

# From Transverse Index Theory To Hopf-cyclic cohomology via the Local Index Formula

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## Refs

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# How it all began?

----- Sometime in 1997, A. Connes and H. Moscovici decided to apply their local index formula to a transverse index problem for foliations.

$$(V, F) \rightsquigarrow (A, H, D) \rightsquigarrow ch(A, H, D) \in HC^*(A)$$

$D \in H_n$  (Hopf) algebra of transverse diff.  
oprs. on  $A$

natural action  $H_n \times A \rightarrow A$

$$\chi: H_n^{\otimes k} \rightarrow \underset{\mathbb{C}}{\text{Hom}}(A^{\otimes (k+1)}, \mathbb{C}) \quad \forall k \geq 0$$

$$\chi(h_1 \otimes \dots \otimes h_k)(a_0, a_1, \dots, a_k) =$$

$$\sim (a_0 h_1(a_1) \dots h_k(a_k))$$

$$ch(A, H, D) \in \text{Im } \chi$$

C.-M. problem: promote  $\{H_n^{\otimes k}\}_{k \geq 0}$  into a cyclic complex (module) s.t.  $\chi$  becomes

$$\chi : HC^*(H_n) \rightarrow HC^*(A)$$

Compute:

$$HC^*(H_n) \simeq \text{[redacted]} H_{Lie}^*(\alpha_n)$$

Thus obtain characteristic classes of foliations as a classical object (Gelfand-Fuchs cocycles)

some details

$(V, F)$  codim  $n$  foliation

$M = \bigcup_i U_i \times \{i\}$  complete transversal

$\Gamma$  = holonomy pseudo group (local diffeos of  $M$ )

$F(M)$  = frame bundle of  $M$  (since no inv. metric on  $M$ )

lift  $\Gamma$ -action to  $F M$  and form the smooth étale groupoid

$F(M) \rtimes \Gamma$  translation

$A = A_{FM} = C_c^\infty(F \rtimes \Gamma)$  groupoid

linear comb's. of  $f|U_\varphi^*$   $f \in C_c^\infty(\text{Dom } \tilde{\varphi})$

$$f_1|U_{\varphi_1}^* \cdot f_2|U_{\varphi_2}^* = f_1 \cdot (f_2 \circ \tilde{\varphi}_1)|U_{\varphi_2 \varphi_1}^*$$

different transversals  $\rightsquigarrow$  Morita equivalent algebras  $A_{FM}$

choose: flat transversals,  $M$  is flat

$M = \mathbb{R}^n$ , w flat connection on  
 $F(\mathbb{R}^n)$

$H = L^2(F(M), \text{vol}_{F(M)})$  canonical

$\text{vol}_{F(M)} = \bar{g}^{ij} dy^i dx^j$  Diff<sup>+</sup>-inv.

volume form

Action of  $A_{FM}$  on  $H$ :

$$((f \circ \varphi^*)(\beta))(p) = f(p)\beta(\varphi(p))$$

operator  $D$  defined by  $D[D] = Q$ ,

$Q$  is a "signature type" operator.

we are actually on:  $P(M) = F(M)/SO(n)$

Sections of  $P(M)$  = Riemannian metrics on  $M$

$$Q = (d_V^* d_V - d_V d_V^*) \oplus (d_H + d_H^*)$$

Connes-Moscovici local index formula  
applied to  $(A, \mathcal{H}, D)$

$$\varphi_n(a^0, \dots, a^n) =$$

$$\sum_K c_{n,K} \{ a^0 [D, a']^{(K_1)} \cdots [D, a^n]^{(K_n)} \}_{D=1}^{-n-2(K_1)}$$

$$T^{(K)} = \nabla^{(T)}, \quad \nabla^{(T)} = D^2 T - T D^2$$

Connes-Moscovici Hopf algebra  $\mathcal{H}_n$  = alg. of  
"diff. oprs" of

$$A_{FM} = A_n = C_c^\infty(F\mathbb{R}^n) \rtimes \text{Diff}^+(\mathbb{R}^n)$$

$$\mathcal{H}_n \subset \text{End}_{\mathbb{C}}(A_n)$$

generated by

- vertical vector fields  $\{y_i^j; i, j = 1, \dots, n\}$

corresponding to a basis of  $gl(n, \mathbb{R})$

$$y_i^j (f u_\varphi^*) = (y_i^j f) u_\varphi^*$$

Since actions of  $\Gamma$  and  $GL^+(n, \mathbb{R})$  on  $FM$   
commute  $\Rightarrow$

