Preliminary Comparisons of Physical Experiments of Waves on Deep Water with Perturbed Solutions of NLS

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* Joint work with Joe Hammack, Diane Henderson, and Harvey Segur. All physical experiments conducted by Hammack and Henderson.

Experiments

PENNSTATE

1855

Experimental Features Observed

Description	Data
Persistence	Images, Contour maps
Connecting leg between cells	Contour maps
Oscillations in nodal region	x-time series in nodal region
Dips in crestlines	y-time series
Time varying width of nodal region	Contour maps

Theory

NLS

$$i\psi_t + \alpha\psi_{xx} + \beta\psi_{yy} + \gamma|\psi|^2\psi = 0$$

- $\psi = \psi(x, y, t)$ is a complex-valued function that represents the envelope of an underlying carrier wave.
- x represents a slow horizontal spatial coordinate perpendicular to the paddles.
- y represents a slow horizontal spatial coordinate parallel with the paddles.
- t represents a slow time or propagation distance.
- $\alpha < 0$, $\beta > 0$, $\gamma < 0$ are determined by the experimental parameters.

A Solution

A 1-D traveling wave solution of NLS

$$\psi(y,t) = \phi(y)\mathrm{e}^{i\lambda t} = \sqrt{-2\frac{\beta}{\gamma}}\, bk\,\mathrm{sn}(by,k)\mathrm{e}^{i\lambda t}, \quad \mathbf{(1)}$$

where

$$\lambda = -\beta b^2(1+k^2),$$

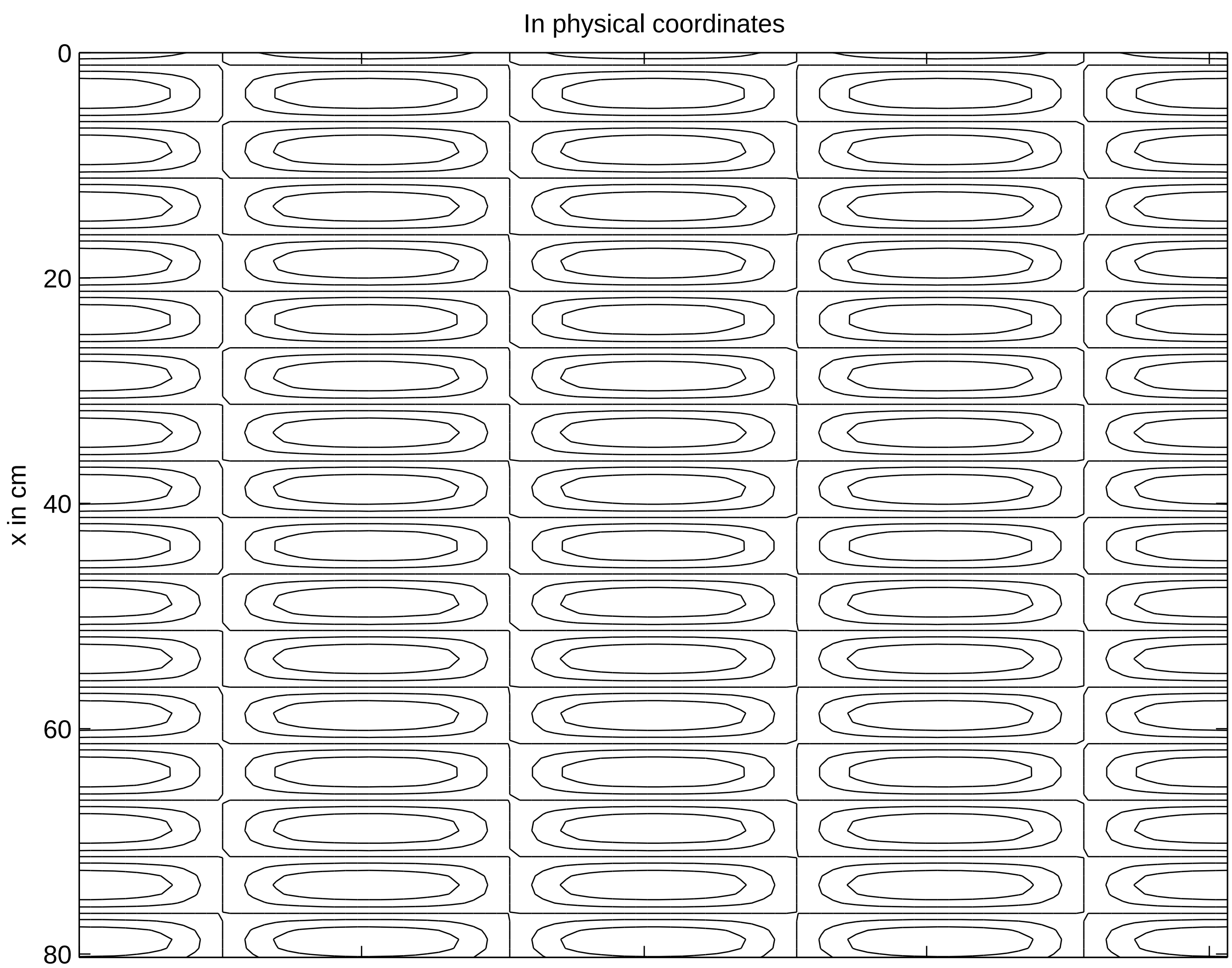
and $k \in [0,1]$ and b are free parameters.

