



$|u(x,t=40)|^2$
1.2
1
0.8
0.6
0.4
0.2
0
0
50
100
150
200
250
x
$|u(x,t=40)|^2$
10^0
10^{-2}
10^{-4}
10^{-6}
0
50
100
150
200
250
x


Max_x(|u(x,t)|^2)
t
|u(x,t=3)|^2
x
|u(k,t=3)|^2
k

MI for the DNLS

- The DNLS is of the form:

$$i\hbar\frac{\partial\psi_n}{\partial t} = -K(\psi_{n-1}+\psi_{n+1}) + (\epsilon_n + U \mid \psi_n \mid^2)\psi_n \tag{24}$$

- The MI analysis can be carried out:

$$(\omega - 2K\sin(k)\sin(q))^2 = 8K\cos(k)\sin^2(\frac{q}{2})\left[2K\cos(k)\sin^2(\frac{q}{2}) + U|\psi_0|^2\right].\tag{25}$$

- Hence for $\pi/2 < k < 3\pi/2$, the MI condition can be realized:

$$U|\psi_0|^2 > -2K\cos(k)\sin^2(q/2) \tag{26}$$

A Finishing Touch: The Experiment !

- An Indirect Experiment to Induce MI: Instantaneously Displace Trap.
- For $\psi_j = \sqrt{n_j} e^{i\phi_j}$, define Collective Coordinates $\phi_{j+1}(t) - \phi_j(t) = \Delta\phi(t)$ (Quasi-Momentum $\rightarrow < k >$) and $\xi = \sum_j j n_j$ (Center of Mass).
- ODE Reduction $\rightarrow$ Josephson Equations

$$\hbar \frac{d}{dt}\xi(t) = 2K \ \sin \Delta\phi(t) \tag{27}$$

$$\hbar \frac{d}{dt}\Delta\phi(t) = -2 \ \Omega \ \xi(t). \tag{28}$$

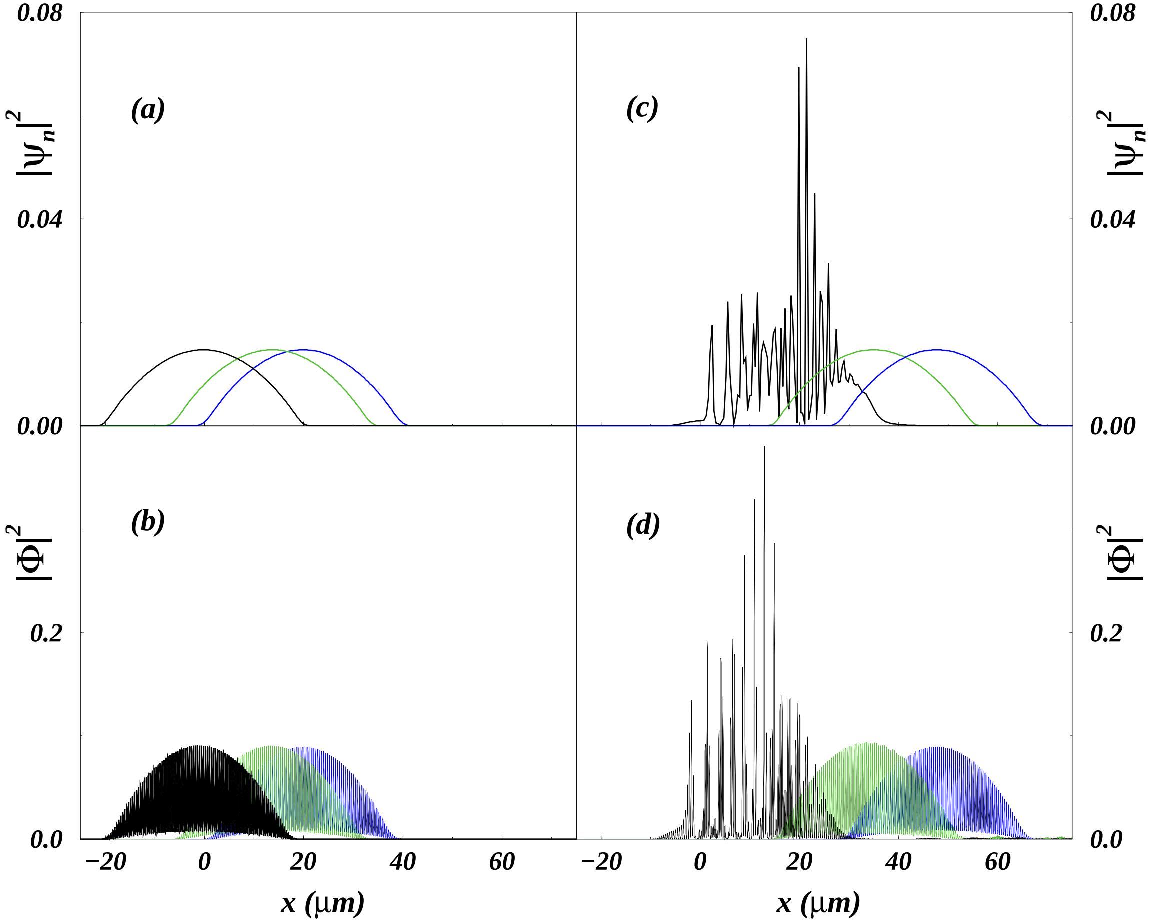
- A Pendulum Equation: Unstable for $\Delta\phi = \pi/2$ and

$$\xi > \xi_{cr} = \sqrt{\frac{2K}{\Omega}}. \tag{29}$$

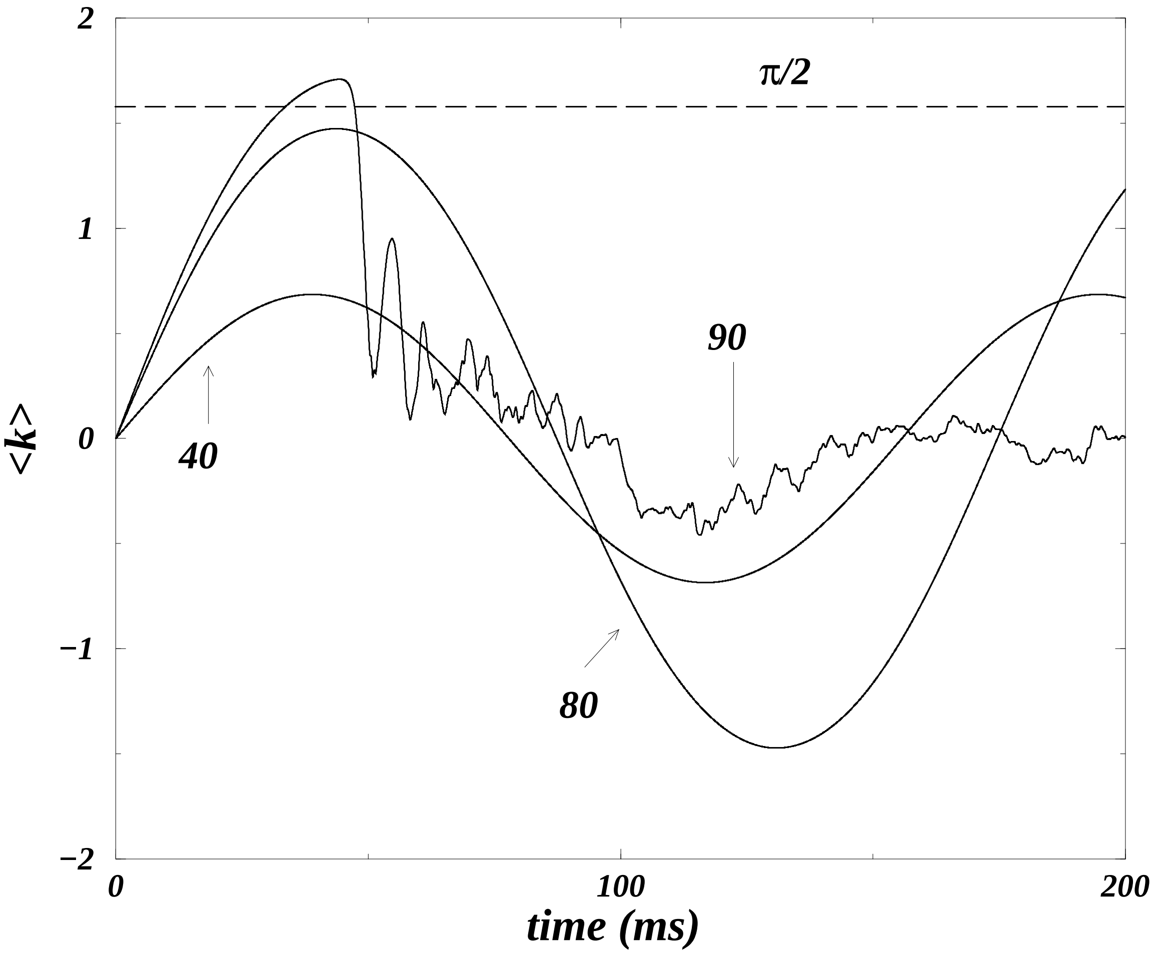
- Also use an Order Parameter for probing Loss of Coherence