$$y_i^j(ab) = y_i^j(a)b + a y_i^j(b)$$

$$\forall a, b \in A_n$$

- horizontal vector fields  $X_k$ ,  $k=1, \dots, n$

$$X_k(fU_\varphi^*) = X_k(f)U_\varphi^*$$

$$X_i(ab) = X_i(a)b + aX_i(b) + \sum_{j,k} S_{ij}^k(a) Y_k^j(b)$$

$$S_{ij}^k(fU_\varphi^*) = \gamma_{ij}^k fU_\varphi^*, \quad \forall a, b \in A_n$$

$\gamma_{ij}^k \in C^\infty(F\backslash X, \Gamma)$  defined by

$$\varphi^* \omega_j^i - \omega_j^i = \sum \gamma_{jk}^i \theta^k$$

$\omega$  = flat connection form

$\theta$  = fundamental form on  $F\mathbb{R}^n$

$$S_{ij}^k(ab) = S_{ij}^k(a)b + aS_{ij}^k(b)$$

$$\text{Let } S_{ab, i_1, \dots, i_r}^c = [x_{i_r} \cdots [x_{i_1} - S_{ab}^i] \cdots] \quad r \geq 1$$

These are multiplication operators of the form

$$T(fU_\varphi^*) = h f U_\varphi^*, \quad h \in C^\infty(F\mathbb{R}^n \backslash X, \Gamma), \text{ hence they commute.}$$

Let

$$h_n = \sum c_i y_i^i \oplus \sum c_j x_j \oplus \sum_{ab, i, \dots, j} c_{ij} s_{ab}^i$$

Obviously, is a Lie algebra. Let

$$H_n = U(h_n)$$

algebra of transverse differential op's with constant coefficients.

Natural action  $H_n \otimes A_n \rightarrow A_n$

satisfies a Leibniz rule:

$$h(ab) = \sum h^{(1)}(a) h^{(2)}(b)$$

defines a coproduct  $\forall a, b \in A_n$

$$\Delta: H_n \rightarrow H_n \otimes H_n$$

$$\Delta(h) = \sum h^{(1)} \otimes h^{(2)}$$

$$\Delta y_i^i = y_i^i \otimes 1 + 1 \otimes x_i^i$$

$$\Delta x_i = x_i \otimes 1 + 1 \otimes x_i + \sum \delta_{ji}^k \otimes y_k^i$$

$$\Delta \delta_{ij}^k = \delta_{ij}^k \otimes 1 + 1 \otimes \delta_{ij}^k$$

+  $\Delta$  is an algebra map

Antipode of  $H_n$

$$S(y_i^j) = -y_i^j$$

$$S(x_k) = -x_k + \delta_{kb}^c y_c^b$$

$$S(\delta_{ab}^c) = -\delta_{ab}^c$$

$S^2 \neq id$ , but  $\exists$  a character

$\delta: H_n \rightarrow \mathbb{C}$  s.t.  $S_\delta^2 = id$ , with

$$S_\delta = \delta * S \quad \text{twisted antipode}$$

$$S_\delta(h) = \sum S(h^{(1)}) S(h^{(2)})$$

where  $\Delta(h) = \sum h^{(1)} \otimes h^{(2)}$ .

$H_1$  by generators and relations

$$H_1 = \mathcal{U}\{x, y, \delta_n; n \geq 1\}$$

$$[y, x] = X, [y, \delta_n] = n\delta_n, [x, \delta_n] = \delta_{n+1},$$

$$[\delta_k, \delta_l] = 0$$

$$\Delta y = y \otimes 1 + 1 \otimes y, \quad \Delta \delta_n = \delta_n \otimes 1 + 1 \otimes \delta_n,$$