Experimental Data

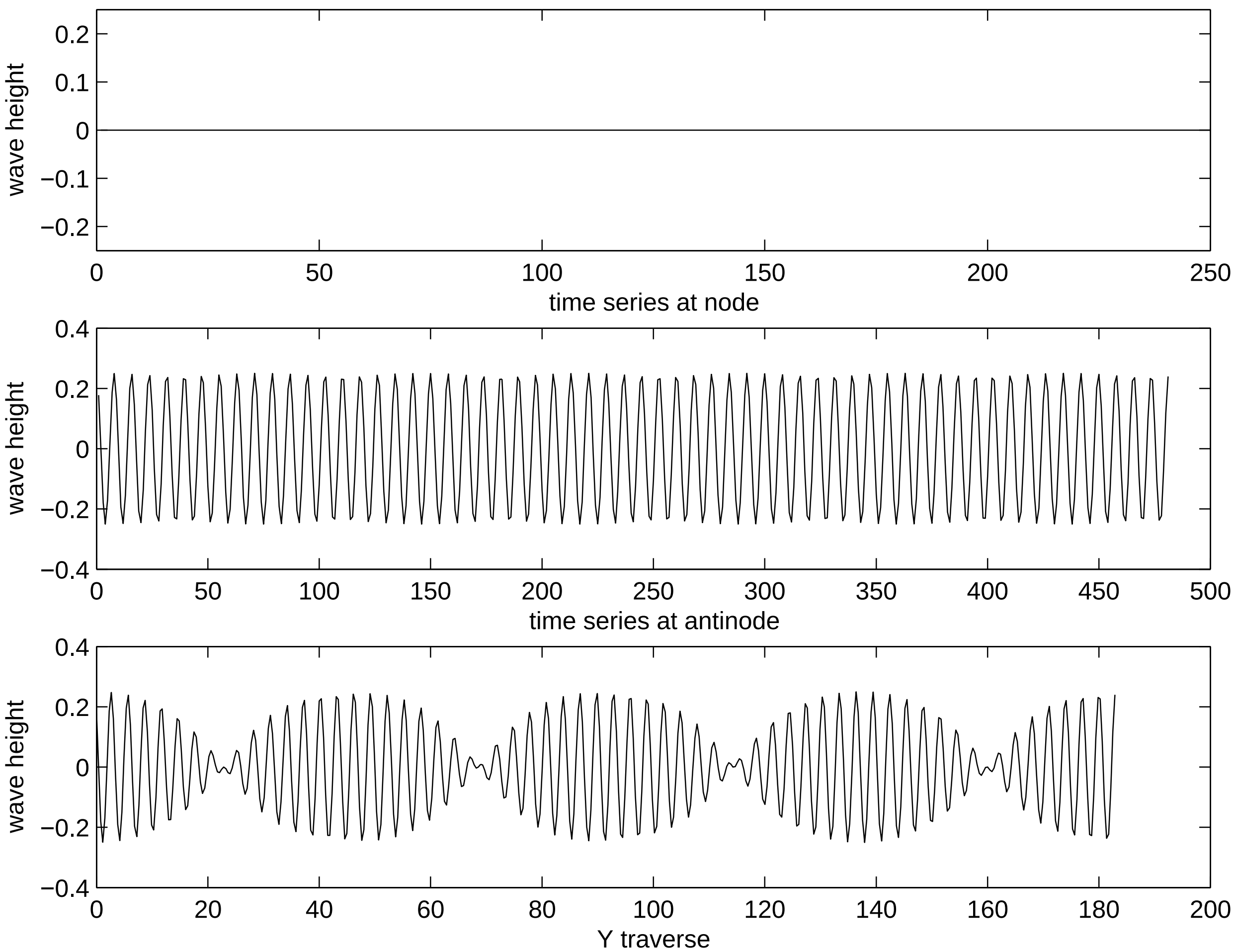
Data from a physical experiment.

