Annuitization and Asset Allocation with HARA Utility

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A speculator is a man who, if he dies at the right time, has a rich widow.

Christina Stead, *House of All Nations*.

Outline: It's all in the Timing

- Annuitization puzzle
- Option to delay: CRRA case
- Option to delay: HARA case
- Numerical examples

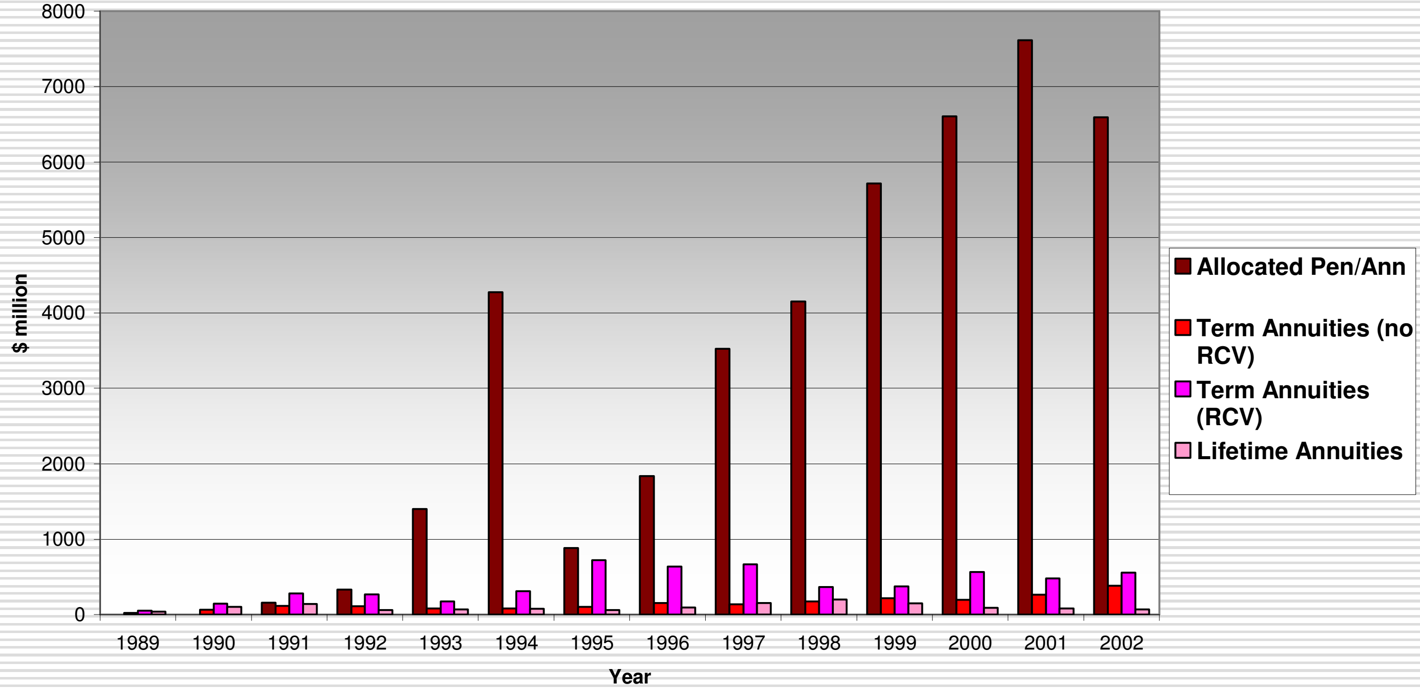
Annuitization Puzzle

- Theory (Yaari 1965): life annuities are valuable
- Observation: Voluntary annuitization is weak.
- Why?
  - bequests, family support
  - shocks to health
  - government social security provision
  - provider loadings
  - adverse selection

Australian income streams

- Age Pension
  - Universal
  - Means-tested
- Life and life-expectancy annuities
  - Voluntary
  - Tax-preferred
  - Fixed income
- Phased withdrawal ‘Allocated pensions’
  - Voluntary
  - Commutable
  - Variable income