$$\Delta x = x \otimes 1 + 1 \otimes x + \delta_n \otimes y$$

$$s(y) = -y, \quad s(x) = -x + y, \quad s(\delta) = -\delta$$

Structure of  $H_1$  bicrossed product decomposition

$$H_1 = \mathcal{U}(\text{Lie}(G_1)) \bowtie \mathcal{U}(\text{Lie}(G_2))^*$$

Coming from

$$\text{Diff}(\mathbb{R}) = G_1 \cdot G_2$$

$G_1$  = "ax+b" - group

$$G_2 = \{\varphi \in \text{Diff}(\mathbb{R}); \varphi(0) = 0, \varphi'(0) = 1\}$$

modular character of "ax+b" - group  $\Rightarrow$

$$\delta: \mathcal{U}(\text{Lie}(G_1)) \rightarrow \mathbb{R}$$

Unique extension to a character

$$S: H \rightarrow \mathbb{R}$$

$$(S(\delta_n) = 0, \text{ since } n\delta_n = [y, \delta_n])$$

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- trivial check:  $\tilde{S}^2 = id$

$$\tilde{S}(x) = (S * S)(x) = -x + y$$

$$\tilde{S}\tilde{S}(x) = \tilde{S}(-x + y) = x - y + y = x$$

$\tilde{S}$   $\Rightarrow$   $S$ -twisted antipode

- why they are important?

A trace on  $A_1 = C_c^\infty(FM) \rtimes \Gamma$

$$\tau(fU_\varphi^*) = \begin{cases} \int_F vol & \varphi = I \\ 0 & \text{otherwise} \end{cases}$$

$\tau$  is "invariant" under  $H_1$ -action, modulo the character  $S$

## Formal definitions

- $H$ -algebra ( $H$ -module algebra in Hopf lingo)

$$H \otimes A \rightarrow A$$

$$h_1(h_2a) = (h_1h_2)(a)$$

$$h(ab) = \sum h^{(1)}(a) h^{(2)}(b)$$

$$h(1) = \varepsilon(h)$$

$$\bullet H = \mathfrak{U} \oplus \mathfrak{g}$$

$\mathfrak{U} \oplus \mathfrak{g}$ -algebra  $\Leftrightarrow \mathfrak{g}$  acts by derivations on  $A$

$$\Delta x = x, \otimes 1 + 1 \otimes x$$

$$\therefore x(ah) = x(a)b + a x(b)$$

$$\bullet H = \mathfrak{C} \oplus \mathfrak{G}$$

$\mathfrak{C} \oplus \mathfrak{G}$ -algebra  $\Leftrightarrow \mathfrak{G}$  acts by automorphisms on  $A$

$$\Delta g = g \otimes g$$

$$\therefore g(ah) = g(a)g(b)$$

## Invariant Trace

$$\tau: A \rightarrow \mathbb{C} \quad \text{Trace} +$$

$$\tau(h(a)) = \varepsilon(h)\tau(a)$$

$$\bullet H = \mathfrak{U} \oplus \mathfrak{g} \quad \varepsilon(x) = 0 \quad \checkmark$$

- $\delta$ -invariant Trace

$$\tau(h(a)) = \delta(h) \tau(a)$$

$\delta: H \rightarrow \mathbb{C}$  character

- $\delta$ -invariant  $\sigma$ -Trace

$$\tau(h(a)) = \delta(h) \tau(a)$$

$$\tau(ba) = \tau(\sigma(a)b)$$

$\sigma \in H$  grouplike :  $\Delta^\sigma = \sigma \otimes \sigma$

- Integration by parts formula

$$\tau(h(a)) = \varepsilon(h) \tau(a) \iff$$

$$\langle h(a), b \rangle = \langle a, sh(b) \rangle,$$

with  $\langle a, b \rangle := \tau(ab)$

$$\tau(h(a)) = \delta(h) \tau(a) \iff$$

$$\langle h(a), b \rangle = \langle a, \tilde{s}(h)(b) \rangle,$$

where  $\tilde{s} = \delta * s$  Twisted antipode

$$\tilde{s}(h) = \sum s(h^{(1)}) s(h^{(2)})$$

- modular pair in involution

$$H, (\delta, \sigma)$$

$$\delta(\sigma) = 1$$

modular condition

$$\tilde{S}^2 = \text{Ad}_{\sigma}$$

involutive property

For C.-M. Hopf algebra  $H$ ,  $\delta$  as before;  $\sigma = 1$

Ribbon & coribbon Hopf algebras as well as compact quantum groups have plenty of modular pairs in involution (Connes-Moscovici)