- Carrier wave parameters
  - ➨ $a_0 = 0.250cm$, $\kappa_0 = 0.626cm^{-1}$
  - ➨ $\alpha = -0.0988$, $\beta = 0.268$, $\gamma = -2.041$
- Solution parameters (dimensionless)
  - ➨ $b = 0.157$ and $k = \sqrt{0.8}$
  - ➨ $\phi(y) = 0.0710\ \mathrm{sn}(by, k)$

Unperturbed Surface



Unperturbed Surface



Stability Analysis

Consider perturbed solutions with structure

$$\psi_p(x, y, t) = (\phi(y) + \epsilon u(x, y, t) + i\epsilon v(x, y, t))\mathrm{e}^{i\lambda t},$$

where

- ϵ is a small real parameter,
- $u(x, y, t)$ and $v(x, y, t)$ are real-valued functions.

Substituting ψ_p into NLS, linearizing and separating into real and imaginary parts gives

$$-\lambda u + 3\gamma\phi^2 u + \beta u_{yy} + \alpha u_{xx} = v_t,$$

$$-\lambda v + \gamma\phi^2 v + \beta v_{yy} + \alpha v_{xx} = -u_t.$$

Without loss of generality, assume that $u(x, y, t)$ and $v(x, y, t)$ have the forms

$$u(x, y, t) = U(y, \rho)\mathrm{e}^{i\rho x - \Omega t} + c.c.,$$

$$v(x, y, t) = V(y, \rho)\mathrm{e}^{i\rho x - \Omega t} + c.c.,$$

where

- ρ is a real constant,
- Ω is a complex constant,
- U and V are complex-valued functions,
- *c.c.* denotes complex conjugate.

This leads to the eigenvalue problem

$$\begin{aligned} \lambda U - 3\gamma\phi^2 U + \alpha\rho^2 U - \beta\partial_y^2 U &= \Omega V, \\ \lambda V - \gamma\phi^2 V + \alpha\rho^2 V - \beta\partial_y^2 V &= -\Omega U. \end{aligned} \qquad *$$

In order to establish that **(1)** is unstable, we must find a solution of * that corresponds to an Ω with negative real part.

We require U and V to be periodic with the same period as ϕ.

Small-ρ Limit

In the small-ρ limit, * can be solved asymptotically. This establishes that **(1)** is unstable with respect to the "neck" mode:

$$U_n(y,\rho) = O(\rho)$$

$$V_n(y,\rho) = \phi(y) + O(\rho)$$

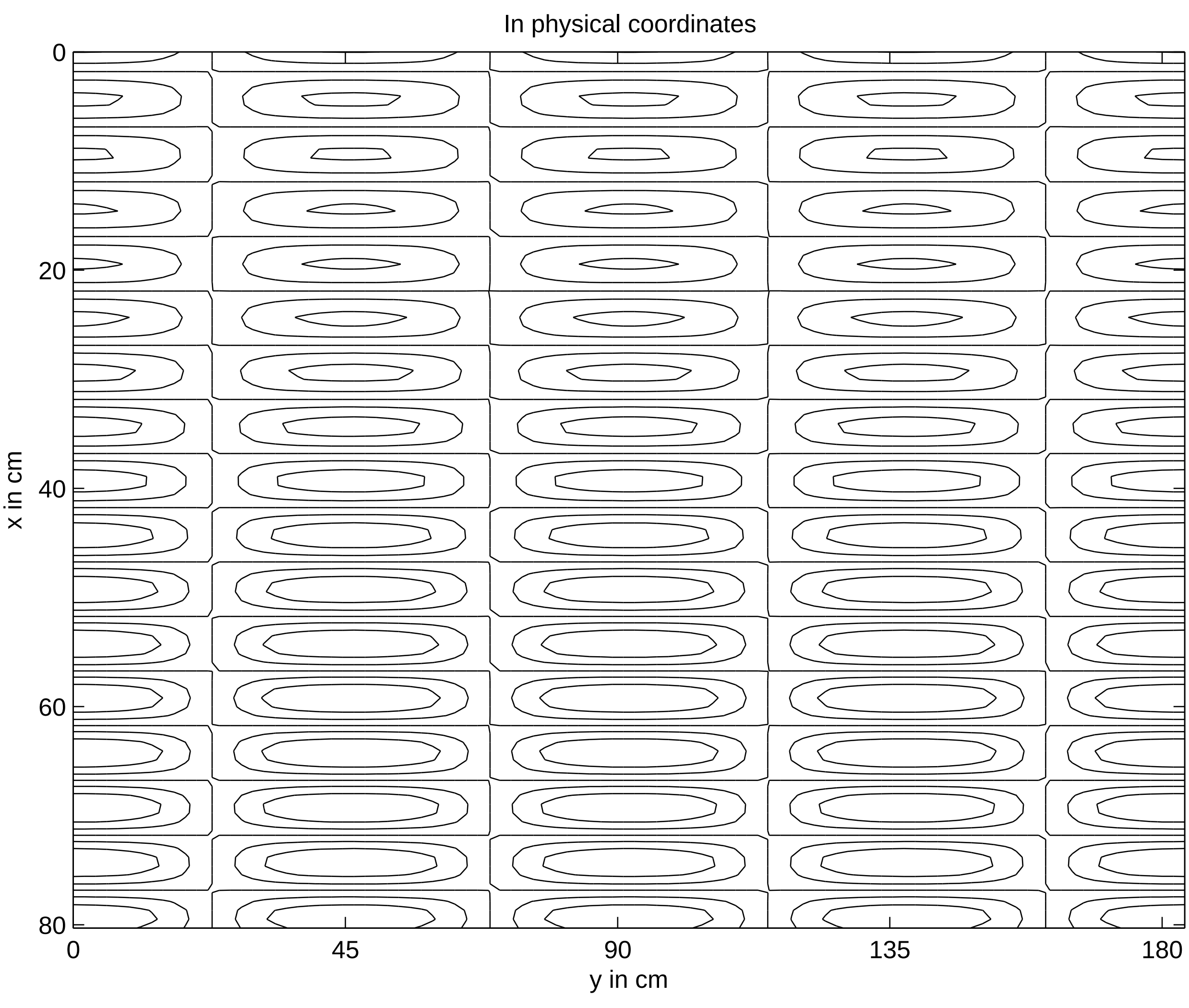
$$\Omega_n^2 = -\alpha\beta b^2\rho^2\omega_{1n} + O(\rho^3)$$

