

On the classification of simple
 C^* -algebras which are
inductive limits of
continuous-trace C^* -algebras

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Setting

Definition A C^* -algebra A is a stably (isomorphic) AI algebra if $A \otimes \mathbb{K}$ is an AI algebra.

Remark There are similar definitions for stably AF or AH algebras.

Remark Any AI algebra is also a stably AI algebra.

Proposition *Any stably AF algebra is necessarily an AF algebra.*

Moreover we have the following result:

Proposition Any (simple) stably AI algebra can be realized as a hereditary subalgebra of a (simple) AI algebra.

Proof

Assume A is a simple stably AI algebra. Hence $A \otimes \mathbb{K}$ is a simple AI algebra. A is isomorphic to the cut down:

$$A \cong \tilde{e_{11}}(A \otimes \mathbb{K})\tilde{e_{11}}$$

which is a hereditary subalgebra of the simple AI algebra $A \otimes \mathbb{K}$.

Real Rank of C^* -algebras

Def The Real Rank of a C^* -algebra A is the smallest integer, $RR(A)$, such that for each m -tuple (x_1, x_2, \dots, x_m) of self-adjoint elements of A , with $m \geq RR(A) + 1$, and any $\varepsilon > 0$ there is a m -tuple (y_1, y_2, \dots, y_m) of self-adjoint elements of A with the properties

$$\sum_{k=1}^m y_k^2 \text{ is invertible}$$

$$\left\| \sum_{k=1}^m (x_k - y_k)^2 \right\| < \varepsilon.$$

Examples of Real Rank 0 C^* -algebras :

von Neumann algebras

UHF algebras

AF algebras, O_m

Examples of non Real Rank 0 C^* -algebras

$C[0,1]$

$M_n(C[0,1])$

AI-algebras which are not AF algebras

some stably AI-algebras

Note: There are much more non Real Rank 0 than Real Rank 0 C^* -algebras.

Analogie: \mathbb{Q} versus $\mathbb{R} \backslash \mathbb{Q}$.

THEOREM (Brown, Pedersen)

If X is a compact Hausdorff space then

$$RR(C(X)) = \dim X$$

Note: Real rank of a C^* -algebra is
a non-commutative analogue
of the dimension

THEOREM

1. A has (HP)
2. A has real rank zero
3. A has (FS)

Let A be a simple AI alg.

with $RR(A) \neq 0$. \Rightarrow

$\Rightarrow \exists H \subseteq A$ s.t. H does not
have an approx. unit consisting of
projections.

H is not a AI alg.

A simple $\Rightarrow H$ full $\Rightarrow H \otimes k \cong A \otimes k$
L.Brown

Hence H is stably AI but not AI.

G.Pedersen : $RR(A) = 0 \Rightarrow RR(A \otimes k) = 0$

$\Rightarrow RR(H) \neq 0$.

Stably isomorphic algebras

Definition Two C^* -algebras A and B are stably isomorphic, denoted $A \sim B$, if $A \otimes \mathbb{K} \cong B \otimes \mathbb{K}$.

Remark The notion of stably isomorphism (or Morita-Rieffel equivalence in the separable case) is an equivalence relation which is weaker than the isomorphism relation. It is easy to see that: $M_2(\mathbb{C}) \sim M_3(\mathbb{C})$ but $M_2(\mathbb{C}) \not\cong M_3(\mathbb{C})$

Using the complete classification result of **I. Stevens** for simple AI algebras we show that the classification of simple stably AI algebras reduces to the classification of hereditary subalgebras of simple AI algebras which are stably isomorphic.

Proposition Let H_1 and H_2 be two hereditary subalgebras of simple AI-algebras A_1 and respectively A_2 . Then if A_1 and A_2 are not stably isomorphic then H_1 is not isomorphic to H_2 . On the contrary: when A_1 is stably isomorphic to A_2 we obtain that H_1 is stably isomorphic to H_2 .