$$\Psi = \sum_j \psi_j \psi_{j+1}^*. \tag{30}$$

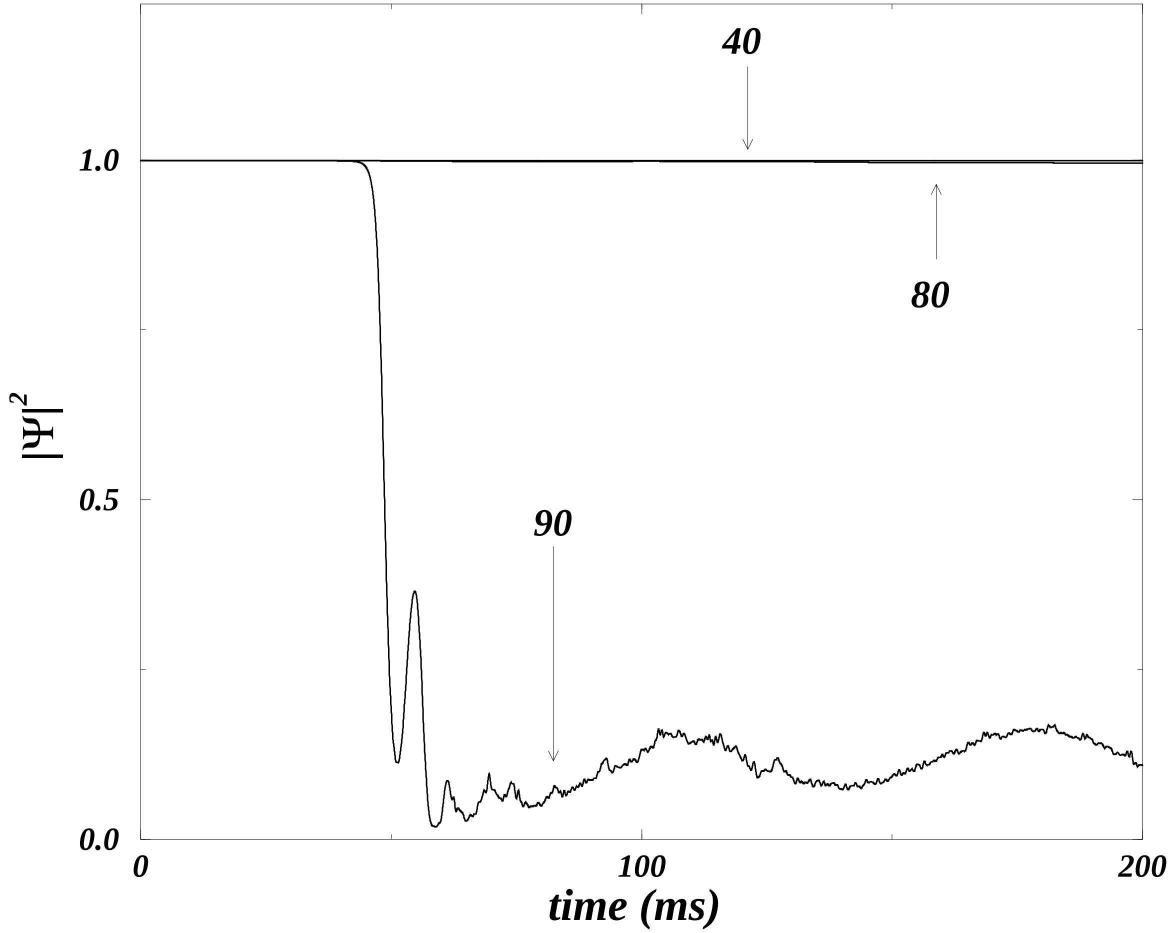
Numerical Results (Part I)



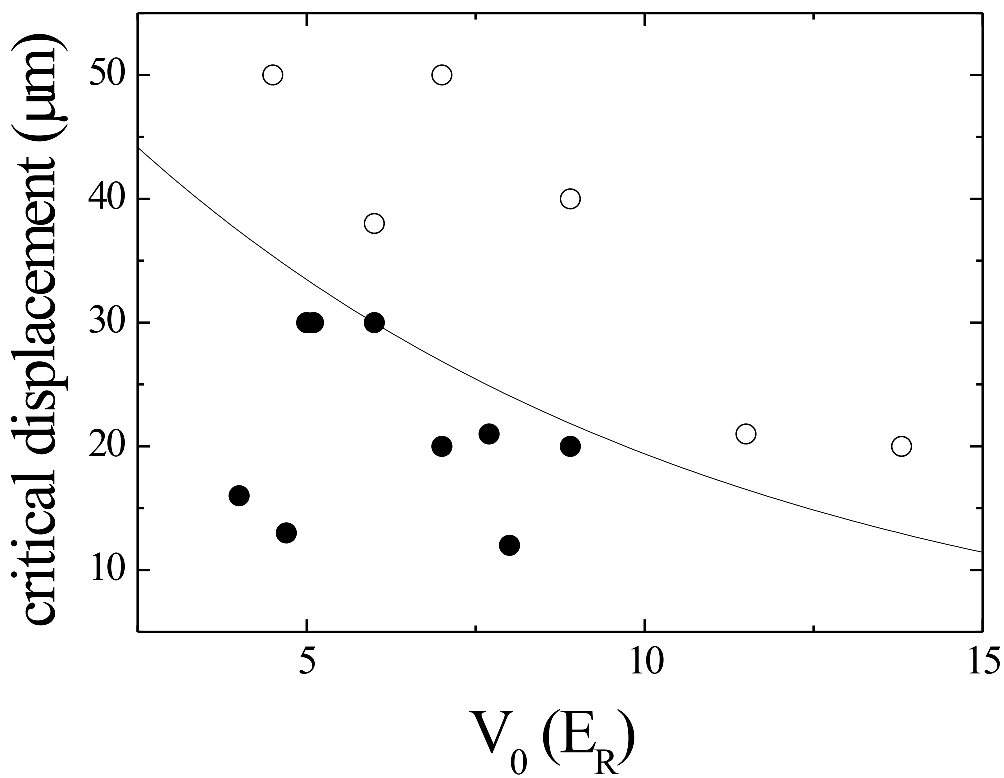
Numerical Results (Part II)



Numerical Results (Part III)



Experimental Results



A 2D Analog: The Snaking Instability

- Consider the 2D Defocusing GP

$$iu_t = -(1/2)\Delta u + |u|^2 u + (1/2)\Omega^2 r^2 u. \tag{31}$$

- Ground State: Thomas Fermi Cloud

$$u_0 = \sqrt{\mu - (1/2)\,\Omega^2 r^2}\, \exp\left(-i\mu t\right) \tag{32}$$

and Ring Dark Solitons:

$$v(r,t) = \cos\varphi(t) \cdot \tanh\xi + i\sin\varphi(t) \tag{33}$$

with $\xi \equiv \cos\varphi(t)\,[r - R(t)]$

- Use again Soliton Perturbation Theory to obtain:

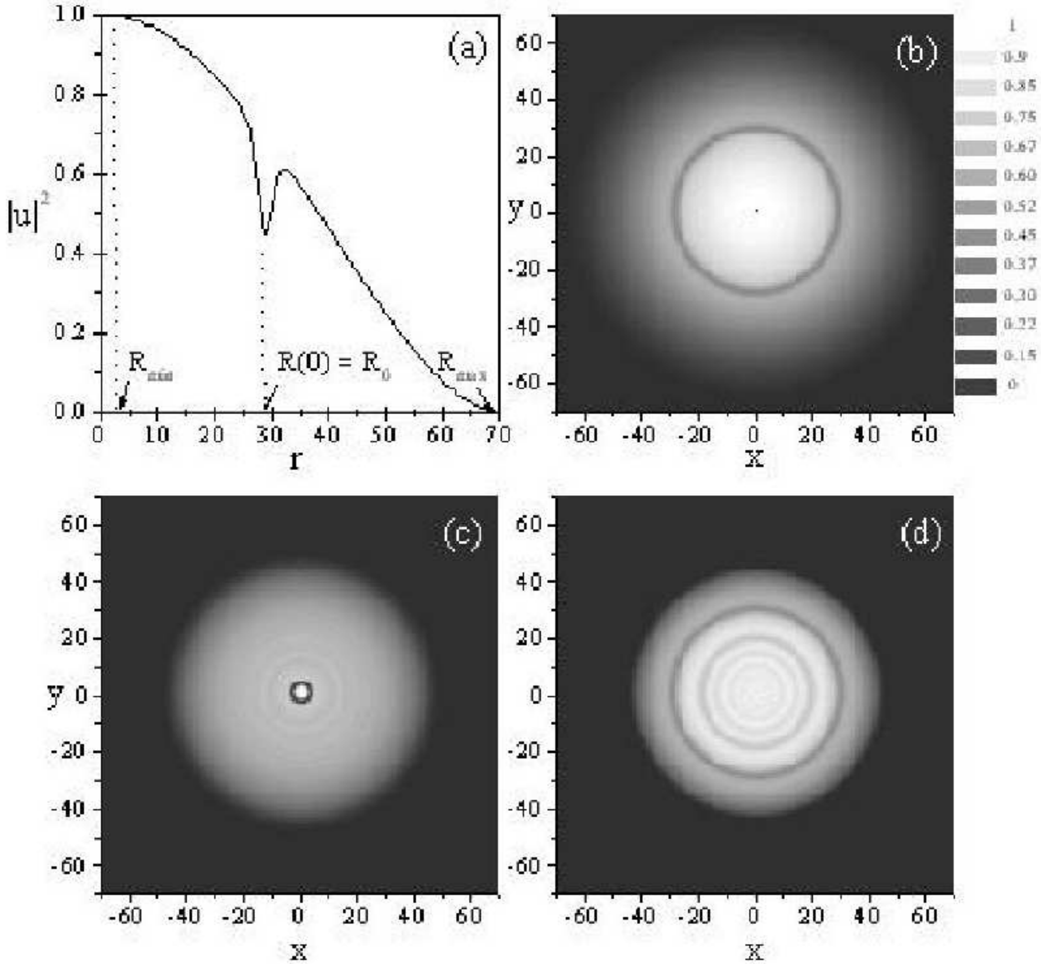
$$\frac{d\varphi}{dt} = -\frac{\cos\varphi}{2\mu}\frac{dW}{dR} + \frac{\cos\varphi}{3\sqrt{\mu}R}, \qquad \frac{dR}{dt} = \sin\varphi\,. \tag{34}$$