where ω_{1n} is a known positive real constant.

Neck Mode

A solution perturbed by the "neck" mode.

$$\rho = 0.0987$$
$$\Omega = -0.00387$$
$$\epsilon = 0.05$$



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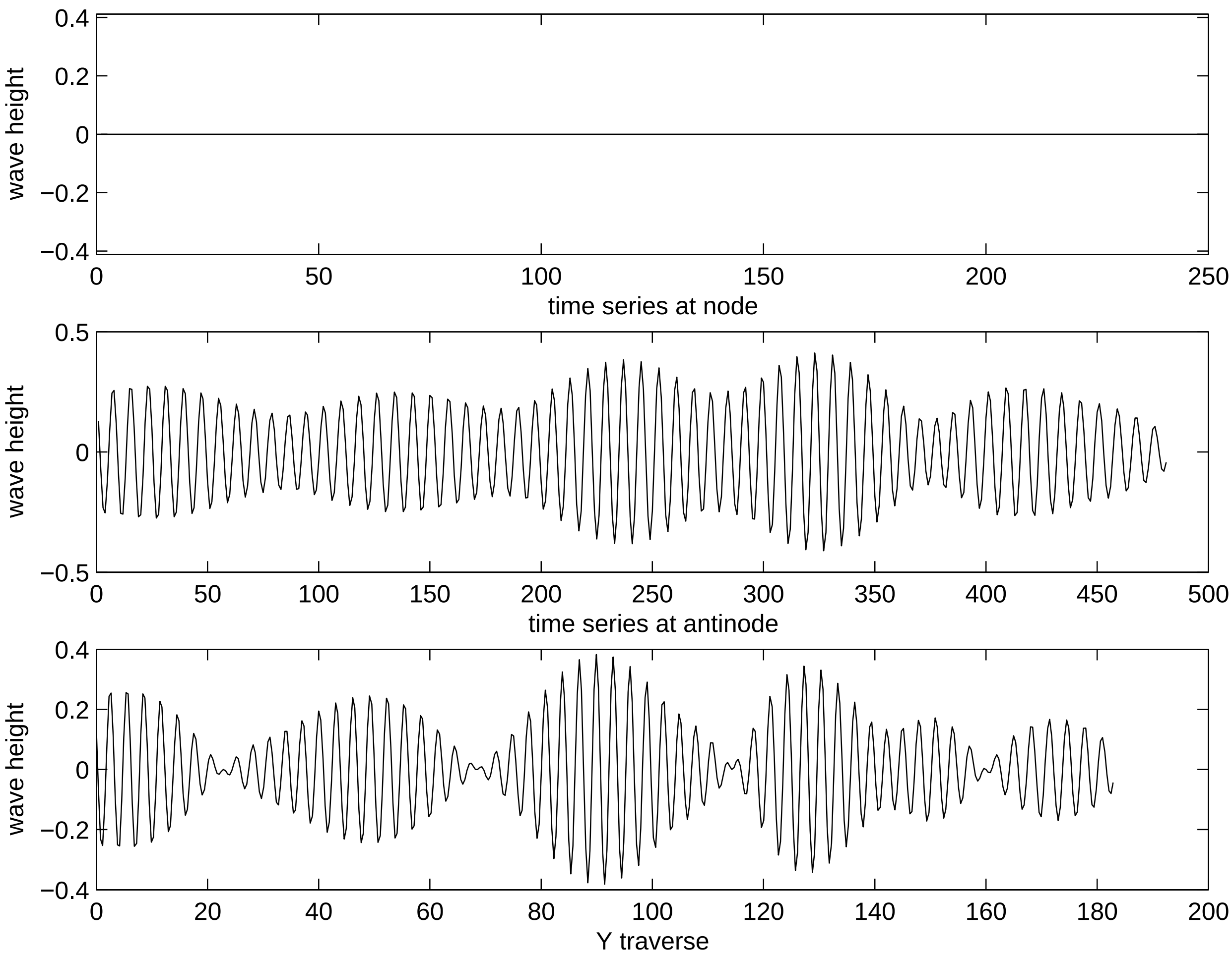


