Computing the Abel transform

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In this talk, you'll hear about

* Riemann theta functions
* Riemann Surfaces
* Abel transforms
* Riemann constants

and more.

Why should you care?

Water Wave Pictures

The following are pictures from actual water waves. Each one of these pictures shows water waves having properties that are very similar to properties of solutions of the KP equation.

Figure 1

Oblique interaction of two nearly solitary waves in shallow water. (Photograph courtesy of T. Toedtemeier.)

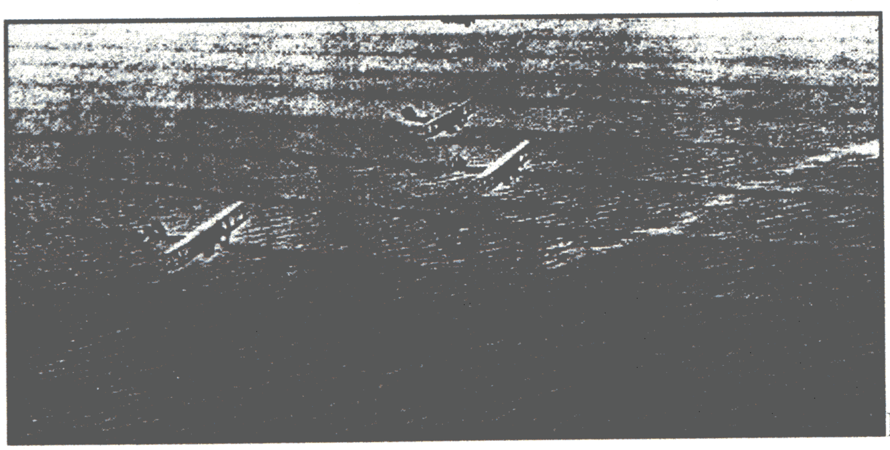


Figure 2

Periodic waves in shallow water. The original caption read: "As they near shallow water close to the coast of Panama, huge deep-sea waves, relics of a recent storm, are transformed into waves that have crests, but little or no troughs. A light breeze is blowing diagonally across the larger waves to produce a cross-chop. Three Army bombers, escorted by a training ship, are proceeding from Albrook Field, Canal Zone, to David, Panama." (Taken from National Geographic 63 (1933).)