$$\frac{d^2R}{dt^2} = \left[-\frac{1}{2}\frac{dW(R)}{dR} + \frac{1}{3R}\right]\left[1 - \left(\frac{dR}{dt}\right)^2\right] \tag{35}$$

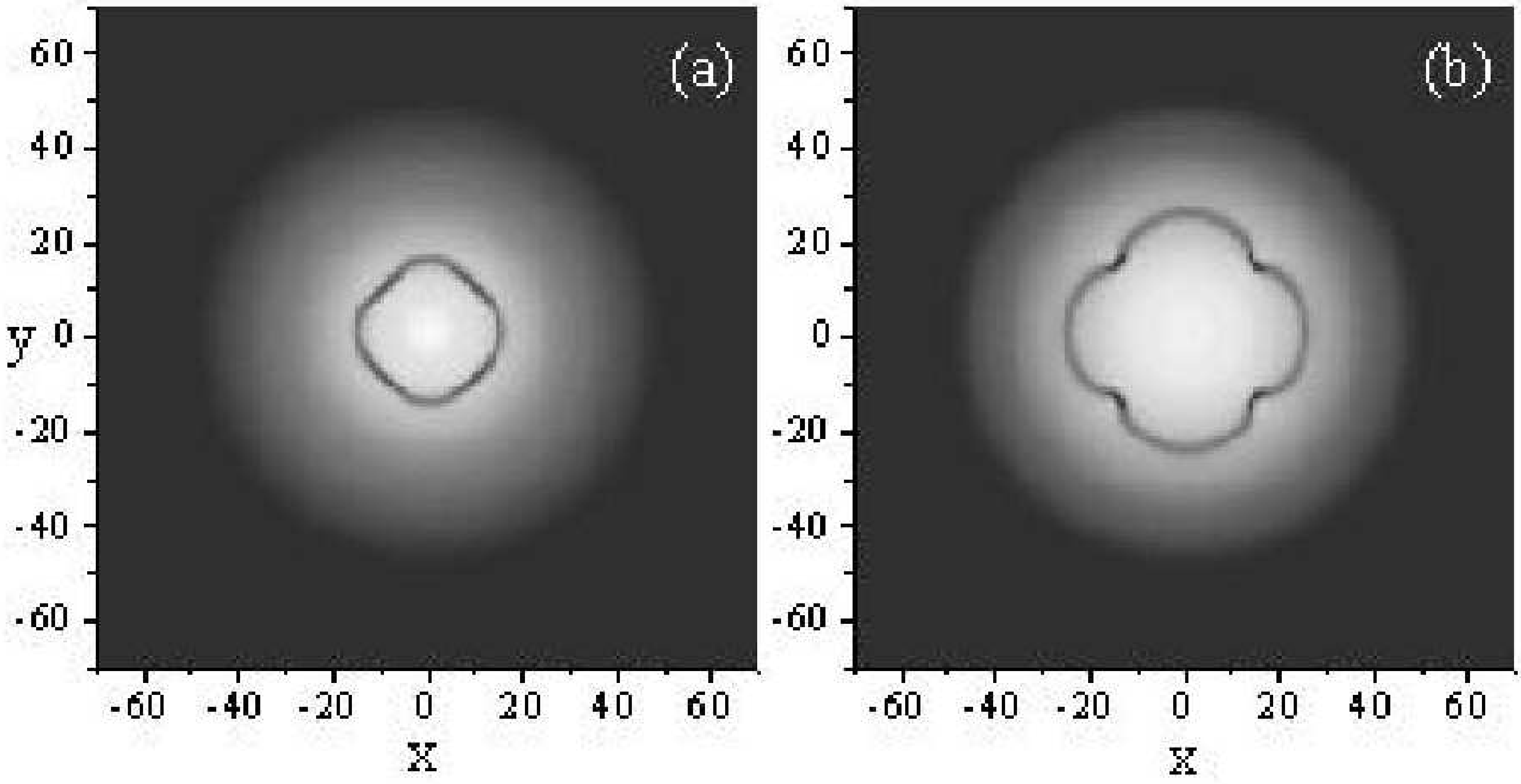
- Motion in Potential: $\Pi(R) = (1/2)(\Omega R)^2 - (1/3)\ln R$.
- Stationary RDS: $R_0 = \sqrt{2/(3\Omega)}$.
- Min-Max Radius: for $\eta \equiv -3W[R(0)]\cos^6\varphi(0)\exp\{-3W[R(0)]\}$:

$$R_{\text{min}} = [-(2/3)w(0,\eta)]^{1/2}\,\Omega^{-1}, R_{\text{max}} = [-(2/3)w(-1,\eta)]^{1/2}\,\Omega^{-1} \tag{36}$$