Connes-Moscovici problem (abstract version)

Given  $H, (\delta, \sigma)$  as above, define a cocyclic module structure on  $\{H^{\otimes n}\}_{n \geq 0}$  s.t. for any  $H$ -algebra  $A$ , the map

$$\chi: H^{\otimes n} \xrightarrow{\cong} \text{Hom}(A^{\otimes(n+1)}, \mathbb{C})$$

$$\chi(h_1 \otimes \dots \otimes h_n)(a^0, \dots, a^n) = \tau(a_0^0 h_1(a^1) \dots h_n(a^n))$$

defines a morphism of cocyclic modules

$$\{H^{\otimes n}\}_{n \geq 0} \longrightarrow \{\text{Hom}(A^{\otimes(n+1)}, \mathbb{C})\}_{n \geq 0}$$

and hence a map between cyclic cohomology groups

$$HC_{(\delta, \tau)}^*(H) \longrightarrow HC^*(A)$$

( $\tau$  is a  $\delta$ -invariant  $\sigma$ -trace on  $A$ ;  
 $x$  depends on  $\tau$ )

Example (Connes 1980)

$$g \rightarrow \text{Der}(A)$$

$\tau: A \rightarrow \mathbb{C}$  invariant trace ( $\tau(x(a)) = 0$

$\forall x, a) \exists$

$$\chi: H_*^{\text{Lie}}(g) \rightarrow HC^*(A)$$

$$x_1 \wedge \dots \wedge x_n \mapsto \phi$$

$$\phi(\alpha_0, \alpha_1, \dots, \alpha_n) =$$

$$\sum_{\sigma \in S_n} \text{sgn}(\sigma) \tau(\alpha_0 x_{\sigma(1)}(\alpha_1) \dots x_{\sigma(n)}(\alpha_n))$$

(Applied to  $A_\theta$ ,  $\tau$ ,  $g = \mathbb{R}^2$ , gives

$$\varphi_1(\alpha_0, \alpha_1) = \tau(\alpha_0 \delta_1(\alpha_1))$$

$$\varphi'_1(\alpha_0, \alpha_1) = \tau(\alpha_0 \delta_2(\alpha_1))$$

$$\varphi_2(\alpha_0, \alpha_1, \alpha_2) = \tau(\alpha_0 (\delta_1(\alpha_1) \delta_2(\alpha_2) - \delta_2(\alpha_1) \delta_1(\alpha_2)))$$

Connes-Moscovici argument:

For

$$\chi : H^{\otimes n} \rightarrow \underset{\mathbb{C}}{\text{Hom}}(A^{\otimes(n+1)}, \mathbb{C})$$

$$\chi(h_1 \otimes \dots \otimes h_n)(a^0, a^1, \dots, a^n) = \tau(a^0 h_1(a') \dots h_n(a^n))$$

To define a morphism of cyclic modules,

differentials  $H^{\otimes n} \rightarrow H^{\otimes(n+1)}$  must be defined via  $\Delta$

$$h(ab) = \sum h^{(1)}(a) h^{(2)}(b)$$

$$\delta_i(h_1 \otimes \dots \otimes h_n) = h_1 \otimes \dots \otimes \Delta h_i \otimes \dots \otimes h_n.$$

Tricky part: ~cyclic operator  $\tau$  on the LHS.

$$\text{RHS: } \tau(a^0 h_1(a')) \xrightarrow{\tau} \tau(a' h_1(a^0))$$

Integration by part formula

$$\tau(h(a)b) = \tau(a s(h)b)$$

$$\Rightarrow \tau(a^0(s h_1)(a')) = \tau(a' h_1(a^0))$$

∴

$$\boxed{t h_i = s h_i}$$

$n=1$

$n=2$ , RHS

$$\tau(a^0 h_1(a') h_2(a^2)) \xrightarrow{\tau} \tau(a^2 h_1(a^0) h_2(a'))$$

$$= \tau(h_1(a^0) \underbrace{h_2(a')}_{a^2})$$

$$= \tau(a^0 sh_1(h_2(a')) a^2)$$

$$= \sum \tau(a^0 (sh_1^{(1)}(h_2(a'))) (sh_1^{(2)}) a^2)$$

Thus, should define  $\tau$  on LHS by

$$\begin{aligned} t(h_1 \otimes h_2) &= \sum s(h_1)^{(1)} h_2 \otimes s(h_1)^{(2)} \\ &= \boxed{\Delta s(h_1) \cdot (h_2 \otimes 1)} \end{aligned}$$

