Absorption:

What is it,

Where do we find it,

and,

What is it good for?

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What are extensions?

$Ext(A, B)$

1. Short exact sequences $0 \to B \to C \to A \to 0$

   or

2. Busby maps: $\tau : A \to \frac{\mathcal{M}(B)}{B}$

Terminology:

trivial ..... τ can be lifted to a homomorphism $\hat{\tau} : A \to \mathcal{M}(B)$

invertible ..... τ can be lifted to a linear, completely positive map, $\hat{\tau} : A \to \mathcal{M}(B)$

essential ..... τ has no kernel.

Extensions form a semigroup

$\tau_1 : A \to \frac{\mathcal{M}(B)}{B}$

$\tau_2 : A \to \frac{\mathcal{M}(B)}{B}$

If there is a copy of O_2 in $\mathcal{M}(B)$, we can define a binary operation:

V_1, V_2 generators of O_2.

Define: $\tau_1 + \tau_2 : a \longmapsto V_1 \tau_1(a) V_1^* + V_2 \tau_2(a) V_2^*$

This operation is canonical up to unitary eq^{n}

..... is associative & commutative up to unitary eq^{n}

Can we make the semigroup into a group?

Standard way to do this:

Grothendieck construction

$(\mathbb{N}, \cdot)$

↓ Groth.

$(\mathbb{Q}^+, \cdot)$

Is not the best approach.

Better: BDF, Kasparov.

Recall: the Grothendieck construction involves weakening the equivalence relation on the semi-group.

i.e. $a = b$ in the group if there is some extension c such that $a+c = b+c$ in the semigroup.

c depends on a and b.

Construction of a group from the semigroup.

Summary:

Grothendieck:	$a=b$ in the group if there is <u>some</u> extension c such that $a+c \sim_u b+c$.
Kasparov	$a=b$ in the group if $a+c \sim b+c$ for a single special extension c called Kasparov's trivial absorbing extension.
Kirchberg	For extensions of purely infinite simple algebras, the semigroup is already a group.
BDF (Voiculescu)	Extensions of K by A are already a group.

Key idea: Absorbing extensions.

Def^n: An extension $\tau: A \to \frac{\mathcal{M}(B)}{B}$ is "trivial" if it lifts to a homomorphism $\hat{\tau}: A \to \mathcal{M}(B)$.

(motivated by Stinespring)

It's not clear that the "trivial" extension is unique, or trivial in the group.

Def^n: An extension τ_1 is absorbing if $\tau_1 + \tau_2 \sim_u \tau_1$ for any "trivial" extension τ_2. (τ_2 trivial)

i.e. An extension is absorbing if "trivial" extensions are algebraically trivial with respect to it.

Results about absorbing extensions usually have interesting consequences for the existence of groups of extensions.

Summary

When working with Ext, we have a choice: use ordinary extensions, and a very coarse equivalence relation,

OR

use absorbing extensions, and a nicer equivalence relation.

(E.K. 1998)

$[\tau]$: $0 \to B \to C \to A \to 0$

TFAE: i) $[\tau]$ absorbs all trivial extensions

ii) $\overline{cBc}$ contains a stable full subalgebra

iii) $h \in B_1$ can be approximated, given $c \in C_1^+$, $\|r\| \leq 1$ by rcr^*.

Corollary: (corona factorization.) If, for any full positive element of the multipliers, $rcr^* = 1$, then every full extension is absorbing.

Remark: B is stable, so $rcr^* = 1_{M(B)}$ for full multiplier elements is equivalent to $rcr^* = 1_{Q(B)}$ for full corona elements.

Why? If we have this in the corona, then, in the multipliers, $rcr^* = 1_{M(B)} + b$.

Now, cut down by an isometry coming from stability: $v^* r c r^* v \approx 1_{M(B)}$.

(K., 2004)

$[\tau]: \quad 0 \to B \to C \to A \to 0$

Assume: Every hereditary kernel in A contains an strictly positive element with connected spectrum.

Factorizability: Let $h \in \overline{BC}$ be positive, and let $\mathrm{spc} = [0,1]$. Then, h can be approximated by rcr^* $\|r\| \leq 1$.

(In the ideal-related case factorizability implies IR · absorption.)

Specialize to case of ordinary absorption.

Corollary: If, for every full element $c \in m(B)^+$ with spectrum $[0,1]$ we have an r with $rcr^* = 1$, $\|r\| \leq 1$, then every full extension is absorbing.

(As before, has equivalent corona formulation.)

What extensions are absorbing?

$0 \to K \to E \to A \to 0$ [Voiculescu]

$0 \to K \otimes C(X) \to E \to A \to 0$ [PPV]

$0 \to$ purely infinite $\to E \to A \to 0$ [Kirchberg]

$0 \to$ type I $\to E \to A \to 0$ [K & N]

(∞-dim'l reps, $\dim B < \infty$).

Applications?

Suppose B is in one of the above classes.

Then, approximate unitary equivalence classes in $\mathcal{M}(B \otimes K)$ are given by a spectral condition:

if $sp\, T_1 = sp\, T_2$ and $sp_e\, T_1 = sp_e\, T_2$

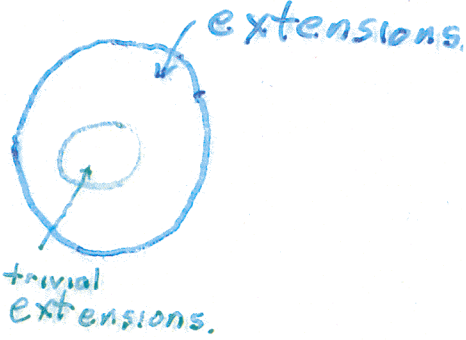
then $T_1 = U T_2 U^* + b$ for some $b \in B$.

Theorem: A separable unital subalgebra of the classic Calkin algebra B/K is equal to its own double relative commutant. [Voiculescu]

By working harder we prove, in the corona of one of these algebras, that the double relative commutant of a hereditary nonunital σ-unital algebra $H \subseteq Q$ is equal to the unitization of H.

EKdS

Summary



Weyl & von Neumann (1930)

Voiculescu (1976)

$\phi: A \to B(H)$

$\phi(A \cap K) = 0$

$Ext(A, K)$

All trivial extensions disappear

Kasparov (1980)

$\phi: A \to M(B \otimes K)$

$\phi(A \cap B \otimes K) = 0$

$A \subset M(B \otimes K)$

$Ext(A, B \otimes K)$

There is a universal trivial extension.

Only have to consider one trivial extension.

Kirchberg (1995)

$\phi: A \to M(B)$

B purely infinite, simple.

$A \subset M(B)$

$Ext(A, B \otimes K)$

B purely infinite, simple.

All trivial extensions disappear.

GE and DK (1998)

$\phi: A \to M(B)$

$\phi(A \cap B) = 0$

$\overline{AB} = B$

A purely large over B.

Gives an if and only if characterization of absorbing extensions

$Ext(A,B)$?

Classification?