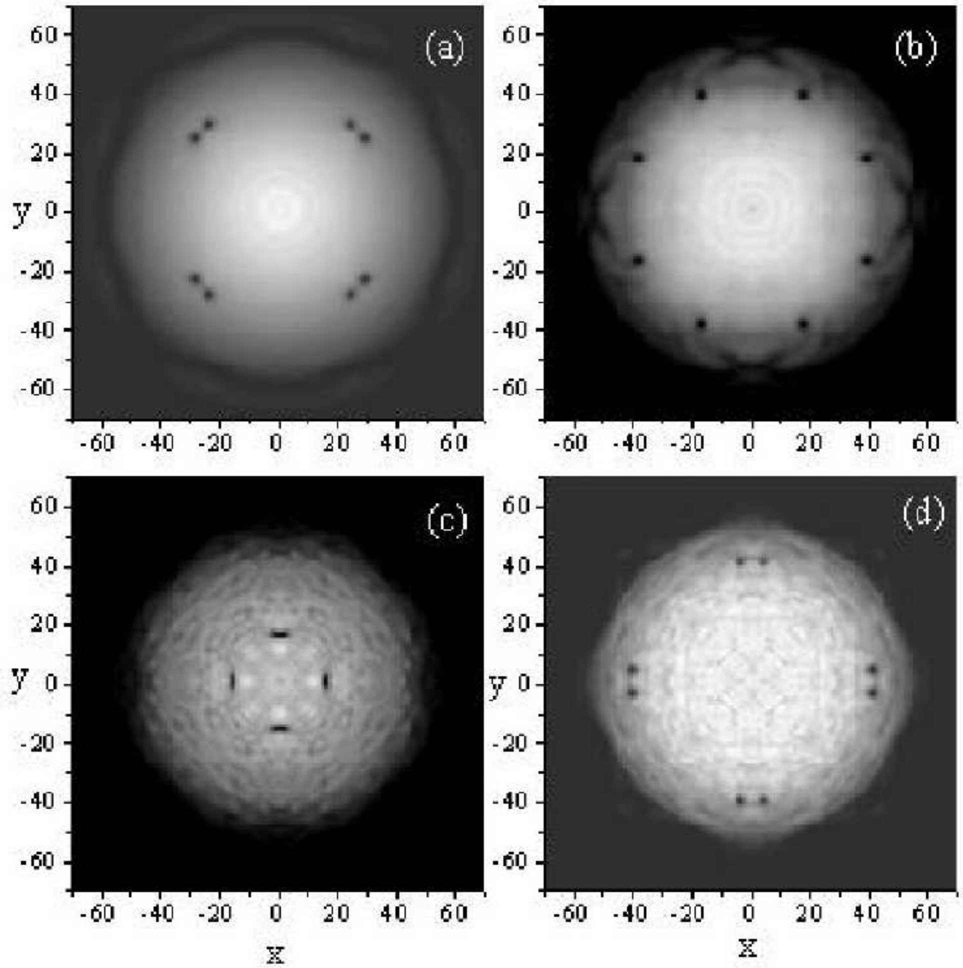
Numerical Results: Oscillating RDS



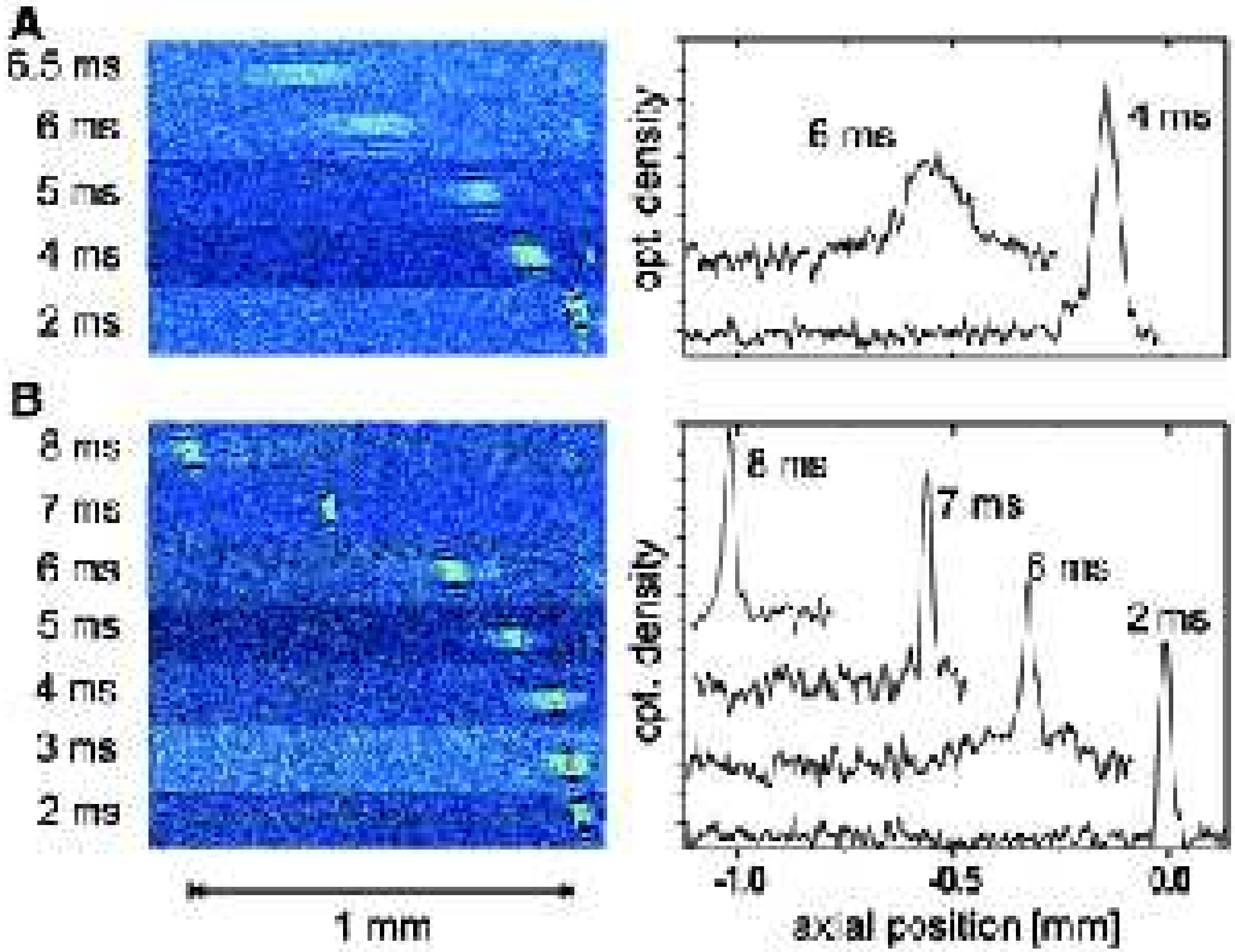
Onset of Snaking: Supercritical Dips



From x's to Crosses (and back !)



(Experimental) Bright Solitons



Dark-in-Bright Solitons: Twisted Modes

- Use Multi-Soliton Ansatz

$$v(x) = \sum_{j=1}^{n} (-1)^j \eta \,\mathrm{sech}\left(\frac{\eta}{\sqrt{2}}(x-\xi_j)\right), \eta \equiv \sqrt{-\mu} \tag{37}$$

in Hamiltonian: $H = \int_{-\infty}^{\infty} \left[|u_x|^2 - \frac{1}{2}|u|^4 + V_0 \cos^2\left(\frac{2\pi x}{\lambda}\right)|u|^2\right] dx.$

- Obtain Effective Potential

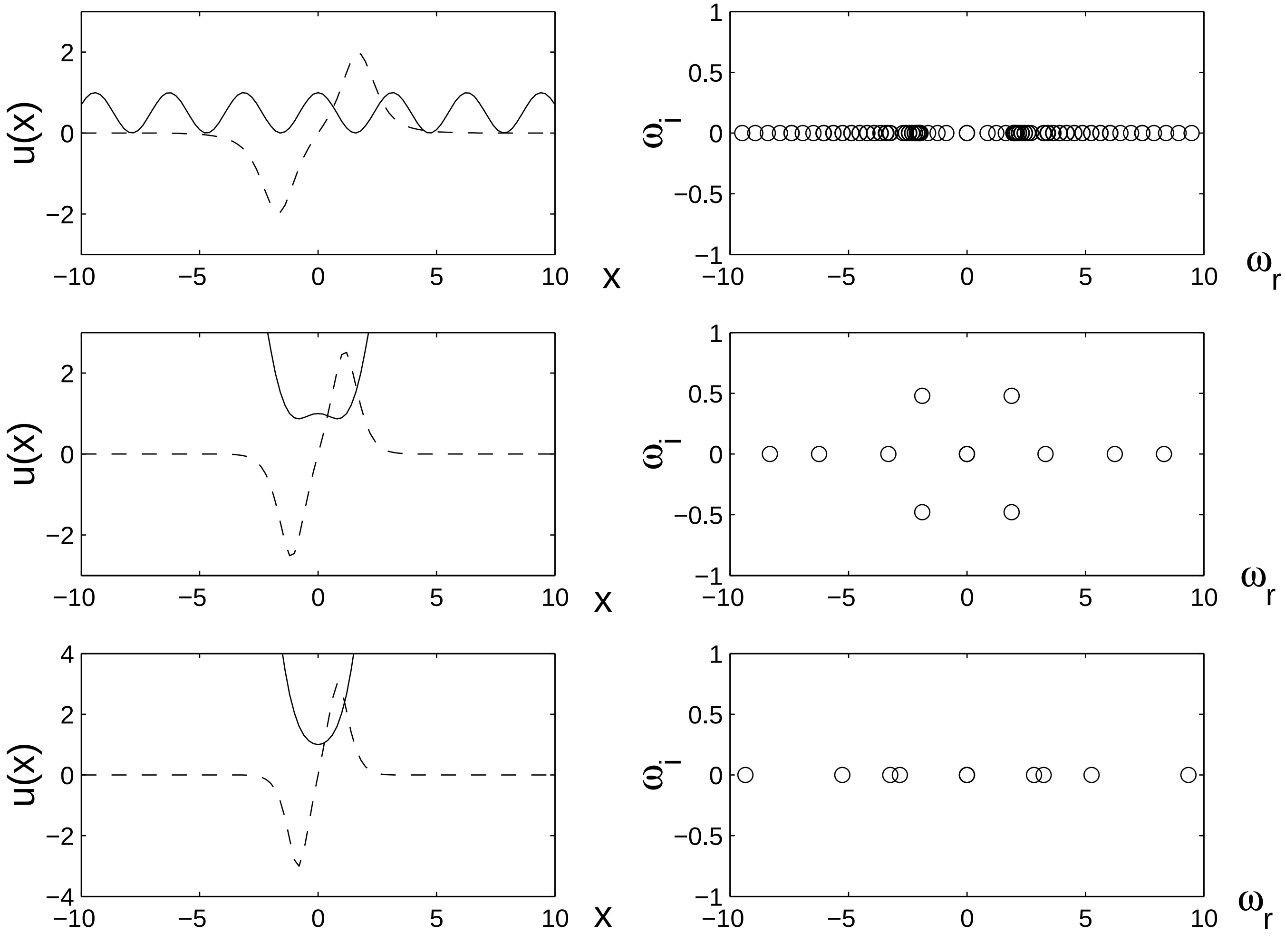
$$V_{\mathrm{eff}}\left(\xi_1, \xi_2; \Delta\phi\right) = -8\eta^3 \cos\left(\Delta\phi\right) \exp\left(-\frac{\eta}{\sqrt{2}}\left|\xi_1 - \xi_2\right|\right)$$

$$+\frac{2\sqrt{2}\pi^2 V_0}{\lambda \sinh\left(\frac{2\sqrt{2}\pi^2}{\lambda\eta}\right)}\left[\cos\left(\frac{4\sqrt{2}\pi}{\lambda}\xi_1\right) + \cos\left(\frac{4\sqrt{2}\pi}{\lambda}\xi_2\right)\right]. \tag{38}$$