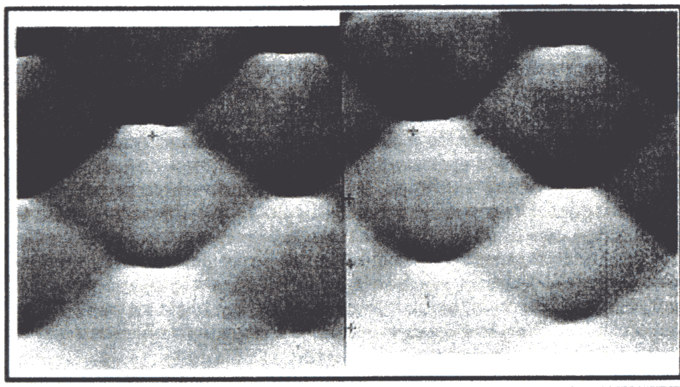


Figure 3a

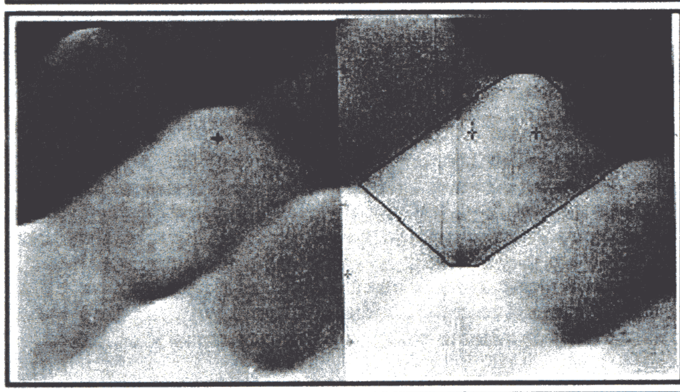


Figure 3b

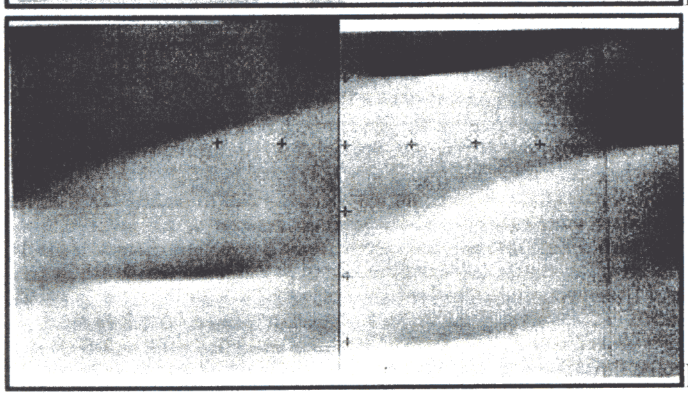


Figure 3c

Mosaic of two overhead photographs, showing surface patterns of waves in shallow water. Each of these waves has a basic hexagonal template; one such hexagon is drawn in the middle figure. (Taken from Hammack, McCallister, Scheffner & Segur, J. Fluid Mechanics 285 (1995) 95-122.)

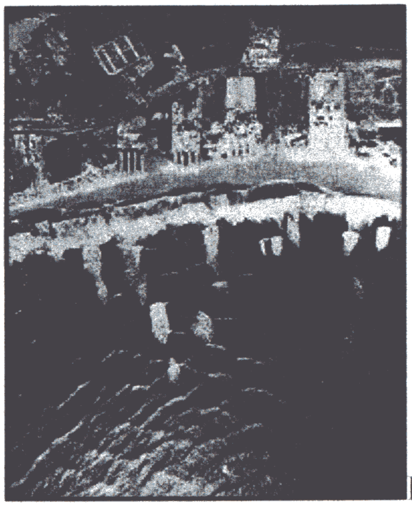


Figure 4

Aerial Photograph of waves off the southern coast of Long Island. The beack is between Lido Beach and Point Lookout, west of Jones Inlet. Beyond the surf zone, the wave patterns are two-dimensional, and approximately periodic. They have flat troughs, sharp crests, and approximately hexagonal shape. (Taken from Hammack, McCallister, Scheffner & Segur, J. Fluid Mechanics 285 (1995) 95-122.)

5

A picture of a storm caused by Hurricane Grace, the "Halloween Storm of 1991". This storm became famous as the storm in the book and movie "The Perfect Storm". Note the hexagonal shape and approximate periodicity of the wave pattern (Picture bravely taken by Carl Miller, from a small plane along the coast of the barrier islands in North Carolina during the storm).

The Kadomtsev-Petviashvili Equation

$$(-4u_t + 6uu_x + u_{xxx})_x + 3\sigma^2 u_{yy} = 0$$

describes waves in shallow water that are

- weakly 2-dimensional
- long (compared to depth)
- small amplitude (compared to depth)
- irrotational
- inviscous

KP has solutions described by Riemann Theta functions:

$$\boxed{u = c + 2\partial_x^2 \ln \theta(\bar{k}x + \bar{\ell}y + \bar{\omega}t + \bar{\phi} \,|\, \bar{\bar{B}})}$$

with: c : constant

$\bar{k}, \bar{\ell}$: wave number vectors

$\bar{\omega}$: frequencies

$\bar{\phi}$: phase

$\bar{\bar{B}}$: Riemann Matrix

θ : Riemann theta function

<u>Note:</u> similar solutions exist for the K-dV equation, Sine-Gordon, Nonlinear Schrödinger, ...

What is Θ ?

$$\Theta(\bar{z}) = \sum_{\bar{m} \in \mathbb{Z}^g} e^{2\pi i\left(\frac{1}{2}\bar{m}.\bar{\bar{B}}.\bar{m} + \bar{m}.\bar{z}\right)} \qquad (*)$$

where $\bar{z} \in \mathbb{C}^g$

$\bar{\bar{B}}$: Riemann Matrix

$\bar{\bar{B}}^T = \bar{\bar{B}}$

$\mathrm{Im}\,\bar{\bar{B}} > 0$

then $(*)$ converges absolutely and uniformly on compact sets of $\mathbb{C}^g \times S^g$