Large-ρ Limit

In the large-ρ limit, * can be solved asymptotically. This establishes that **(1)** is unstable with respect to the "large-ρ" mode:

$$U_l(y,\rho) = \zeta_{11}\sin(\mu y + y_0) + O(\rho^{-2}),$$

$$V_l(y,\rho) = \xi_{11}\sin(\mu y + y_0) + O(\rho^{-2}),$$

$$\Omega_l^2 = \alpha^2 b^2 \omega_{1l} + O(\rho^{-2}),$$

$$\mu^2 = -\frac{\beta}{\alpha}\rho^2 + \lambda - 2\alpha b\sqrt{\omega_{1l}},$$

where ω_{1n} is a known positive constant and ζ_{11}, ξ_{11} and y_0 are constants.

Large-ρ Mode

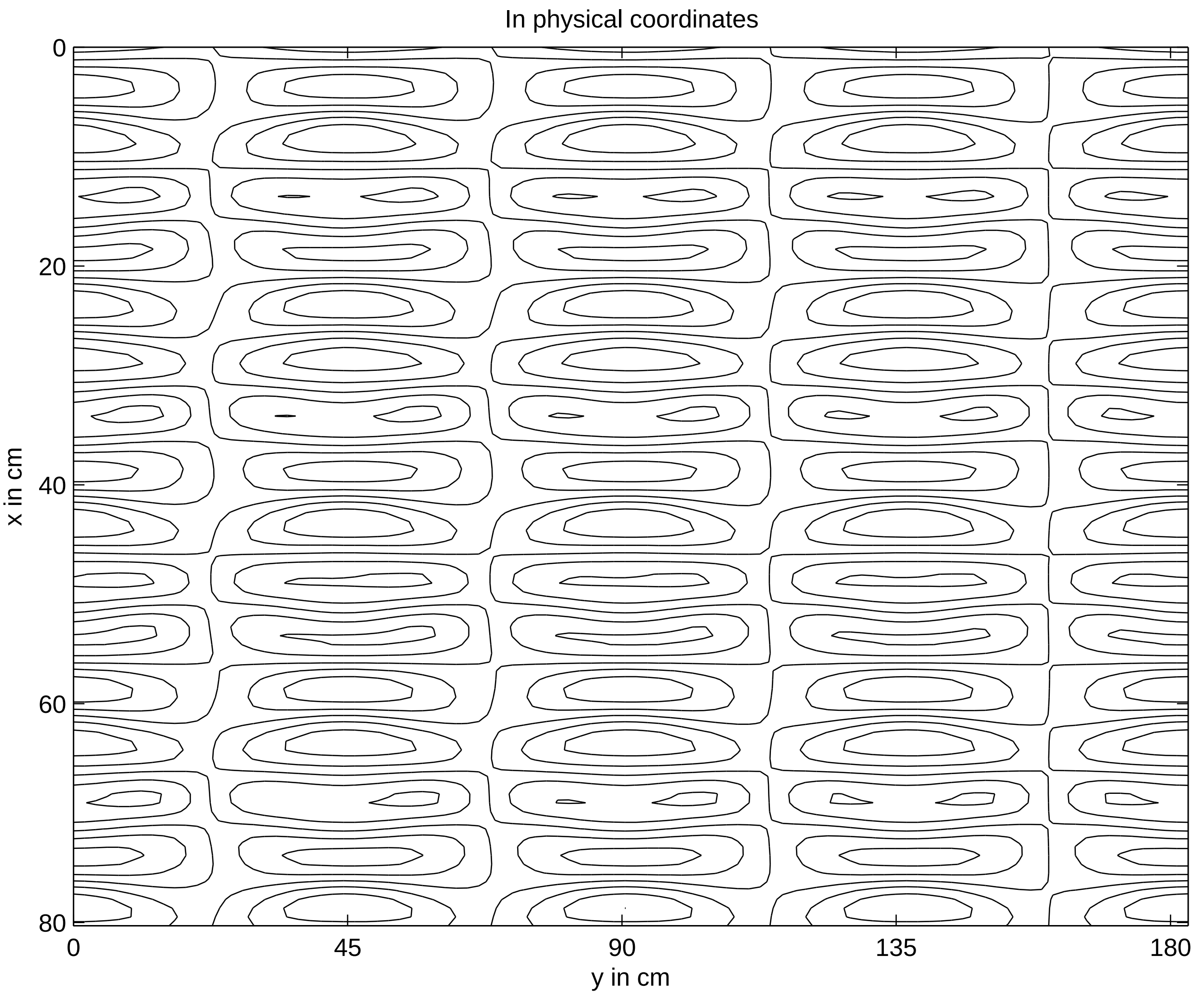
A solution perturbed by the "large-ρ" mode.