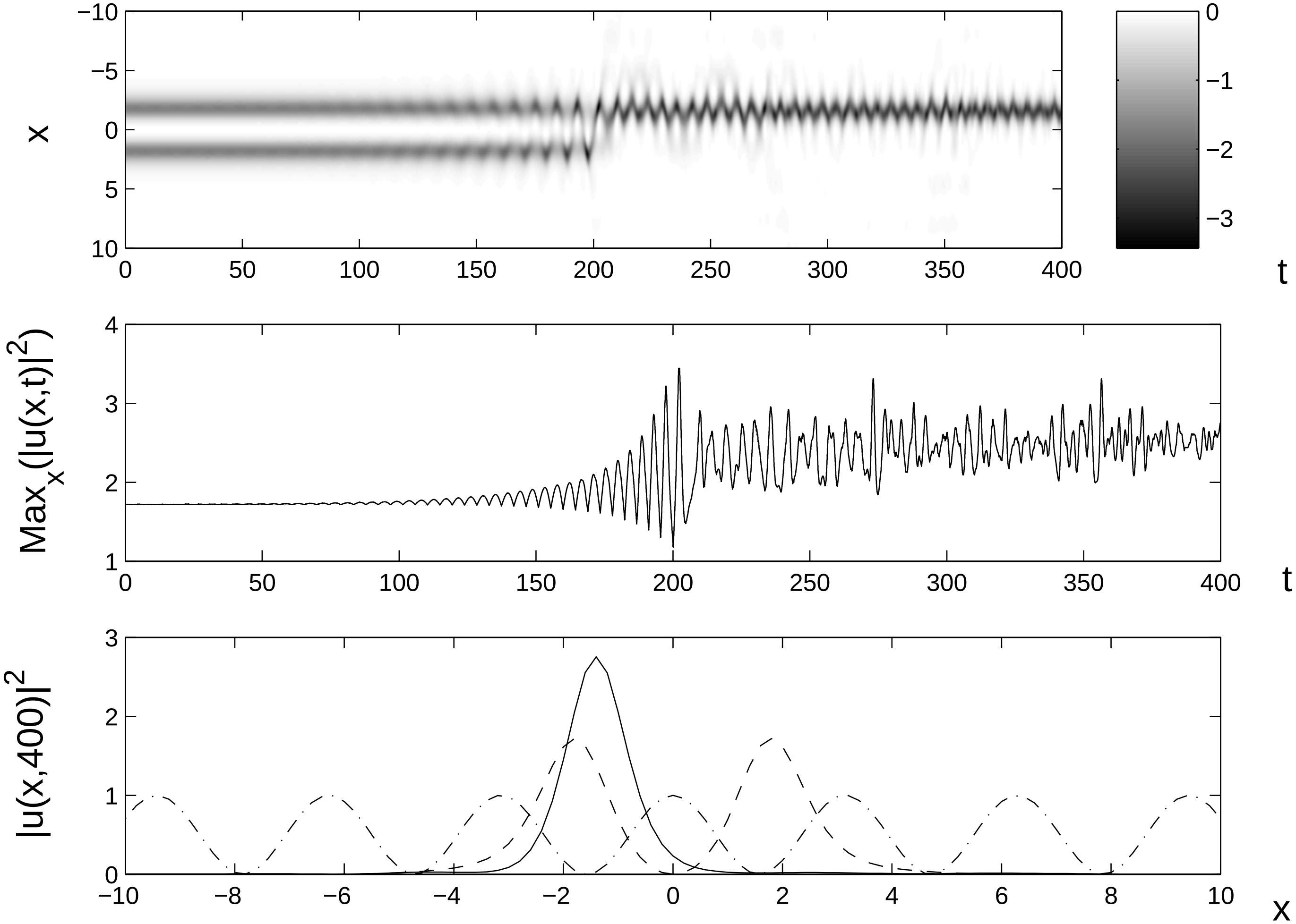
- Also for Magnetic Trap

$$V_{\mathrm{eff}}(\xi_1, \xi_2; \Delta\phi) = -8\eta^3 \cos\left(\Delta\phi\right) \exp\left(-\frac{\eta}{\sqrt{2}}\left|\xi_1 - \xi_2\right|\right) + 4\sqrt{3}\epsilon\left(\xi_1^2 + \xi_2^2\right) \tag{39}$$

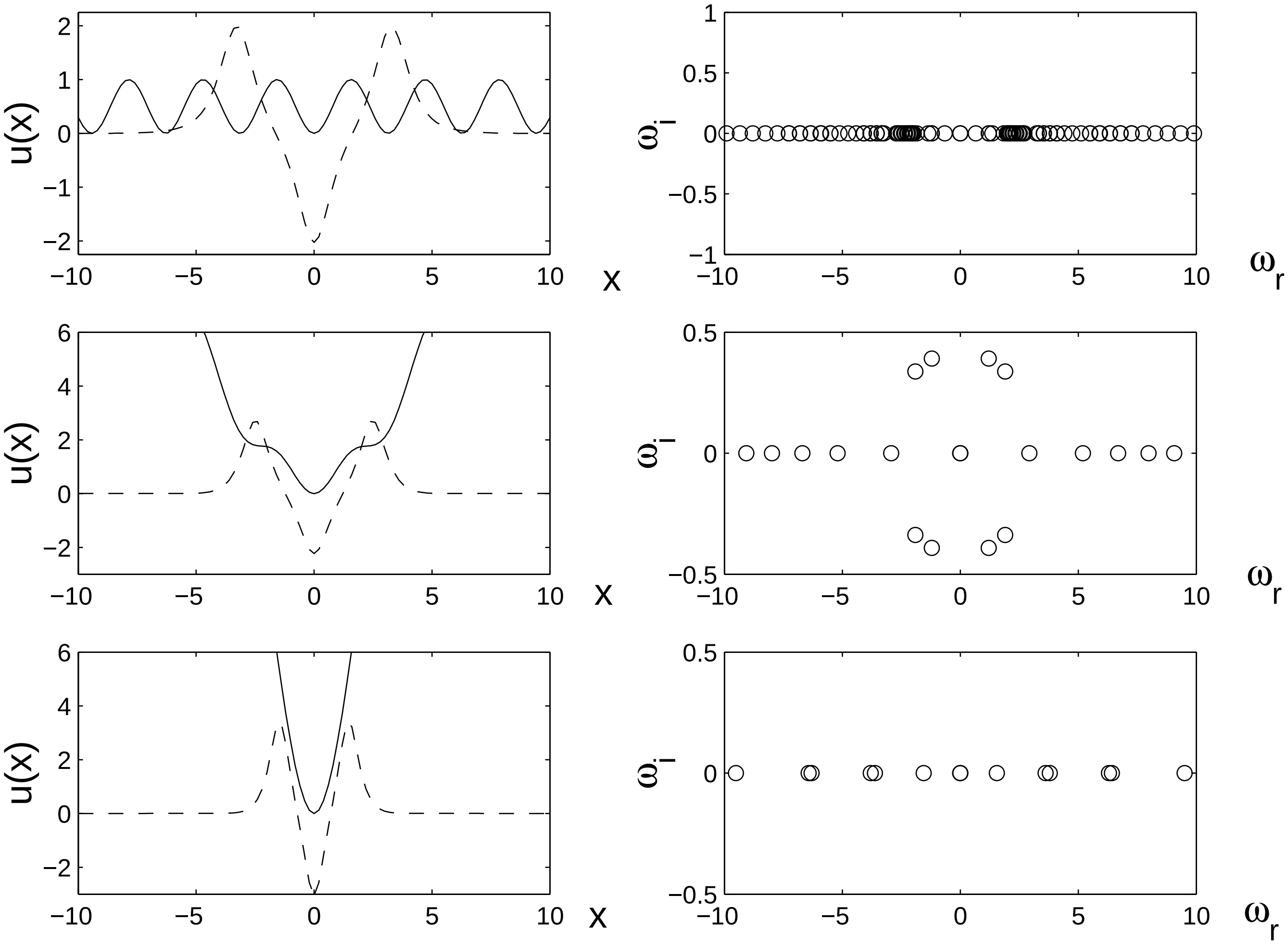
Numerical Results: Twisted Modes



TLMs: Oscillatory Instabilities



Multi-Soliton Solutions



A Dispersion Management Analog: Feshbach Resonance Management

- Modulation of Scattering Length via Feshbach Resonance

$$iu_t = -(1/2)u_{xx} + a(t)|u|^2u + (1/2)\Omega^2x^2u \tag{40}$$

- Notice: Modulational Stability Analysis for

$$u = A_0 \exp[iA_0^2 \int_0^t a(s)ds] \, [1 + \epsilon w(t) \cos(kx)] \tag{41}$$