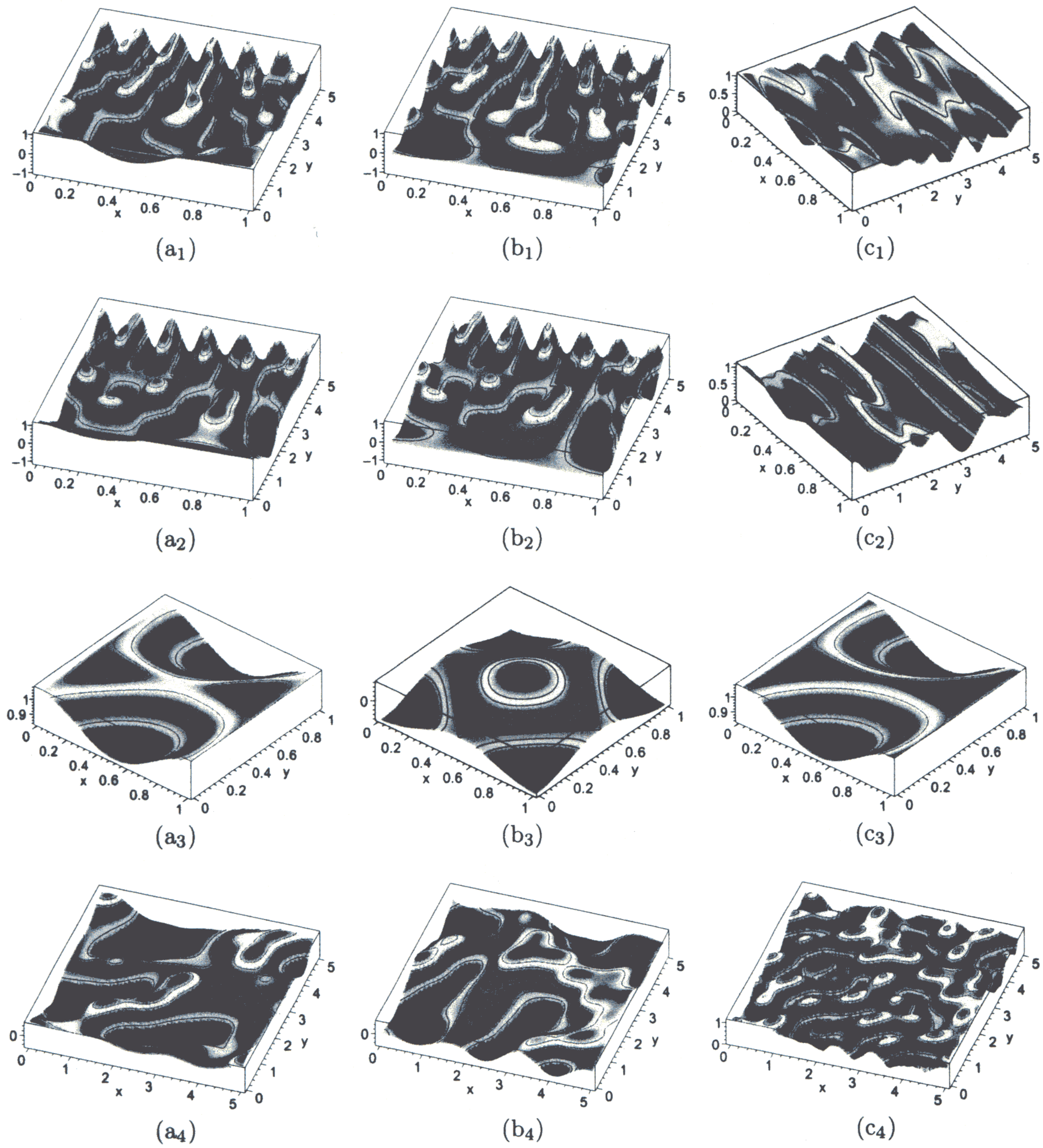


Figure 4: Various views of the oscillatory part of the Riemann theta function parametrized by the Riemann matrix given in (22), with FFE=0.001. All plots are oscillatory parts of $\theta(x+iy, 0|\mathbf{\Omega})$ (index 1), $\theta(0, x+iy|\mathbf{\Omega})$ (index 2), $\theta(x, y|\mathbf{\Omega})$ (index 3), $\theta(ix, iy|\mathbf{\Omega})$ (index 4). Shown are the real part (a), the imaginary part (b) and the absolute value (c).

<u>Today's problem</u>: what is

$$\bar{\phi} \, ?$$

For KP:

$$\bar{\phi} = -\bar{A}(D) - \bar{K}$$

where $\bar{A}$: Abel transform

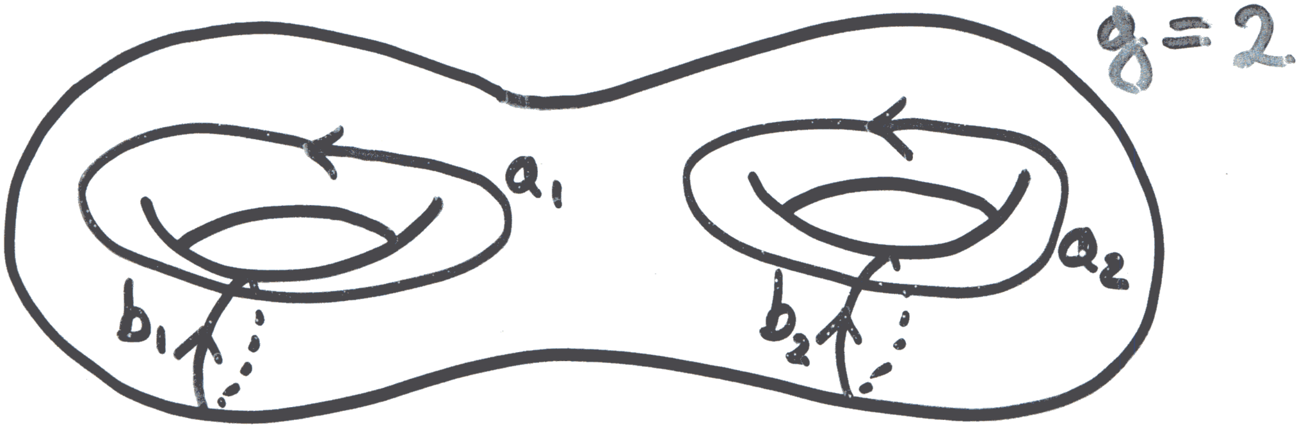
D : Divisor

$\bar{K}$: Riemann Constants

Huh?

What's all of this?

Let Γ be a compact, connected Riemann Surface (i.e. $P(\lambda,\mu)=0$, of genus g



On Γ there are g holomorphic differentials

$$\omega_1, \omega_2, \ldots, \omega_g$$

and $2g$ integration paths

$$a_1, a_2, \ldots, a_g,$$
$$b_1, b_2, \ldots, b_g$$

From these $\Rightarrow$ $\bar{\bar{B}}$, $\bar{k}, \bar{t}, \bar{w}, \bar{c}$

From these and $D = P_1 + \ldots + P_g$ (non special divisor) $\Rightarrow$ $\bar{\phi}$

$$D = P_1 + P_2 + \cdots + P_g$$

$$= \{P_1, P_2, \ldots, P_g\}$$

is a set of g points on Γ

What is $\bar{A}$?

$\bar{A}$: Abel transform

$\bar{A}$ maps Γ to $J(\Gamma)$, the Jacobian of Γ, i.e.

$$J(\Gamma) = \mathbb{C}^g / \{\bar{N} + 2\pi i \bar{\bar{B}} \cdot \bar{M}\}$$

$$A_k(P) = \int_{P_0}^{P} \omega_k$$

P_0 ⟶ fixed reference point

and

$$A_k(D) = \sum_j \int_{P_0}^{P_j} \omega_k = \sum_{j=1}^{g} A_k(P_j)$$

So, we have to integrate on Γ. What's the problem?

When we integrated before:

- we knew which paths to use

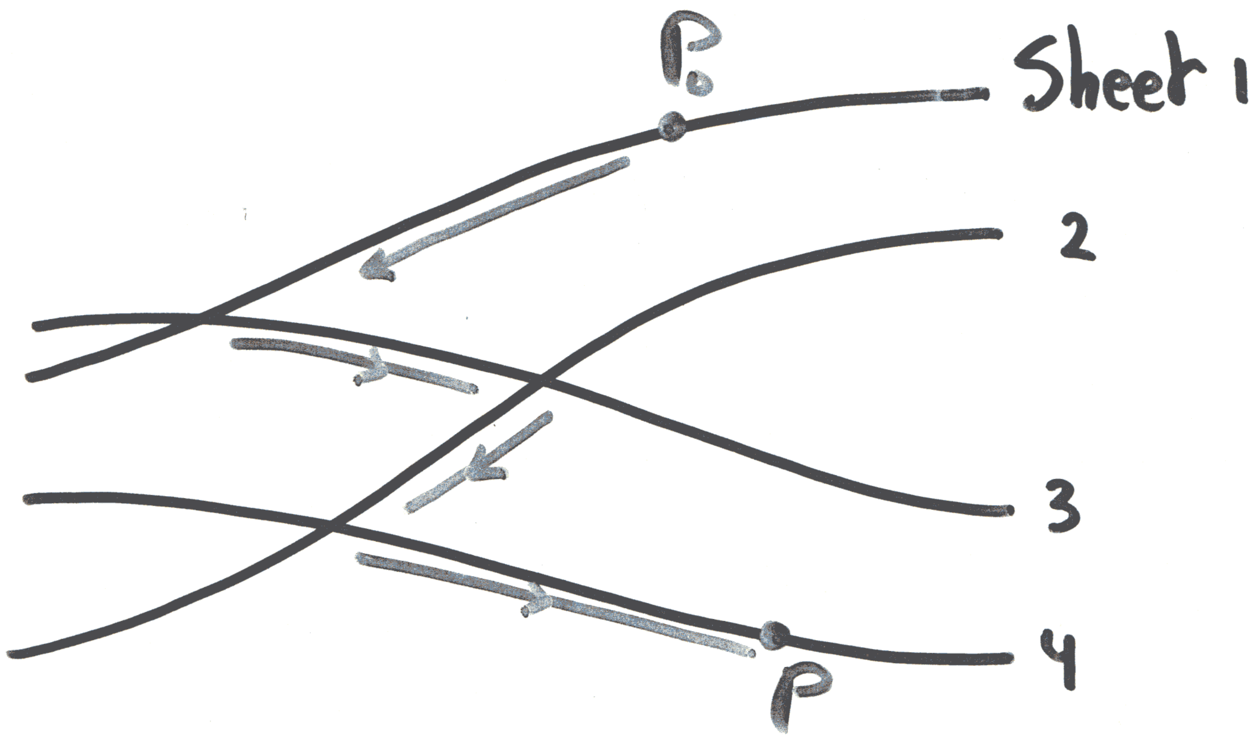
- we stayed far away from branchpoints, singular points, etc. Now, we cannot do this