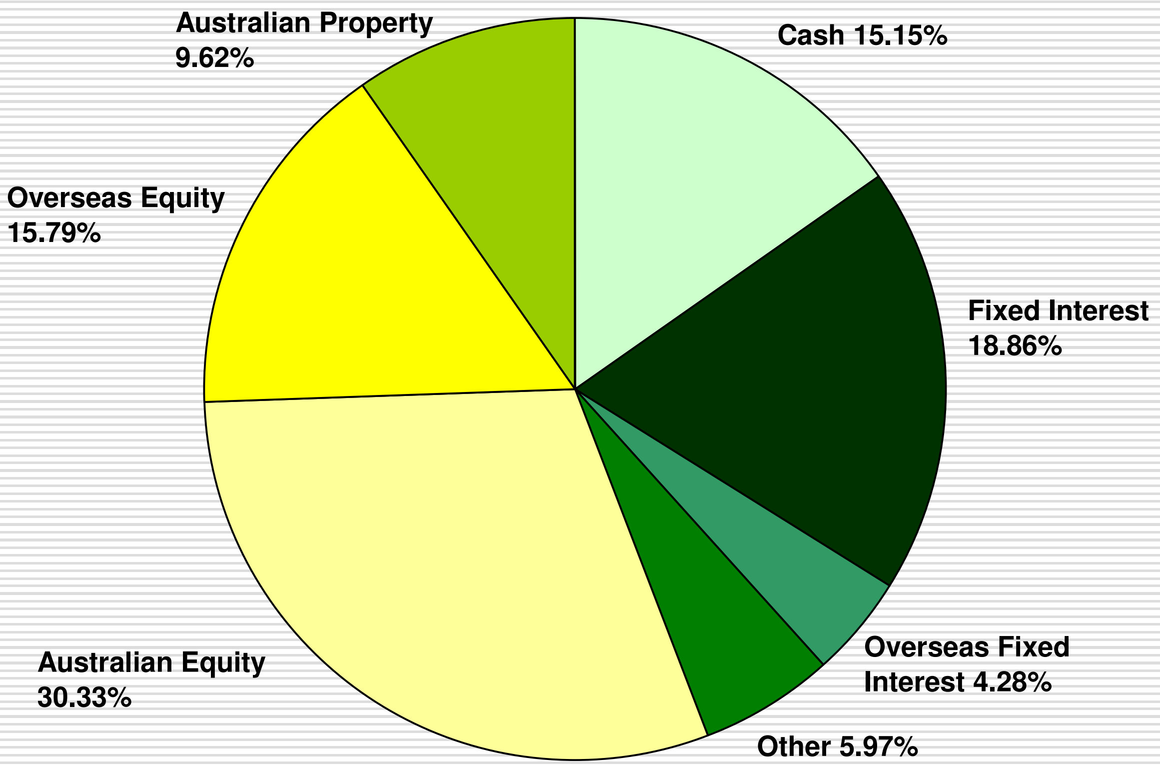
Pensions and Annuities
Eligible Termination Payment Sales
1989-2002



Source: Plan For Life Research

Allocated Pension Funds by asset class

31 December 2002



Source: Plan For Life Research

Life, as it is called, is for most of us one long postponement.

Henry Miller, *The Wisdom of the Heart.*

Optimal Inertia

- irreversibility
- uncertain environment
- opportunity to delay

Asset allocation and annuitization with CRRA preferences

(Milevsky and Young 2002, 2003)

- Merton 1969 + uncertain lifetimes + annuitization
- delaying annuitization usually pays
- diverging survival probabilities means more delay

Why HARA utility?

- Policy and planning practice
- Consumption habits
- Portfolio insurance

The mathematician lives long and lives young; the wings of his soul do not early drop off, nor do its pores become clogged with the earthy particles blown from the dusty highways of vulgar life.

James Joseph Sylvester

The life so short, the craft so long to learn.