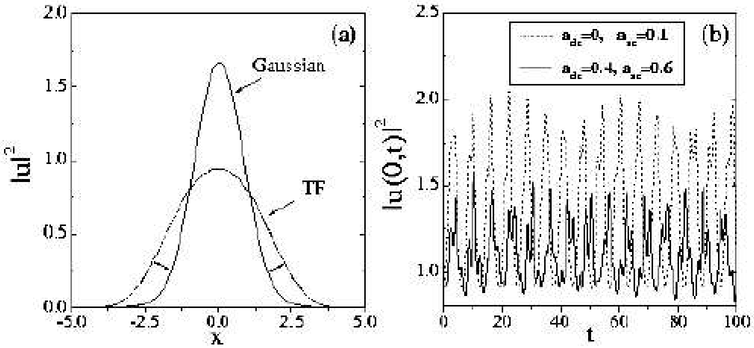
leads to Hill's Equation

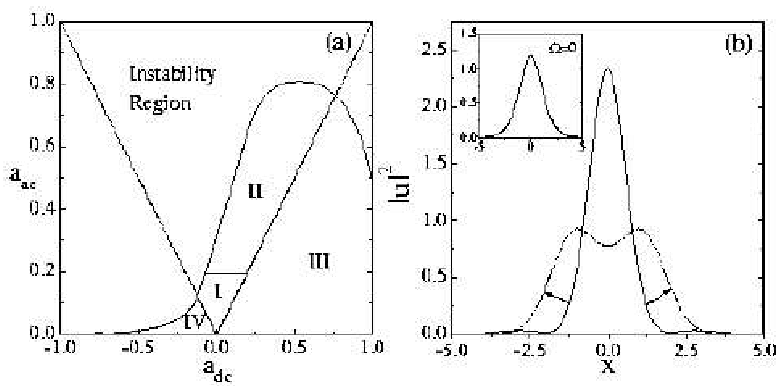
$$\ddot{w}_1 = -k^2 \left[k^2/4 + a(t)\right] w_1 \tag{42}$$

and to the Kronig-Penney model for FRM.