$$\rho = 0.5559$$

$$N = 3$$

$$\Omega = -0.00710$$

$$\epsilon = 0.02$$



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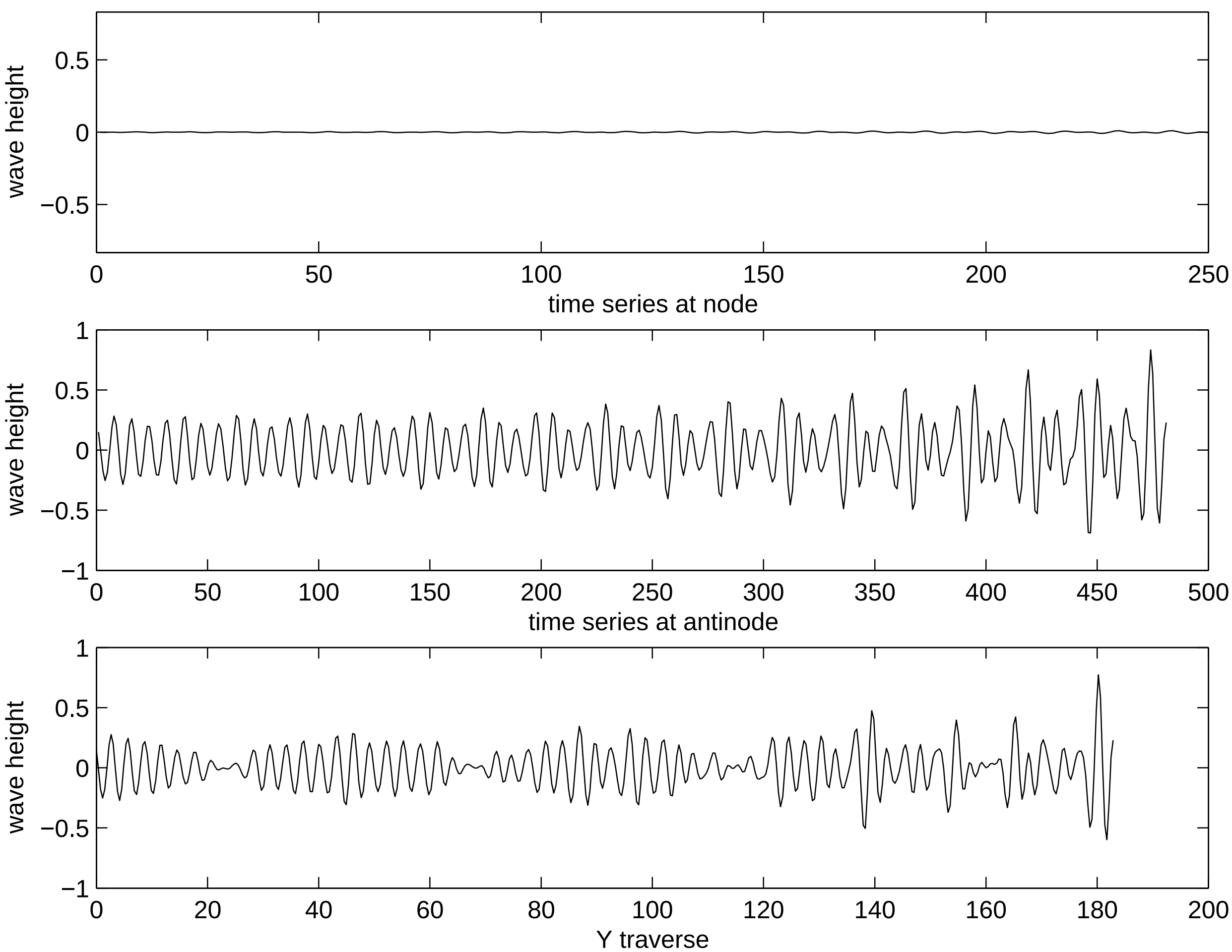
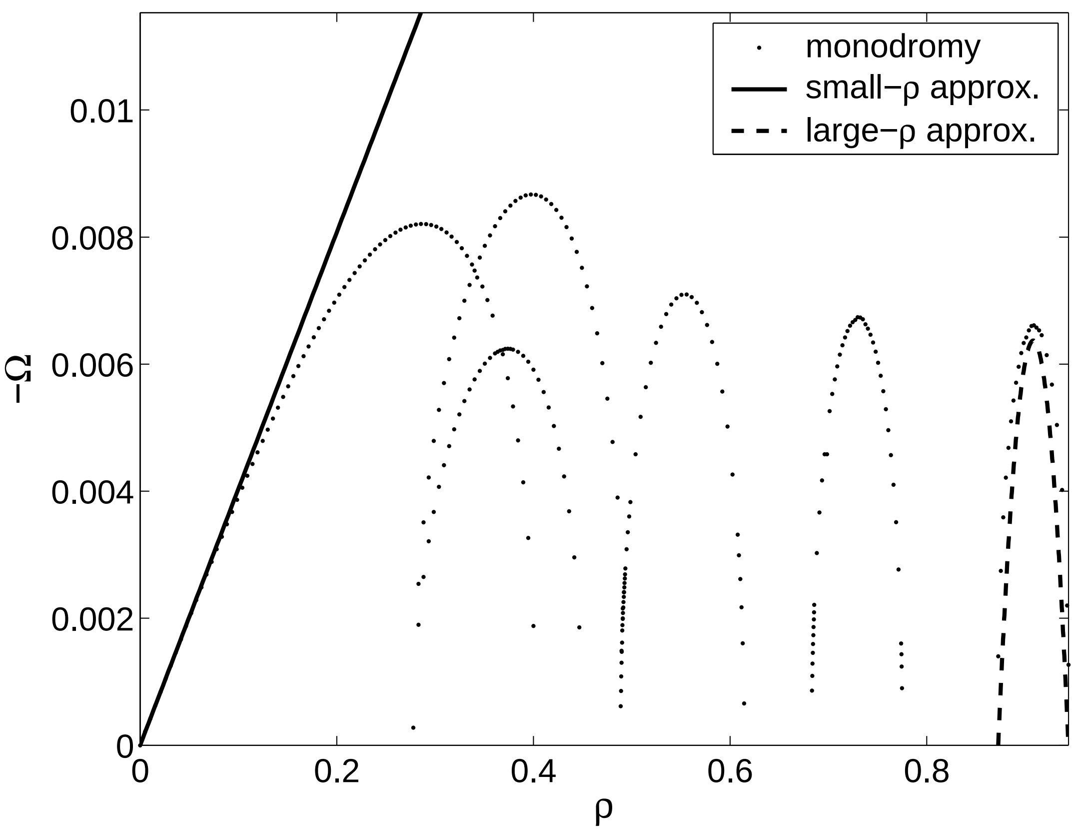