Ideea of proof

$$\begin{array}{c} H_1 \otimes \mathbb{K} \cong A_1 \otimes \mathbb{K} \\ \text{---} \\ H_2 \otimes \mathbb{K} \cong A_2 \otimes \mathbb{K} \end{array} \quad \left. \begin{array}{l} \text{---} \\ \text{---} \end{array} \right\} \Rightarrow H_1 \not\cong H_2$$

Main classification result

$$A = \varinjlim A_i; \quad \text{simple } C^* \text{-algebras}$$
$$B = \varprojlim B_i;$$

A_i, B_i are continuous-trace C^* -algebras
with spectrum $[0, 1]$ or finitely many
closed intervals

$\varphi_0 : D(A) \rightarrow D(B)$ isomorphism

$\varphi_T : (\text{Aff} T^* A, \text{Aff}' A) \rightarrow (\text{Aff} T^* B, \text{Aff}' B)$
isomorphism

$$\widehat{\varphi_0}([\rho]) = \widehat{\varphi_T}([\rho]) \quad \forall \rho \in D(A)$$

$\Rightarrow \exists \varphi : A \rightarrow B$ isomorphism which induces
the given isomorphism

The invariant:

$D(A)$ - the set of Murray-von Neumann equivalence classes of projections on A

$$[\rho] + [q] = [\rho + q] \text{ if } \rho \perp q$$

T^*A - traces on A

$\text{Aff } T^*A = \{f: T^*A \rightarrow \mathbb{R} \mid f \text{ is continuous and linear}\}$

$$\|f\| = \sup \{f(\tau) \mid \tau \in T^*A, \tau(p) = 1\}$$

$$\overline{\text{Aff}'A} = \overline{\{a \in \text{Aff } T^*A : a \geq 0, \|a\| \leq 1\}}$$

$$\hat{a}(\varepsilon) = \varepsilon(a)$$

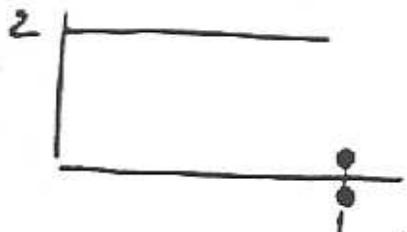
The building blocks:

Def A continuous trace C^* -algebra is a C^* -algebra A with Hausdorff spectrum T such that, for each $t \in T$, there are a neighborhood N of t and $a \in A$ such that $a(s)$ is a rank-one projection, $\forall s \in N$.

Example

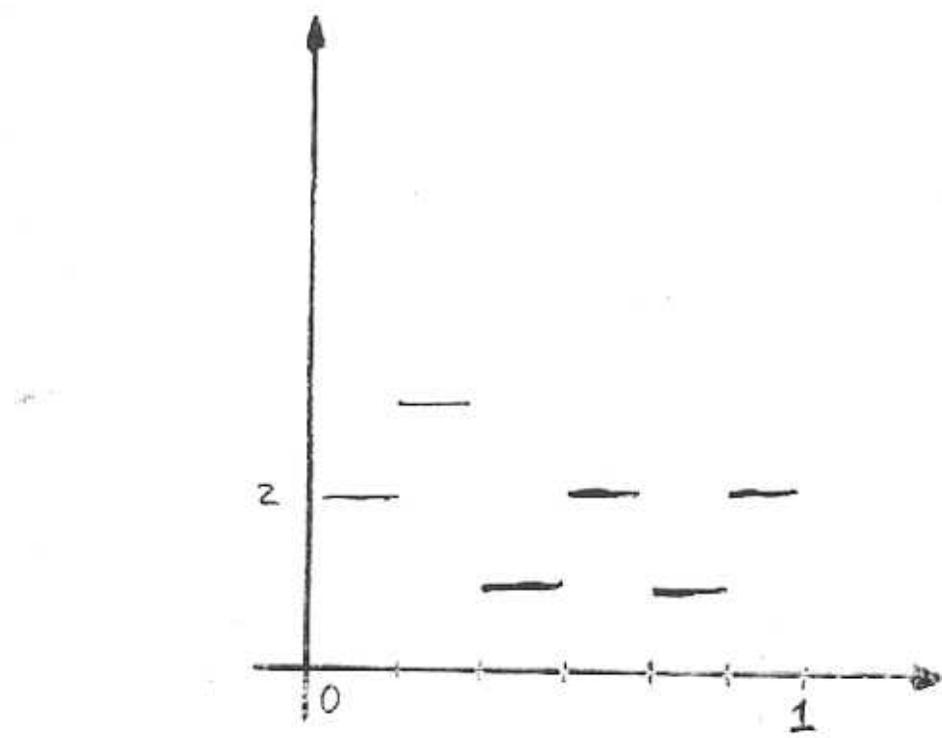
$$1) \begin{pmatrix} C_0([0,1]) & C_0([0,1]) \\ C_0([0,1]) & C([0,1]) \end{pmatrix} \quad P(t) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$2) \{f \in C([0,1], M_2(\mathbb{C})) : f(1) = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \lambda \in \mathbb{C}, \mu \in \mathbb{C}\}$$



