

Corollary:

$$1. \quad K_{\mathbb{Z}}(A_{\mathbb{G}, \rho}) \cong \mathbb{Z}^{s \cdot 2^{n-1}}$$

2. When ρ is nonrational (equivalently, ρ has infinite order),

$A_{\mathbb{G}, \rho}$ has cancellation property for projections;

the order on projections is determined by traces:

$$\tau(p) < \tau(q) \text{ for all tracial states } \tau,$$

$$\Rightarrow p \leq q.$$

(Riffel).

$A_{G, \rho}$ is the section algebra
of ~~the~~ continuous field of
 C^* -algebras over $\text{Spec}(\text{Center of}$
 $A_{G, \rho})$ with fibre algebras

$A_{G', \rho'}$. Here

$$G' = G / Z(G, \rho)$$

$$Z(G, \rho) = \{g \in G : \rho(g, h) = 1 \text{ for all } h\}$$

ρ factors through (G', ρ') .

Up to Morita equivalence, $A_{G, \rho}$
can be trivialized!

$$A_{G,p}, \quad A_{G,p}^{\infty}$$

When G is torsion-free \rightarrow
noncommutative torus.

1. G_{tor} is cyclic in Phillips' work on simple noncommutative tori.
2. Every $A_{G,p}$ is a quotient of some noncommutative torus.
The simple quotients of NC torus are all of the form $A_{G,p}$.
3. For nonrational NC torus,
every proj. f.g. module is a direct sum of modules whose endomorphism algebra is of the form $A_{G,p}$. (Rieffel)

$C^*(G, \sigma)$: the universal C^* -algebra
generated by unitaries U_g satisfying
 $U_g \cdot U_h = \sigma(g, h) U_{gh} = \dots U_h U_g$

$C_{\text{red}}^*(G, \sigma)$: the image for the
left regular rep. on $\ell^2(G)$.

If G is finitely generated and
of polynomial growth, consider
the smooth algebra

$$H^\infty(G, \sigma) = \left\{ \sum_{g \in G} a_g U_g : a_g \in \mathbb{C}, \right. \\ \left. g \mapsto a_g \cdot (1 + \ell(g))^k \text{ is in } \ell^2(G) \text{ for} \right. \\ \left. \text{all } k \in \mathbb{N} \right\} \subseteq C_{\text{red}}^*(G, \sigma).$$

ℓ : word length $\rightarrow k$.

Rieffel, Schwarz, Li
 $A_{G, \rho}^{\infty} \stackrel{\text{C.M.E.}}{\cong} A_{\theta'}^{\infty} \Leftrightarrow \theta = g(\theta')$ for some $g \in \text{Aut}(H, \tau)$

In particular, every noncommutative torus is C.M.E. to one as the tensor product of an ordinary torus and a ^{simple} noncommutative torus.

$A_{G, \rho}^{\infty} \stackrel{\text{C.M.E.}}{\cong} A_{H, \omega}^{\infty}$ Complete Morita equivalence (Schwarz)

$A_{G, \rho}^{\infty} \overset{E}{\overset{A_{H, \omega}^{\infty}}{\text{}}} a \overset{\text{Hermitian}}{\text{Hilbert}} \text{equivalence bimodule.}$

(E has inner products valued in each algebras, satisfying certain conditions)

$A_{G, \rho}^{\infty} \overset{\bar{E}}{\overset{A_{H, \omega}^{\infty}}{\text{}}} a \text{ Hilbert equivalence bimodule.}$

Theorem: For any (G, ρ) , ($n = \text{rank}(G)$) there exists $(\mathbb{Z}^m \times \mathbb{Z}^k, \omega)$ s.t.

1. $A_{G, \rho} \xrightarrow{\text{S.M.E.}} \underbrace{A_{\mathbb{Z}^m \times \mathbb{Z}^k, \omega} \oplus \dots \oplus A_{\mathbb{Z}^m \times \mathbb{Z}^k, \omega}}_s$

2. $n = m+k, \quad s=1 \quad G = \mathbb{Z}_n \times \mathbb{Z}_{n_1} \times \dots \times \mathbb{Z}_{n_r}$

3. $s = |G_{\text{tor}} \cap Z(G, \rho)|$, where (\cdot, \cdot)

$$Z(G, \rho) = \{g \in G : \rho(g, h) = 1 \text{ for all } h \in G\}.$$

4. $\omega((g_1, h_1), (g_2, h_2)) = \omega(h_1, h_2)$.