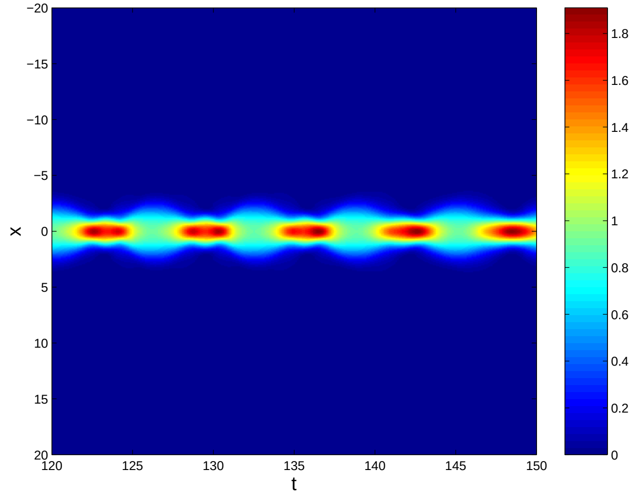
- Robust Bright and Dark FRM solitons

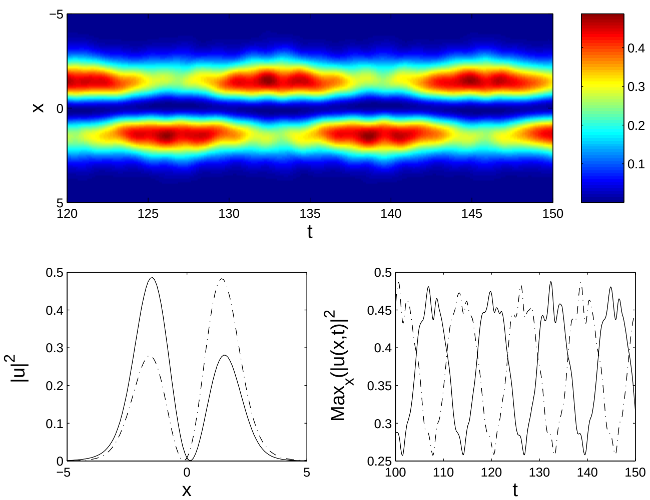
FRM: Phase Diagram



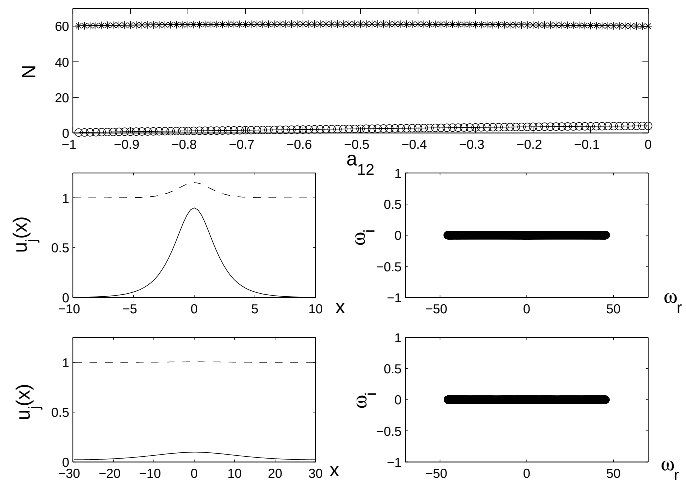


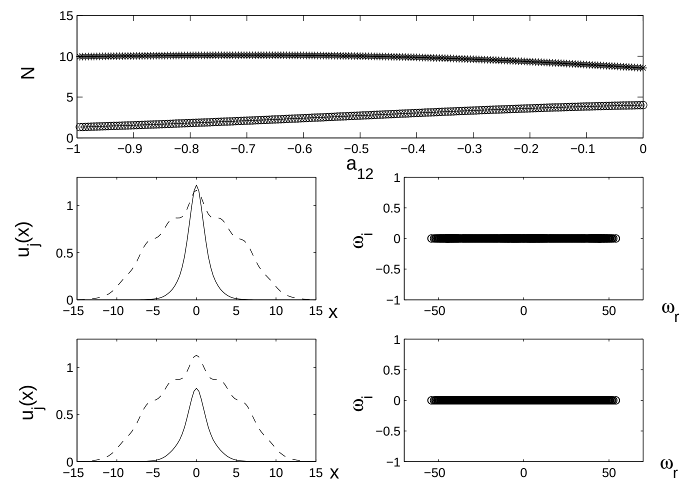
FRM Bright and Dark Solitons



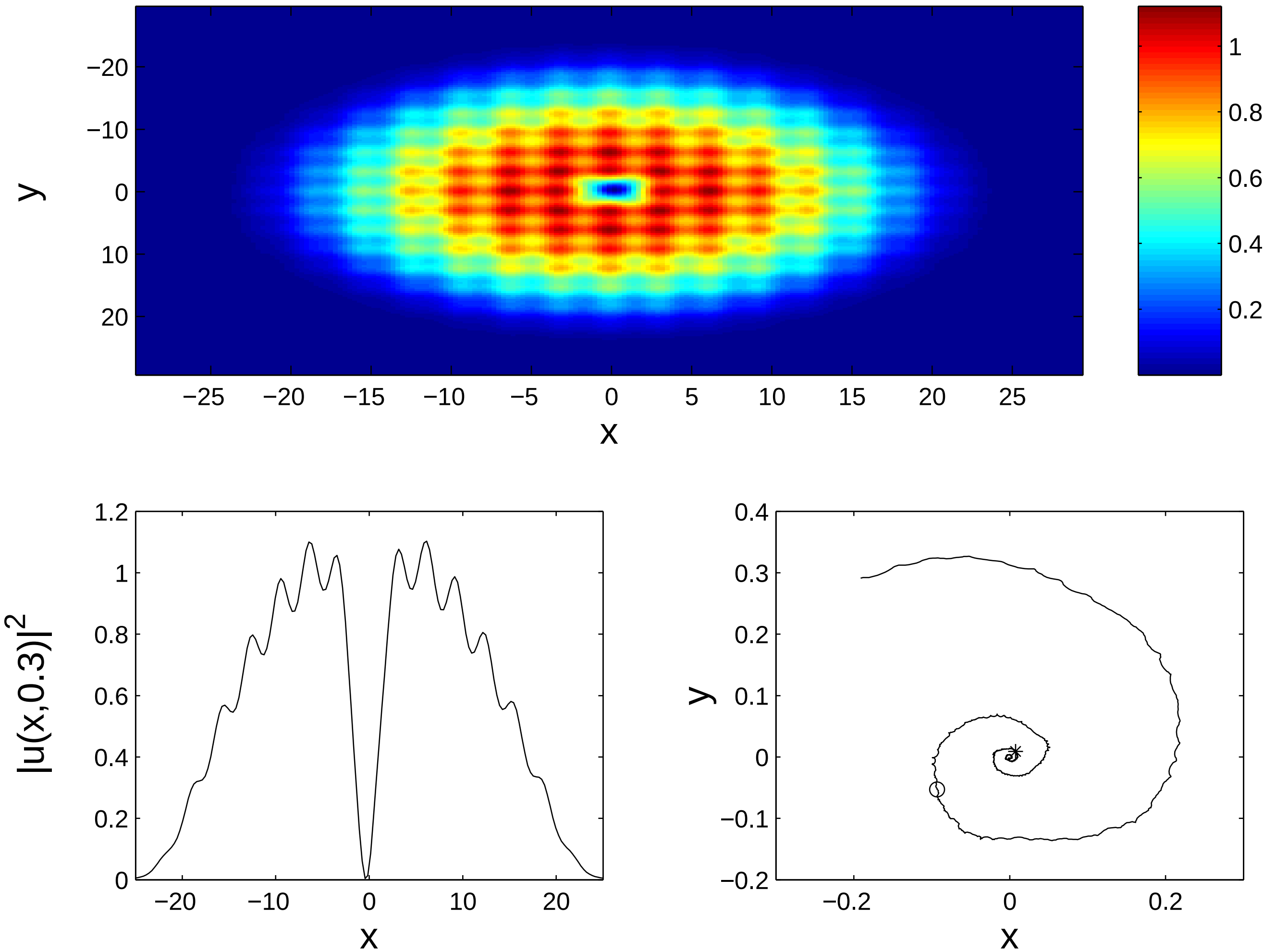


Multi-Components: DWs, Bright-Antidark, Bright-Grey, Dark-Antidark, Dark-Grey





2d Structures of Interest: Vortices



Recent Additions: Part I

Infrared Catastrophes (And How To Avoid Them)

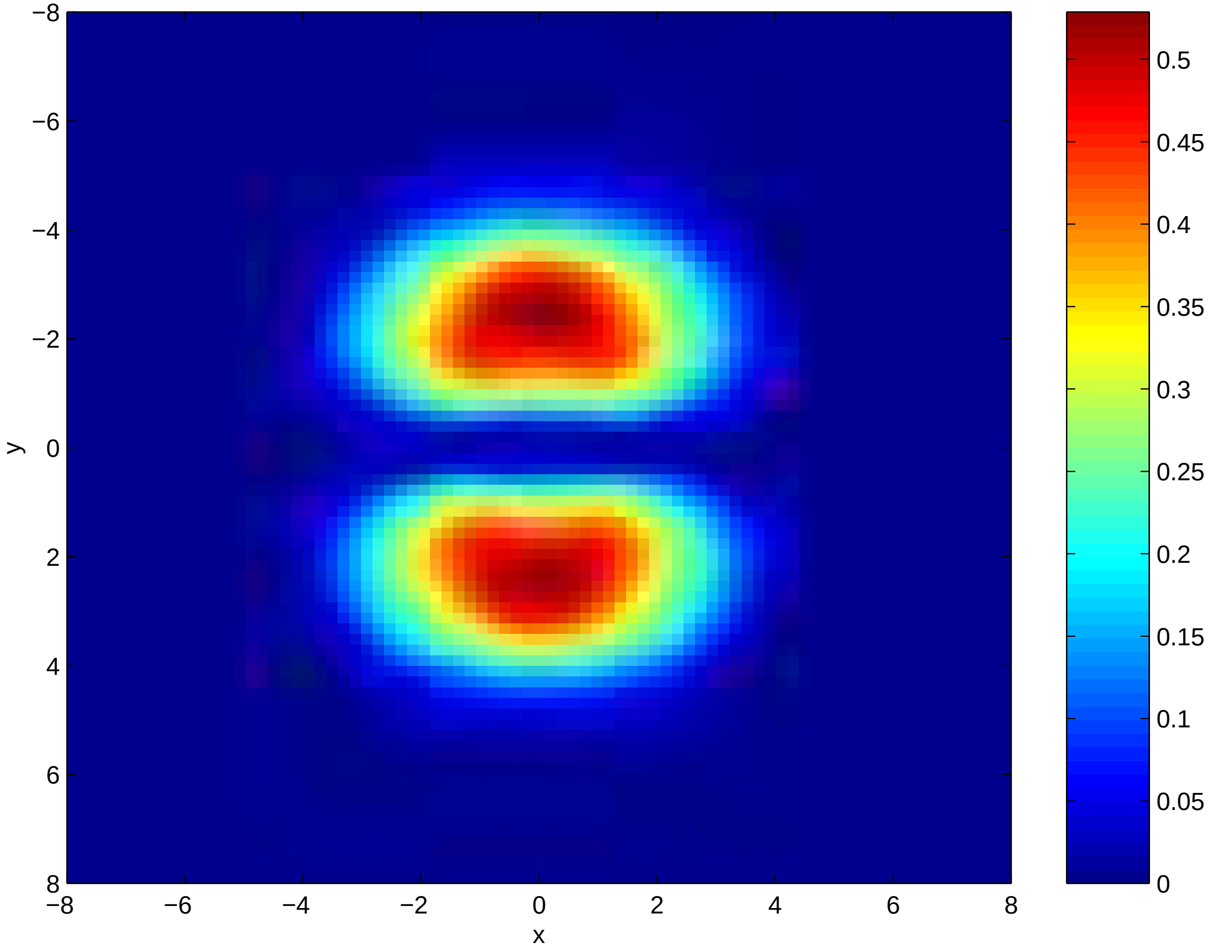
- Modulational Instability

$$\lambda_{BEC} < \lambda_{cr} \Rightarrow \Omega > \Omega_{cr} \equiv 2^{3/2}\frac{(a\mu)^{1/2}u_0}{\pi} \tag{43}$$

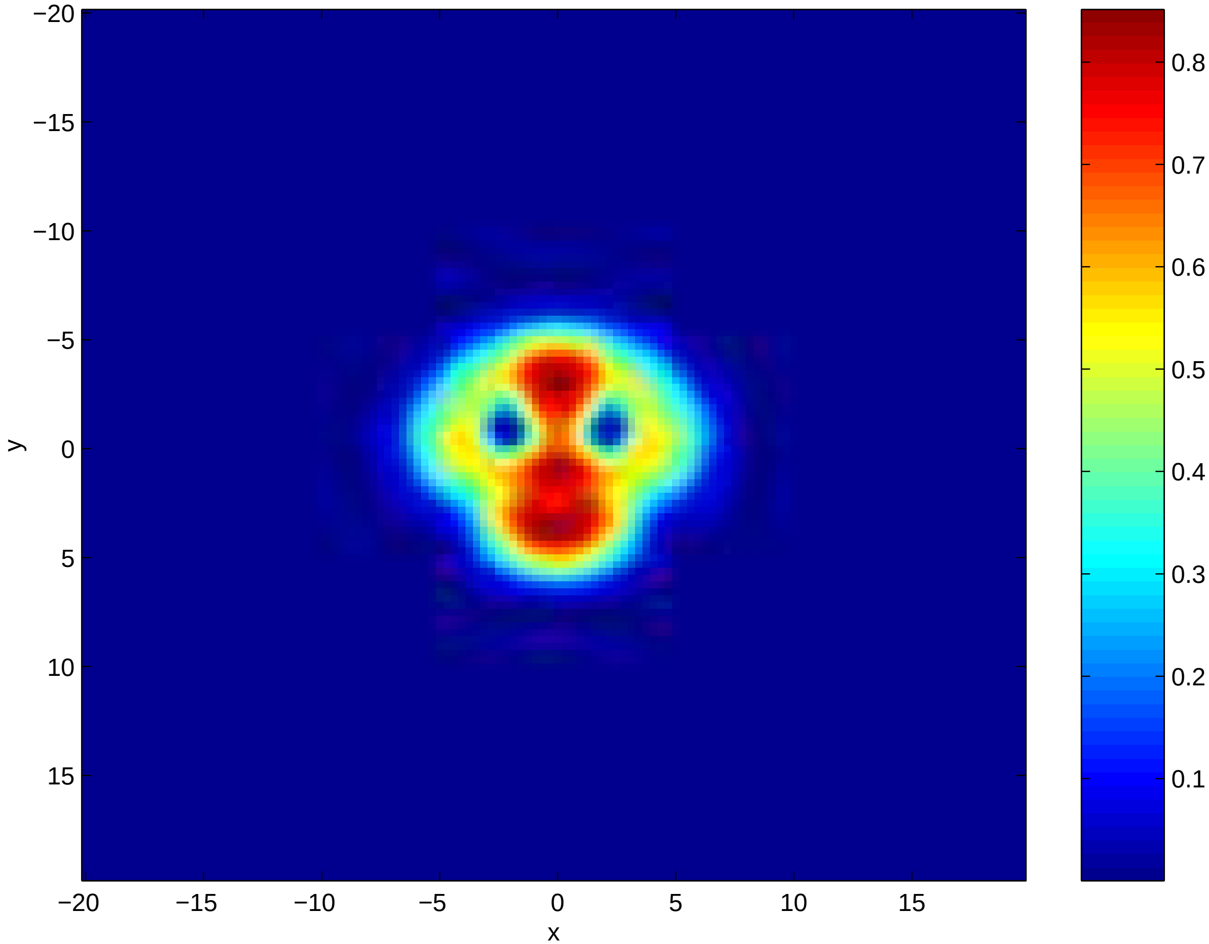
- Transverse Instability

$$\lambda_{BEC} < \lambda_{cr} \Rightarrow \Omega > \frac{\sqrt{2\mu}}{\pi} \tag{44}$$