$$3) \{f \in C([0,1], M_2(\mathbb{C})) : f(1) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \lambda \in \mathbb{C}\}$$

no rank 1 projection



Sketch of the proof

Step 1 introduce "special" continuous-trace C^* -algebras

Let A be a continuous-trace C^* -algebra and \hat{A} its spectrum

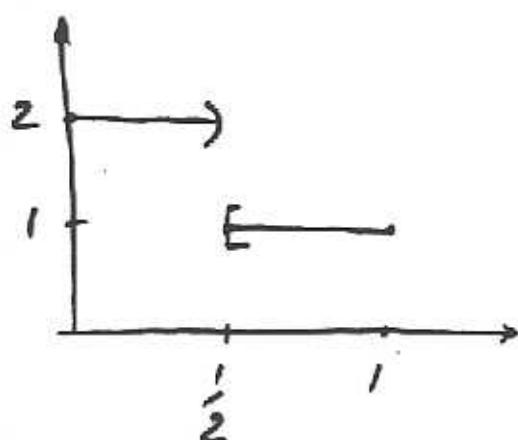
$$\hat{A} \ni [\pi_t] \xrightarrow{d} \dim H_{\pi_t}$$

If d is a finite step function then A is "special"

Note: d is lower semi-continuous

Example:

$$A = \{f \in C([0,1], M_2(\mathbb{C})) : f(t) = \begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix}, t \in [\frac{1}{2}, 1]\}$$



Step 2 Show an isomorphism theorem for
special continuous-trace C^* -algebra

THEOREM Let A and B be two special
continuous-trace C^* -algebras. If $d_A = d_B$ and
 $\hat{A} \xrightarrow{\text{homeo}} \hat{B}$ in a canonical way then
there is an isomorphism

$$A \cong B$$

which preserves the identification of the spectrum.

Note: Dixmier-Douady invariant,
 $\delta(\cdot) \in H^3(\hat{A}, \mathbb{Z})$ is trivial

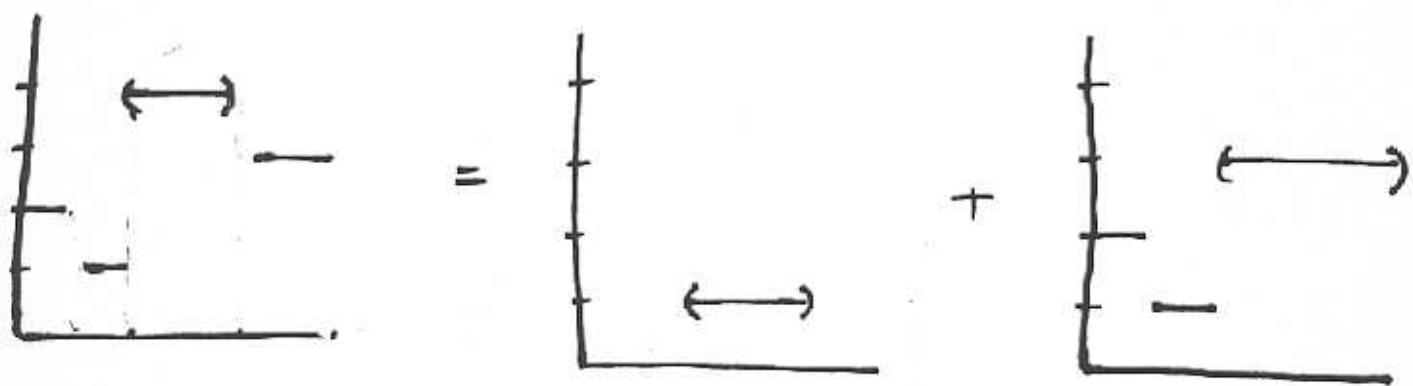
Note: in consequence we can use very
special continuous-trace C^* -algebras.