(In matrix: $\begin{pmatrix} 0 & 0 \\ 0 & \times \end{pmatrix})$

5. $\omega|_{\mathbb{Z}^k \times \mathbb{Z}^k}$ is nondegenerate: $Z(\mathbb{Z}^k, \omega|_{\mathbb{Z}^k, \mathbb{Z}^k}) = 0$.

3. $HP^i(A_{G,p}^\infty) \cong \mathbb{C}^{2^{n-1}}$ for $s=1$.

(algebraic Morita equivalence
is always topological
+ Nest)

4. Simple $A_{G,p}$ is AT algebra.
(Phillips)

Theorem: (Elliott, Li)

$$A_{G,p} \xrightarrow{\text{M.E.}} A_{G',p'}$$

iff they have the same ordered
 K_0 -groups and centers.

(Phillips for simple Nc tori)

~~$A_{G,p} \cong A_{G',p'}$~~ / $K_0(A_0) \cong K_0(A_0')$
ordered

a linear isomorphism $\varphi: \text{Lie}(H) \rightarrow \text{Lie}(G)$
 a connection ∇ .

($\nabla_x: E \rightarrow E$ linear map. $x \in \text{Lie}(H)$)

$$\nabla_x(\pm b) = \nabla_x(\pm) \cdot b + \pm \cdot \nabla_x(b).$$

$$\nabla_x(at) = a \cdot \nabla_x(t) + \nabla_{\varphi(x)}(a) \cdot t.$$

$a \in A_{G,p}^\infty$ $t \in E$, $b \in A_{H,w}^\infty$

∇ has constant curvature:

$$[\nabla_x, \nabla_y] \in \mathbb{C}.$$

$$Z(G, p) \xrightarrow{\text{isom}} Z(H, w)$$

\hat{G} acts on $A_{G,p}$:

$$\alpha_\lambda(\mathcal{U}g) = \langle \lambda, g \rangle \mathcal{U}g.$$

$A_{G,p}^\infty$ = the smooth vectors of α .

$\text{Lie}(\hat{G})$ acts on $A_{G,p}^\infty$ as derivation.

Take a decomposition $G = G_{\text{tor}} \times \mathbb{Z}^n$.

$$a \in A_{G,p}^\infty \Leftrightarrow a = \sum_{\substack{g \in G_{\text{tor}} \\ h \in \mathbb{Z}^n}} a_{g,h} \cdot \mathcal{U}g,h$$

for each $g \in G_{\text{tor}}$, $h \mapsto a_{g,h}$ is
in $\mathcal{S}(\mathbb{Z}^n)$.

$$\text{Lie}(\hat{G}) = \mathbb{R}^n. \quad \chi \in \mathbb{R}^n.$$

$$\delta_\chi(\mathcal{U}g,h) = 2\pi i \cdot \langle \chi, h \rangle \mathcal{U}g,h.$$

dense Fréchet $*$ -subalgebra,
closed under holomorphic functional
calculus and C^∞ -functional calculus
for self-adjoint elements.

G : a finitely generated abelian
group.

$C^*(G, \sigma)$ and $H^\infty(G, \sigma)$ depend
only on $\rho: G \times G \rightarrow \mathbb{T}$

$$\rho(g, h) = \sigma(g, h) \overline{\sigma(h, g)}.$$

a skew-symmetric bicharacter.

$C^*(G, \sigma)$ is the universal C^* -alg
generated by unitaries U_g satisfying

$$U_g U_h = \rho(g, h) U_h U_g.$$

On the C^* -algebras and smooth algebras generated by projective rep. of finitely generated abelian group

G : a group

H : a Hilbert space

A proj. rep. of G : $G \rightarrow \mathcal{U}(H)$
 $g \mapsto \mathcal{U}_g$

s.t. $G \rightarrow \mathcal{U}(H) \rightarrow \mathcal{U}(H)/\mathcal{T}$ is a group homomorphism.

$$\mathcal{U}_g \mathcal{U}_h = \sigma(g, h) \mathcal{U}_{gh}$$

for some map $\sigma: G \times G \rightarrow \mathcal{T}$.
2-cocycle.