Thus: 2 questions:

- how to find our way from $P_0 \to P$

- how to deal with problem points

1. Finding our way around



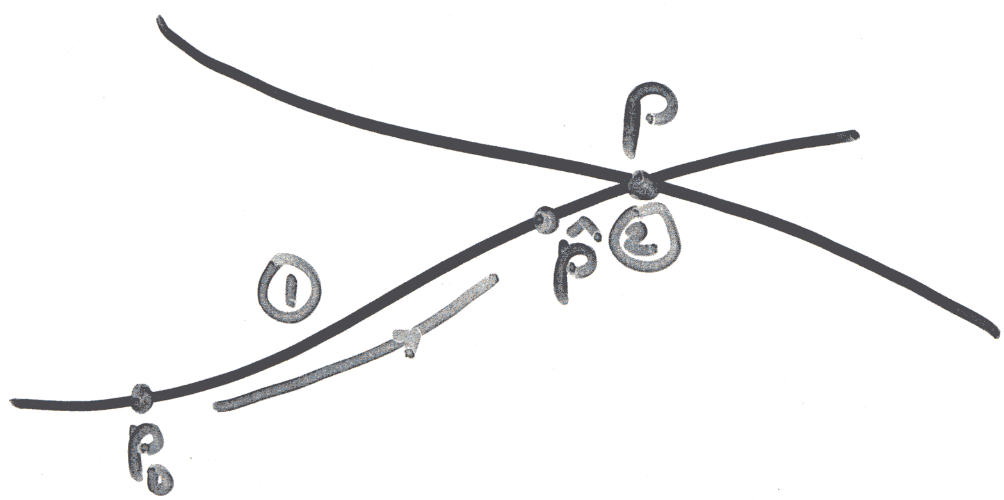
- determine which sheet P_0, P are on
- determine how to move between sheets

(Big combinatorics problem)

2. Problem points

General idea: use puiseux series, integrate away from the problem point

$$\int_{P_0}^{P} \omega_k = \underbrace{\int_{P_0}^{\hat{P}} \omega_k}_{①} + \underbrace{\int_{\hat{P}}^{P} \omega_k}_{②}$$



① As before

② Using puiseux, expand ω_k

Note:

- for branchpoints: OK
- for singular points: need to specify puiseux expansion

What is $\overline{K}$?

$$K_j = K_j(P_0) \qquad \text{(}P_0 \to \text{fixed ref. point)}$$

$$= \frac{1 + B_{jj}}{2} - \sum_{\ell \neq j} \int_{a_\ell} \omega_\ell(P) \int_{P_0}^{P} \omega_j$$

double integral on Γ

!!!

→ very costly

All integration usually done using analytic continuation. Not here.

Conclusion:

* Computing (quasi) periodic solutions of integrable equations requires
  - $\overline{\overline{B}}$ ✓
  - ϑ ✓
  - $\overline{A}$ ✓
  - $\overline{K}$ ✓

* For specific equations:

  $\bar{k}$, t , $\overline{\omega}$, c

$\longrightarrow$ next step

Other applications of Riemann Surfaces:

- "Near Shepherd's Bush, 2000 Beta-minus mixed doubles were playing Riemann Surface tennis"

- "She was watching the semi-finials of the South-American Riemann Surface Tennis Championship"

from: Brave New World
by Aldous Huxley