Decomposition result

Let $d: [0,1] \rightarrow N$ be a integer valued, lower semicontinuous map. Then:

$$d = d_{\max} + (d - d_{\max})$$

$$d_{\max}(x) = \begin{cases} 1 & \text{if } d(x) = \max_{z \in [0,1]} d(z) \\ 0 & \text{otherwise} \end{cases}$$



By induction: $d = \sum_{i \in F} d_i$

$$\text{im}(d_i) \in \{0,1\}$$

Very special building blocks

$$\begin{bmatrix} \mathcal{C}_0(A_1) & & & \\ & \mathcal{C}_0(A_2) & \dots & \\ & & \ddots & \\ & & & \mathcal{C}_0(A_n) \end{bmatrix} \subseteq M_n \otimes \mathbb{C}[0,1]$$

has dimension function $d = \sum_{i=1}^n d_{\max_i}$

$$\begin{bmatrix} \mathcal{C}_0(A_1) & \mathcal{C}_0(A_1) & \dots & \mathcal{C}_0(A_1) \\ & \mathcal{C}_0(A_2) & \dots & \mathcal{C}_0(A_2) \\ & & \ddots & \\ \mathcal{C}_0(A_1) & \mathcal{C}_0(A_2) & \dots & \mathcal{C}_0(A_n) \end{bmatrix} \subseteq M_n \otimes \mathbb{C}[0,1]$$

PROPOSITION Special continuous-trace C^* -algebras have a finite presentation

$$G = \{P, x_i^k, i = \overline{1, n}, k \in \{1, \dots, c\}\}$$

$$\mathcal{R} = R_1 \cup R_2 \cup R_3$$

$$R_1 = \{P = P^* = P^2, \|x_i^k\| \leq 2, \|x_i^{k+1}\| \leq 1\}$$

$$R_2 = \{x_i^k x_j^\ell = 0, \dots\}$$

$$R_3 = \{xx^* = f(y), f: [0, 1] \rightarrow \mathbb{R}, f \geq 0, 0 \leq y \leq 1\}$$

PROPOSITION Special continuous-trace C^* -algebras have stable relations

Step 3 Properties of special continuous-trace C^* -algebras

- a) Existence + Uniqueness theorems hold
use i. Stevens results for very special
continuous trace C^* -algebras
- b) Finitely presented
 - finitely generated
 - finitely many relations
- c) Relations are stable

Step 4 approximate a continuous-trace C^* -algebra
with special continuous-trace C^* -algebra

Main tool: A decomposition result for the
projection-valued function

Main idea: Any open set of \mathbb{R} is the
increasing union of finitely many open
intervals.

$$A_1 \xrightarrow{\varphi_{12}} A_2 \rightarrow \dots \xrightarrow{\lim} A_i = A$$

$$\vdots \quad \vdots$$

$$A_1^K \quad A_2^K$$

Step 5 we stable relations to modify maps
(T. Loring)

$$A_1^K \xrightarrow{\varphi_{12}} A_2$$

$$\varphi_{12} \sim \text{UI}$$

$$\tilde{\varphi}_{12} \text{ close to } \varphi_{12}$$

$$A_2^K$$

$$A_1 \rightarrow A_2 \rightarrow \dots \xrightarrow{\lim} A_i = A$$

$$\vdots \quad \vdots$$

$$A_1^K \xrightarrow{\varphi_{12}} A_2^K \dots \xrightarrow{\lim} A_i^K$$

Step 6 use all previous ideas to obtain
an approximate intertwining diagram

$$\begin{array}{ccccccc} A_1^K & \longrightarrow & A_2^K & \longrightarrow & \dots & \longrightarrow & \varinjlim A_i^K = A \\ \downarrow & \nearrow & \downarrow & & & & \vdots \\ B_1^K & \longrightarrow & B_2^K & \longrightarrow & \dots & \longrightarrow & \varinjlim B_i^K = B \end{array}$$

By Elliott argument we conclude that
 $A \cong B$

Note: the algebras that are classified, are
simple stably AI
(i.e. $A \otimes K$ is AI-algebra)

Question: Do we classify all stably A_i algebras?

Remark:

if the algebra, that I classify, has an approximate unit consisting of projections then is AH algebra (Ai algebra)

if the algebra, that I classify, does not have an approximate unit consisting of projections then is ASH algebra and not AH algebra



use the classification result and the norm map

$$\mu: \tilde{T}A \rightarrow \mathbb{R}$$

$$\begin{aligned}\mu(\tau) &= \|\tau\| \quad \text{if } \tau \text{ is bounded} \\ &= +\infty \quad \text{if } \tau \text{ is unbounded}\end{aligned}$$

Range of the invariant

Remark The invariant considered in the isomorphism theorem was $(K_0, D, \lambda, (AffT^+, Aff'))$. For the range of the invariant we consider an equivalent form of the invariant:
 $((K_0, D, \lambda, (T^+, \mu, \mathfrak{S}))$

Theorem Suppose that G is a simple countable dimension group, V is the cone associated to a metrizable Choquet simplex S , $\lambda : S \rightarrow Hom^+(G, R)$ is a continuous affine map with its range dense and sends extreme rays in extreme rays, and $\mu : S \rightarrow (0, \infty]$ any affine lower semicontinuous map. Then $[G, (V, S), \lambda, \mu]$ is the Elliott invariant of some simple non-unital algebra stably AI algebra.

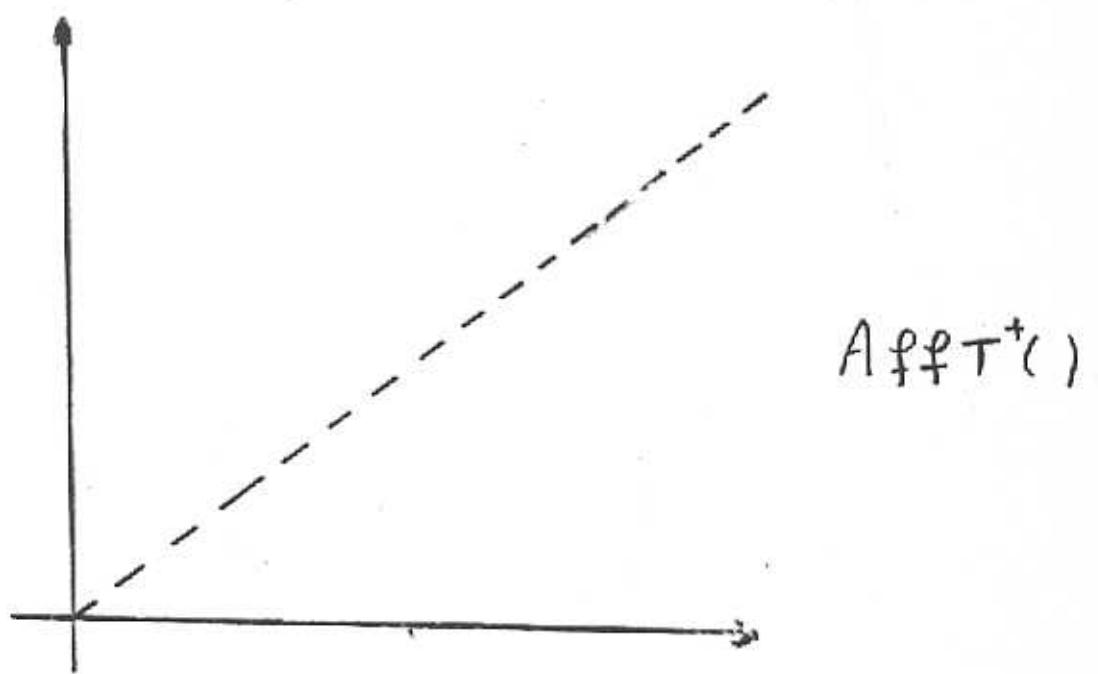
Proposition Let A be a simple AI algebra and μ its norm map. Then the following is true:

$$\mu(v) = \sup\{v(d) : d \in D\}$$

Theorem Let A be a simple stably AI algebra. A necessary and sufficient condition for A to be a simple AI algebra is:

$$\mu(v) = \sup\{v(d) : d \in D\}$$

Corollary If a simple stably AI algebra is approximately homogeneous then it is a simple AI algebra.



Non AI-algebras which are stably AI algebras

We can impose a necessary and sufficient condition on the invariant which will allow us to construct a simple stably AI algebra which is not a simple AH algebra but it is an ASH algebra:

$$\mu(v) \neq \sup\{v(g) : g \in D\}$$