general case, RHS

$$\tau(a^0 h_1(a') h_2(a^2) \dots h_n(a^n)) \xrightarrow{\tau}$$

$$\tau(a^n h_1(a^0) h_2(a') \dots h_n(a^{n-1}))$$

$$= \tau(h_1(a^0) h_2(a') \dots h_n(a^{n-1}) a^n)$$

$$= \tau(a^0 sh_1(h_2(a') \dots h_n(a^{n-1}) a^n))$$

$$= \sum \tau(a^0 (sh_1^{(1)}(h_2(a')) \dots (sh_1^{(n+1)}) a^{n+1}))$$

Should define  $\tau$  on LHS by

$$t(h_1 \otimes \dots \otimes h_n) = \sum_{n=1}^{\infty} s(h_1)^{(1)} h_2 \otimes s(h_1)^{(2)} h_3 \otimes \dots \otimes s(h_1)^{(n)}$$

This works iff  $S^2 = \text{id}$ . In general, use a modular pair in involution  $(\delta, \sigma)$ , (corresponds to a  $\delta$ -invariant  $\sigma$ -trace), use

$$t(h_1 \otimes \dots \otimes h_n) = \Delta^{n-1} S(h_1) \cdot (h_2 \otimes h_3 \otimes \dots \otimes h_n \otimes 0)$$

**Thm (Connes-Moscovici)** Given  $H$ ,  $(\delta, \sigma)$  as above, the following formulas define a cocyclic module. Furthermore, for any  $H$ -algebra  $A$ , and any  $\delta$ -inv.  $\sigma$ -trace on  $A$ , the map  $\chi$  is a morphism of cocyclic modules.

Remark: the argument leading to a formula for  $t$  is not enough. For  $H_n$  it is enough thanks to injectivity of  $\chi$ .

$$\mathcal{H}_{(\delta, \sigma)}^{\natural, 0} = k, \quad \mathcal{H}_{(\delta, \sigma)}^{\natural, n} = \mathcal{H}^{\otimes n}, \quad n \geq 1$$

The coface, codegeneracy and cyclic operators  $\delta_i, \sigma_i, \tau$  are defined by

$$\begin{aligned}\delta_0(h_1 \otimes \cdots \otimes h_n) &= 1_{\mathcal{H}} \otimes h_1 \otimes \cdots \otimes h_n \\ \delta_i(h_1 \otimes \cdots \otimes h_n) &= h_1 \otimes \cdots \otimes \Delta(h_i) \otimes \cdots \otimes h_n \\ \delta_{n+1}(h_1 \otimes \cdots \otimes h_n) &= h_1 \otimes \cdots \otimes h_n \otimes \sigma \\ \sigma_i(h_1 \otimes \cdots \otimes h_n) &= h_1 \otimes \cdots \otimes \epsilon(h_{i+1}) \otimes \cdots \otimes h_n \\ \tau(h_1 \otimes \cdots \otimes h_n) &= \Delta^{n-1} \tilde{S}(h_1) \cdot (h_2 \otimes \cdots \otimes h_n \\ &\quad \otimes \sigma).\end{aligned}$$

$\delta(\sigma) = 1$

$\tilde{S}^2 = \text{Ad}_\sigma$        $(\delta, \sigma)$  modular pair  
"in involution"

The cohomology groups  $HP_{(\delta,\sigma)}^{\bullet}(\mathcal{H})$  are so far computed for the following Hopf algebras. For quantum universal enveloping algebras no examples are known except for  $U_q(sl_2)$  that we recall below.

1.  $\mathcal{H} = \mathcal{H}_n$  Connes-Moscovici Hopf algebra,

$$HP_{(\delta,1)}^n(\mathcal{H}) \cong \bigoplus_{i=n \pmod{2}} H^i(\mathfrak{a}_{\mathbb{R}}, \mathbb{C})_k$$

where  $\mathfrak{a}_{\mathbb{R}}$  is the Lie algebra of formal vector fields on  $\mathbb{R}^K$ .