The Dipole



The "Monkey-Pole"



Recent Additions: Part II

Averaged Dynamics for Feshbach Resonance Management

- FRM for Strong Nonlinearity Management

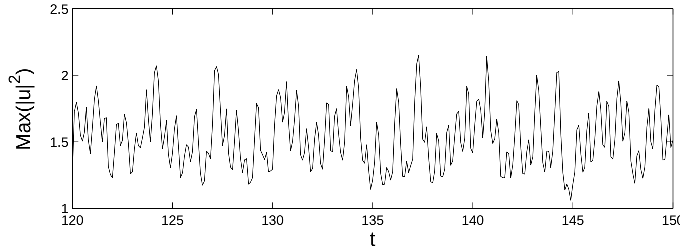
$$iu_t = -u_{xx} + \Gamma(t)|u|^2 u + V(x)u \tag{45}$$

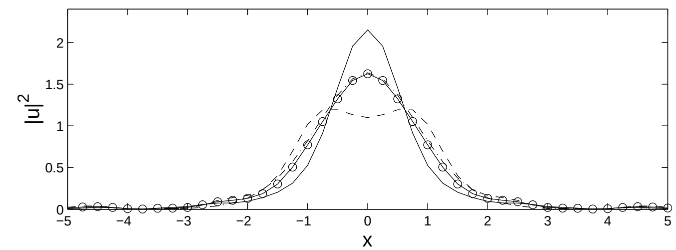
- For $\Gamma(t) = \gamma_0 + \frac{1}{\epsilon}\gamma(\tau), \tau = \frac{t}{\epsilon}$, obtain:

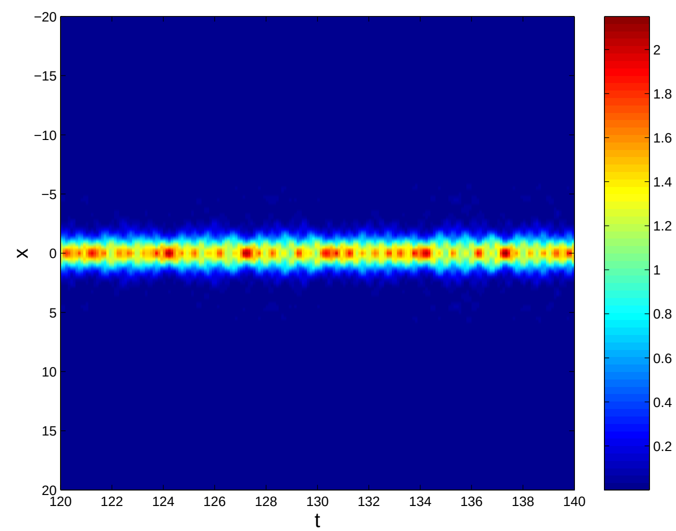
$$iu_t = -\,u_{xx} + \gamma_0 |u|^2 u + V(x)u + \mu u \left(|u|_x^2\right)^2 + i\nu_1 u \left(\bar{u}u_x - \bar{u}_x u\right)_x \tag{46}$$

- Use Standing Wave Ansatz $u(x,t) = \phi(x)e^{i\omega t}$,to obtain:

$$-\phi'' + \omega\phi + V(x)\phi + \gamma_0\phi^3 + 4\mu(\phi')^2\phi^3 = 0. \tag{47}$$


Max(|u|^2)
t


|u|^2
x


x
t

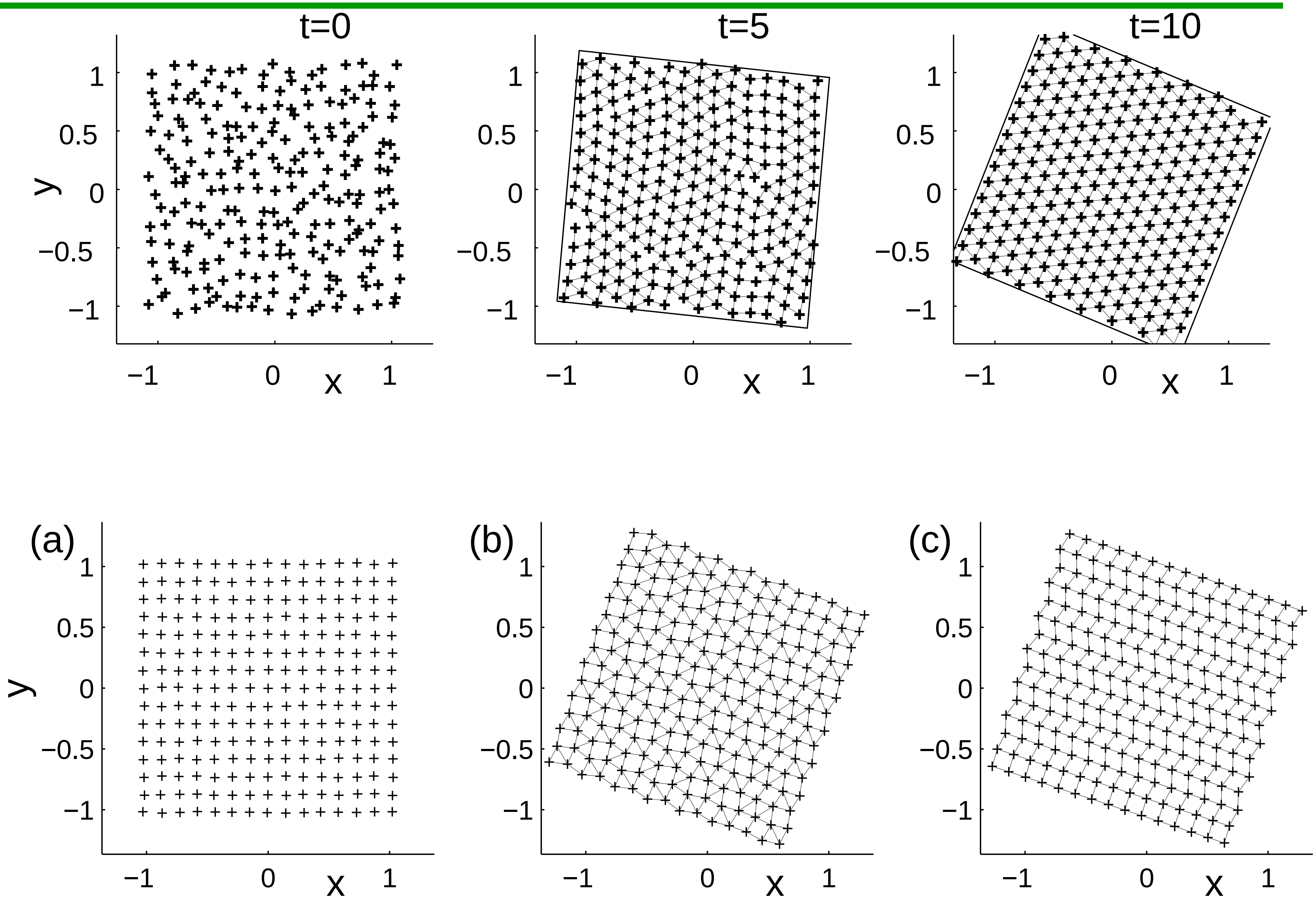
Recent Additions: Part III

Vertex Lattices and their Structural Transitions

- Parrinello-Rahman Molecular Dynamics

$$\mathcal{L} = \frac{1}{2}\sum_{n} M\,(G_{11}\,\dot{\xi}_n^2 + 2G_{12}\,\dot{\xi}_n\dot{\nu}_n + G_{22}\,\dot{\nu}_n^2) - p_{ext}S + \frac{1}{2}W(\dot{a}_x^2 + \dot{a}_y^2 + \dot{b}_x^2 + \dot{b}_y^2)$$
$$- \sum_{m,n} V(r_{mn}, \psi_{mn}). \qquad (48)$$

- Identify Ground State as Triangular Configuration
- Identify also Richer Pattern Landscape: Rhomboidal and Star Patterns


t=0
t=5
t=10
(a)
(b)
(c)
x
y
−1
0
1
0.5
−0.5

Conclusions

- BEC is a new experimentally controllable playground for nonlinear waves
- Bright and Dark Solitons in Attractive and Repulsive BECs have been identified Analytically, Numerically and Experimentally.
- Depending on Trapping, 1, 2 and 3d GP with Parabolic or Periodic Potentials (or even Discrete) are the relevant Models for the Condensate.
- Old tricks (Modulational and Snaking Instability) have been used to generate Solitons and Vortices.
- Other old tricks (DM, TLMs, Multiple Components) have been adapted to generate new beasts !.

Future Challenges

- Vortices and Vortex Lattices
- 2d-3d FRM Solitons
- Multiple-Components
- Genuinely “Discrete” Structures
- Avoiding Collapse: Can the OL or FRM Help ?
- Thermal Cloud Fluctuations: Including the Non-Condensate
- Examining Collapse: the Dynamic Renormalization Framework