Hippocrates

Asset choice

- Risky asset:

$$dS(t) = \alpha S(t)dt + \sigma S(t)dz$$

- Risk-free asset:

$$dB(t) = rB(t)dt$$

- Actuarial present value of life annuity:

$$\overline{a}_x = \int_0^\infty e^{-rt}\, {}_t p_x dt$$

$${}_t p_x = e^{-\int_0^t \lambda_{x+s} ds}$$

After annuitizing

- Post-annuitization consumption stream:

$$C_T = \frac{W_T}{\bar{a}^o{}_{x+T}}$$

- Boundary condition:

$$V(W,T,T) = \bar{a}^b_{x+T} \frac{\left(\frac{W}{\bar{a}^o_{x+T}} - \hat{C} \right)^{1-\gamma}}{1-\gamma}$$

Before Annuitization

- Problem at time zero:

$$\max_{C_t,\Pi_t,T} E\left[\int_0^T e^{-rt}({}_t p_x^b)\frac{(C_t-\hat{C})^{1-\gamma}}{1-\gamma}dt+V(W,0,T)\right]$$

- Subject to:

$$dW_t=[rW_t+(\alpha-r)\Pi_t-C_t]dt+\sigma\Pi_t dz_t$$

Change of variable

- Value function given T:

$$V(w,t;T) \equiv \max_{C_s,\Pi_s} E_t[\int_t^T e^{-r(s-t)} {}_{s-t}p_{x+t}^b \frac{(C_s-\hat{C})^{1-\gamma}}{1-\gamma} ds + e^{-r(T-t)} {}_{T-t}p_{x+t}^b \overline{a}_{x+T}^b \frac{\left(\frac{W_T}{\overline{a}_{x+T}^o}-\hat{C}\right)^{1-\gamma}}{1-\gamma} \mid W_t = w]$$

- Escrow fund:

$$\hat{W}_t \equiv \frac{\hat{C}}{r}(1-e^{r(t-T)}) + \hat{C}\,\overline{a}_{x+T}^o e^{r(t-T)}$$

$$\tilde{W} \equiv W - \hat{W}, \qquad \tilde{C} \equiv C - \hat{C}$$

- Wealth constraint:

$$d\tilde{w}_s = [r\tilde{w}_s + (\alpha - r)\Pi_s - \tilde{C}_s]ds + \sigma\Pi_s dz_s$$

Solution

- HJB Equation

$$(r+\lambda_{x+t}^{b})V = V_t + \max_{\Pi}[\frac{1}{2}\sigma^2\Pi^2 V_{\tilde{w}\tilde{w}} + (\alpha - r)\Pi V_{\tilde{w}}] + r\tilde{w}V_{\tilde{w}} + \max_{\tilde{C}\geq 0}(-\tilde{C}V_{\tilde{W}} + \frac{\tilde{C}^{1-\gamma}}{1-\gamma})$$

- Boundary condition

$$V(\tilde{w},T,T) = \overline{a}_{x+T}^{b}\frac{1}{1-\gamma}(\frac{\tilde{w}}{\overline{a}_{x+T}^{o}})^{1-\gamma}$$

- Solution

$$V(\tilde{w},t;T) = \frac{1}{1-\gamma}\tilde{w}^{1-\gamma}.k^{-\gamma}(t)$$

Optimal annuitization date

- At T=t, and b=o, CRRA

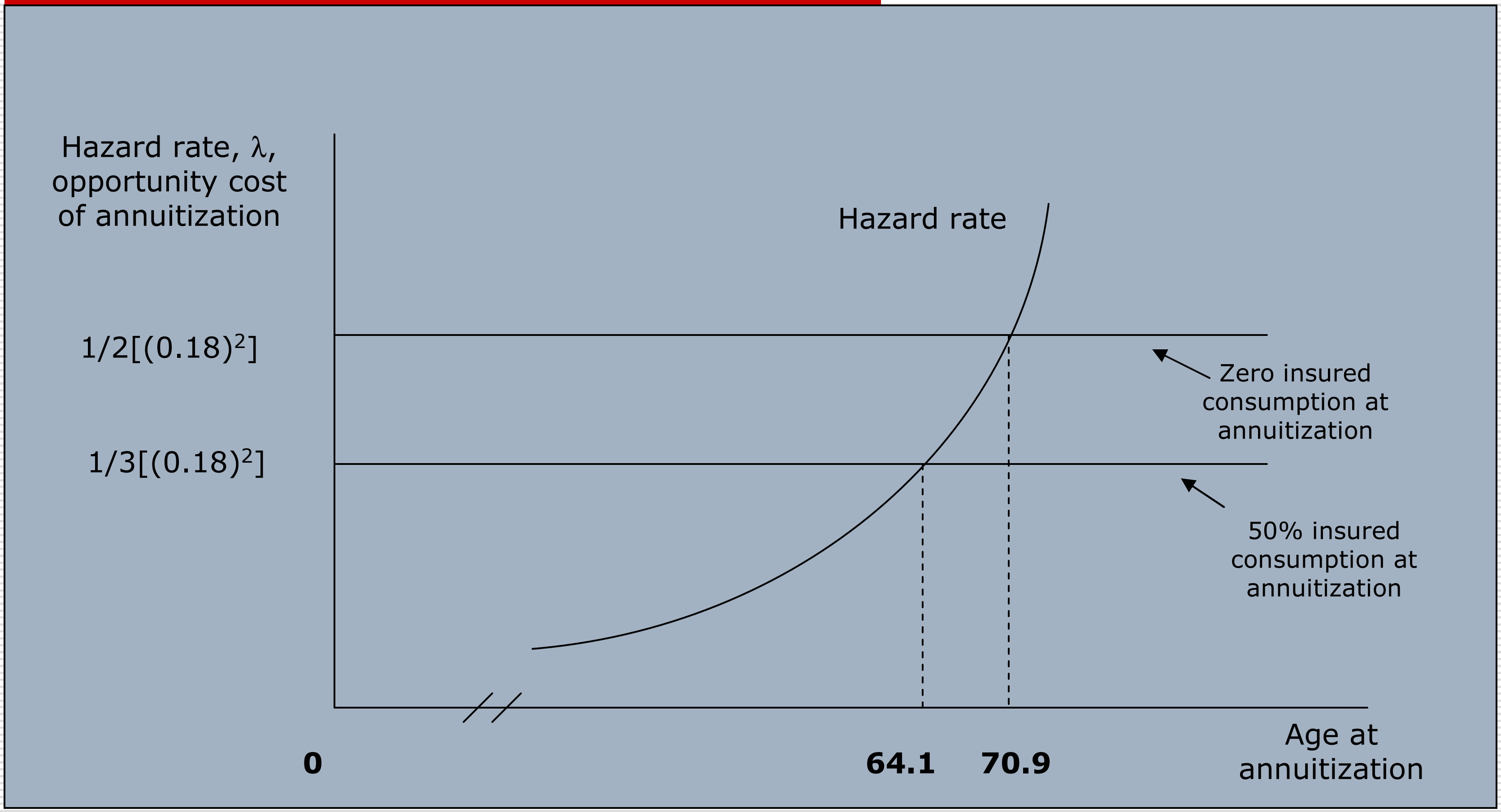
$$\left.\frac{\partial V}{\partial T}\right|_{t=T}^{b=o} \propto \delta - (r + \lambda_{x+T})$$

- HARA

$$\left.\frac{\partial V}{\partial T}\right|_{t=T}^{b=o} \propto \delta - (r + \lambda_{x+T}[1 + \frac{\hat{W}_T}{\tilde{W}_T}])$$

$$\delta = r + \frac{(\alpha - r)^2}{2\sigma^2\gamma}$$

Optimal age at annuitization with and without a consumption floor



Approximate Optimal Age at Annuitization

Male (Female)

	RRA	*Sharpe ratio*	
$\gamma=1$		**0.18**	**0.30**
Zero floor	1	70.9 (76.1)	80.9 (84.6)
50% floor	2	64.1 (70.3)	74.1 (78.8)
$\gamma=2$			
Zero floor	2	64.1 (70.3)	74.1 (78.8)
50% floor	4	57.4 (64.5)	67.4 (73.0)

Conclusions

- Protecting floor consumption brings forward annuitization
- Extensions:
  - Numerical estimation of pre-annuitization paths?
  - Partial annuitization?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