General-ρ Results

Monodromy (Floquet theory) can be used to find periodic solutions of * corresponding to negative Ω for arbitrary values of ρ.



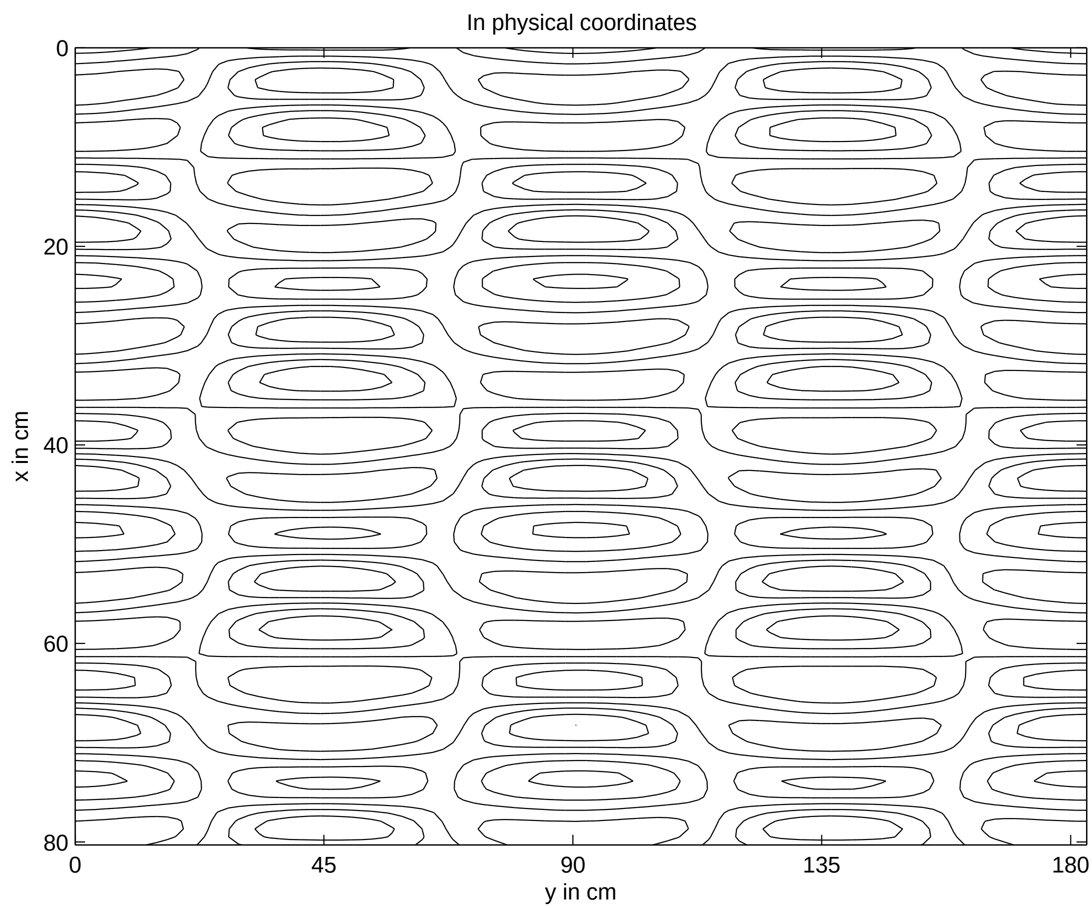
Even Mode

A solution perturbed by the "even" mode.

$$\rho = 0.3975$$

$$\Omega = -0.008673$$

$$\epsilon = 0.05$$



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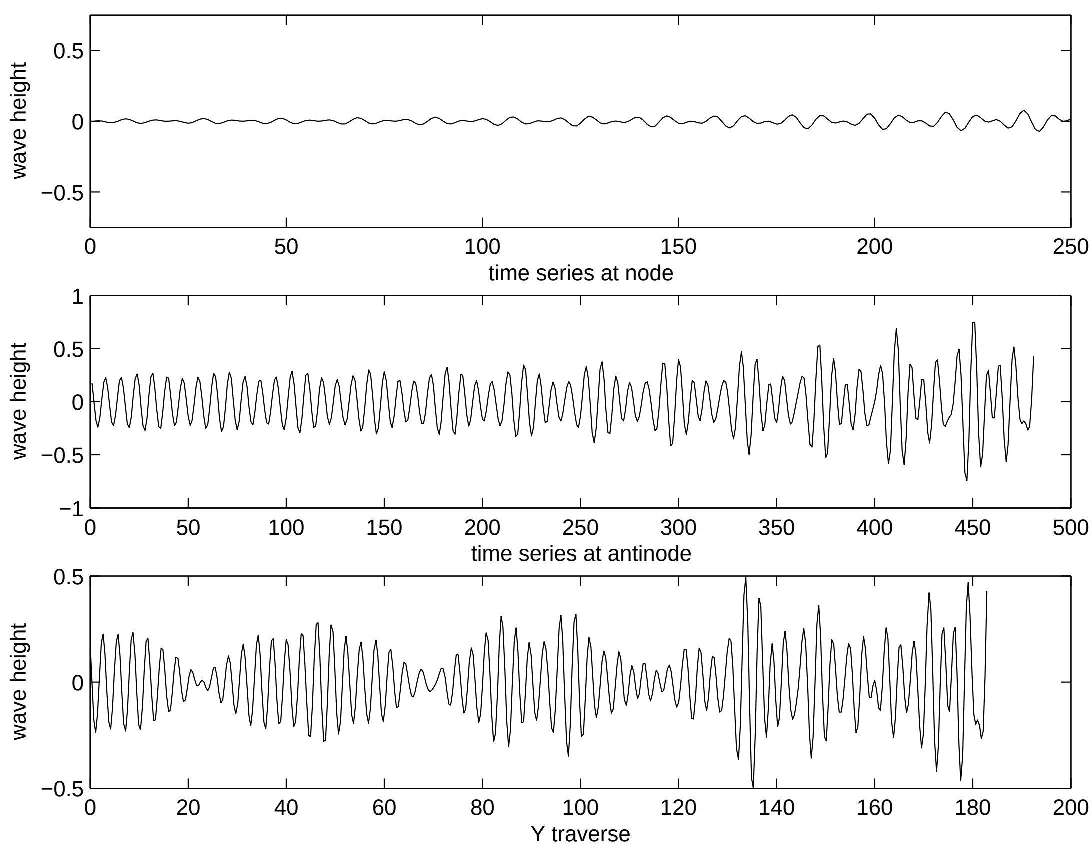

Mathematical Features Observed

Description	Instability
Connecting leg between cells	Even, Large-ρ
Oscillations in nodal region	Even, Large-ρ
Dips in crestlines	Even, Large-ρ
Time varying width of nodal region	All