2.  $\mathcal{H} = U(\mathfrak{g})$  enveloping algebra of a Lie algebra  $\mathfrak{g}$ ,

$$HP_{(\delta,1)}^n(\mathcal{H}) \cong \bigoplus_{i=n \pmod{2}} H_i(\mathfrak{g}, \mathbb{C}_{\delta})$$

3.  $\mathcal{H} = \mathbb{C}[G]$  the coordinate ring of a nilpotent affine algebraic group  $G$ ,

$$HP_{(\epsilon,1)}^n(\mathcal{H}) \cong \bigoplus_{i=n \pmod{2}} H^i(\mathfrak{g}, \mathbb{C}),$$

1, 2, 3 are proved by C.-M. 10

3 admits a nice generalization:

prop. (M.K. & B. Rangipour). The periodic cyclic cohomology of a commutative Hopf algebra is given by

$$HP_{(E,1)}^n(H) = \bigoplus_{i=n(\text{mod } 2)} H^i(H, k)$$

( $H^i$  = cohomology of the coalgebra  $H$ )

- Apply to  $H = \mathbb{C}[G]$  affine algebraic group

$$HP_{(E,1)}^n(\mathbb{C}[G]) = \bigoplus_{i=n(\text{mod } 2)} H^i(G, \mathbb{C})$$

continuous cohomology of  $G$  based on regular functions on  $G$ . When  $G$  nilpotent

$$\text{Van-Est} \Rightarrow \text{RHS} = \bigoplus_{i=n(\text{mod } 2)} H^i(g, \mathbb{C}).$$

where  $\mathfrak{g} = \text{Lie}(G)$ .

4 If  $\mathcal{H}$  admits a normalized left Haar integral,

$$HP_{(\delta,\sigma)}^1(\mathcal{H}) = 0, \quad HP_{(\delta,\sigma)}^0(\mathcal{H}) = k.$$

$$h_1 \otimes h_2 \otimes \dots \otimes h_n \mapsto (f(h_1)h_2 \otimes \dots \otimes h_n) \leftarrow \text{homotopy}$$

A linear map  $f : \mathcal{H} \rightarrow k$  is called a normalized left Haar integral if for all  $h \in \mathcal{H}$ ,  $f(h) = f(h^{(1)})h^{(2)}$  and  $f(1) = 1$ . It is known that a Hopf algebra defined over a field admits a normalized left Haar integral if and only if it is cosemisimple. Compact quantum groups and group algebras are known to admit normalized Haar integral in the above sense. In the latter case  $f : kG \rightarrow k$  sending  $g \mapsto 0$  for all  $g \neq e$  and  $e \mapsto 1$  is a Haar integral. Note that  $G$  need not to be finite. In this regard, we should also mention that there are interesting examples of finite-dimensional non-cosemisimple Hopf algebras defined as quantum groups at roots of unity. Nothing is known about the cyclic (co)homology of these Hopf algebras.

**5** (Crainic) If  $\mathcal{H} = U_q(sl_2(k))$  is the quantum universal algebra of  $sl_2(k)$ , we have ,

$$HP_{(\epsilon,\sigma)}^0(\mathcal{H}) = 0, \quad HP_{(\epsilon,\sigma)}^1(\mathcal{H}) = k \oplus k.$$

**6** Let  $\mathcal{H}$  be a commutative Hopf algebra. The periodic cyclic cohomology of the cocyclic module  $\mathcal{H}_{(\epsilon,1)}^\natural$  can be computed in terms of the Hochschild homology of coalgebra  $\mathcal{H}$  with trivial coefficients.

### Dual theory: cyclic homology of Hopf algebras

Our cyclic module as a simplicial module is exactly the Hochschild complex of  $\mathcal{H}$  with coefficients in  $k$  where  $k$  is an  $\mathcal{H}$ -bimodule as above. So if we denote our cyclic module by  $\widetilde{\mathcal{H}}_{\natural}^{(\delta,\sigma)}$ , we have  $\widetilde{\mathcal{H}}_{\natural_n}^{(\delta,\sigma)} = \mathcal{H}^{\otimes n}$ , for  $n > 0$  and  $\widetilde{\mathcal{H}}_{\natural_0}^{(\delta,\sigma)} = k$ . Its faces and degeneracies are as follows: