ON THE UNDERSTANDING OF THE BLOW-UP FOR CRITICAL NLS

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We study blow up dynamics of solutions to the L^2 critical non linear Schrödinger equation

$$\begin{cases} iu_t = -\Delta u - |u|^{\frac{4}{N}}u, & (t,x) \in [0,T) \times \mathbf{R}^N \\ u(0,x) = u_0(x), & u_0 : \mathbf{R}^N \to \mathbf{C} \end{cases} \tag{1}$$

with $u_0 \in H^1 = \{u, \nabla u \in L^2\}$. Dimension $N = 2$ is physically relevant.

(NLS) is an infinite dimensional Hamiltonian system with energy space H^1 without any space localization property.

It is, together with the critical GKdV Equ. the only example where blow up is known to occur. (suspected for many equations: Zakharov, Yang-mills, Wave maps, Euler,...)
For (NLS), the proof of existence of blow up solutions is based on energy constraints and the existence of a "conformal" invariance. In particular, *It is not constructive. No qualitative information of any type on the blow up dynamics.*

The natural questions regarding blow up dynamics (related to how? why?) are
-Does there exist a universal blow up speed, or are there several possible regimes? Among these regimes, which ones are stable, generic?
-Does there exist a universal space time structure for the formation of singularities?
-What are the related mathematical problems?

1 HAMILTONIAN STRUCTURE, CRITICALITY

Local well posedness and Hamiltonian structure in the energy space

Local well posedness in time in energy space H^1: 80's Ginibre, Velo. (Kenig Ponce Vega and of J. Bourgain in the periodic setting). THEORY OF OSCILLATORY INTEGRALS.

For $u_0 \in H^1$, there exists $0 < T \leq +\infty$ such that $u(t) \in \mathcal{C}([0,T), H^1)$ and either $T = +\infty$, we say the solution is global, or $T < +\infty$ and then $\limsup_{t \uparrow T} |\nabla u(t)|_{L^2} = +\infty$, then we say the solution blows up in finite time (concentration in L^2).

Symmetries in the energy space H^1: If $u(t,x)$ solution, then are sol.

- Space-time translation invariance: $u(t+t_0, x+x_0)$.
- Phase invariance: $u(t,x)e^{i\gamma}$.
- Scaling invariance: $\lambda^{\frac{N}{2}} u(\lambda^2 t, \lambda x)$.
- Galilean invariance: $u(t, x-\beta t)e^{i\frac{\beta}{2}(x-\frac{\beta}{2}t)}$.

The pseudo-conformal symmetry holds in $\Sigma = H^1 \cap \{xu \in L^2\}$ (not H^1):

$$v(t,x) = \frac{1}{|t|^{\frac{N}{2}}} \overline{u}(\frac{1}{t}, \frac{x}{t}) e^{i\frac{|x|^2}{4t}}.$$

L^2 appears to be critical space.

Invariants in the energy space H^1:

- L^2-norm:
$$\int |u(t,x)|^2 = \int |u_0(x)|^2; \tag{2}$$

- Energy:
$$\frac{1}{2}\int |\nabla u(t,x)|^2 - \frac{1}{2+\frac{4}{N}}\int |u(t,x)|^{2+\frac{4}{N}} = E(u_0); \tag{3}$$

- Momentum:
$$Im\left(\int \nabla u\overline{u}(t,x)\right) = Im\left(\int \nabla u_0\overline{u_0}(x)\right). \tag{4}$$

In virial space Σ, the pseudo conformal symmetry, energy conservation for v imply the so called virial identity (monotonicity of quantity not in H^1):

$$\frac{d^2}{dt^2}\int |x|^2|u(t,x)|^2 = 16E(u_0). \tag{5}$$

Variational characterization of the ground state and global solutions

Sharp criterion of smallness for global existence of solutions related periodic solutions (variational tools M. Weinstein 80's).
There exists a unique non zero radial positive solution to

$$\begin{cases} \Delta Q - Q + Q^{1+\frac{4}{N}} = 0 \\ Q(r) \to 0 \ \text{ as } \ r \to +\infty. \end{cases} \tag{6}$$

Variational characterization of Q gives the Gagliardo-Nirenberg inequality with sharp constant:

$$\forall v \in H^1, \quad E(v) \geq \frac{1}{2} \int |\nabla v|^2 \left(1 - \left(\frac{|v|_{L^2}}{|Q|_{L^2}} \right)^{\frac{4}{N}} \right). \tag{7}$$

Cor: If $|u_0|_{L^2} < |Q|_{L^2}$ then the solution $u(t)$ to (1) is global and bounded in H^1.

H^1 symmetries of (1) yield a three parameter family of global solutions:

$$W_{\lambda_0, x_0, \gamma_0}(t, x) = \lambda_0^{\frac{N}{2}} Q(\lambda_0 x + x_0) e^{i(\gamma_0 + \lambda_0^2 t)} \tag{8}$$

which satisfy: $E(W) = 0, \quad Im(\int \nabla W \overline{W}) = 0, \quad \int |W|^2 = \int Q^2$.

The L^2 criticality implies that the Hamiltonian invariants do not see the size of the different solitary waves.

Blow up for large data: the virial identity

We now turn to the super critical case in mass $|u_0|_{L^2} > |Q|_{L^2}$.

Let then $u_0 \in \Sigma$ with $E(u_0) < 0$, the corresponding solution $u(t)$ to (1) satisfies the virial identity (5):

$$\frac{d^2}{dt^2} \int |x|^2 |u(t,x)|^2 dx = 16E(u_0),$$

and force the solution to blows up in finite time.

- No description of the blow up dynamic.

- Argument instable by H^1 perturbation.

The case of critical mass, $|u_0|_{L^2} = |Q|_{L^2}$.

Weinstein's criterion for global solutions is sharp. Applying the pseudo-conformal transformation to $W(t,x) = Q(x)e^{it}$,

$$S(t,x) = \frac{1}{|t|^{\frac{N}{2}}} Q(\frac{x}{t}) e^{i\frac{|x|^2}{4t} - \frac{i}{t}} \tag{9}$$

is a solution to (1) with the following properties:
-$S(t)$ has critical mass: $|S(t)|_{L^2} = |Q|_{L^2}$.
-$S(t)$ blow up at $t = 0$ with rate $|\nabla S(t)|_{L^2} \sim \frac{1}{|t|}$ as $t \to 0$.
-The singularity corresponds to formation of a Dirac mass:

$$|S(t)|^2 \rightharpoonup \left(\int Q^2\right) \delta_{x=0} \text{ as } t \to 0. \tag{10}$$

Dynamical characterization of $S(t)$

Theorem: (F.Merle) Let $u_0 \in H^1$ with $|u_0|_{L^2} = |Q|_{L^2}$ and assume $u(t)$ blows up in finite time, then up to the set of H^1 symmetries of (1) $u(t) = S(t)$.

Energy constraints imply Dirac Mass blow-up (no dispersion). A set of elliptic estimates on none dispersive solution implies regularity and expo decay). Successful for KdV...

Explicit construction of blow up solutions

Remark first that the following lower bound on blow up rate always holds from scaling considerations:

$$|\nabla u(t)|_{L^2} \geq \frac{C}{\sqrt{T-t}}. \tag{11}$$

There are two results of construction of blow up solutions with a prescribed dynamic:

- Bourgain and Wang ($N = 1, 2$) solutions $u(t)$ which blow up in finite time T and behave locally like explicit blow up solution $S(t)$.

$$|\nabla u(t)|_{L^2} \sim \frac{1}{T-t} \ \text{ as } \ t \to T. \tag{12}$$

(Stability up to codimension G perturbation in H^G).

- Numerical simulations, Formal picture in the past thirty years, Sinai $|\nabla u(t)|_{L^2} \sim \frac{1}{(T-t)^{(2/3)}}$, Zakharov $|\nabla u(t)|_{L^2} \sim \sqrt{\frac{|\log T-t|}{T-t}}$, finally Landman, Papanicolaou, Sulem, Sulem suggest solutions (stable by perturbation of the initial data) which blows up as

$$|\nabla u(t)|_{L^2} \sim \sqrt{\frac{\log(|\log T-t|)}{T-t}}. \tag{13}$$

Anomalous rate of blow up (13) is to be seen as a double log correction to the scaling estimate. For $N = 1$, G. Perelman proves the existence of one solution which blows up according to (13) and its stability in some space strictly included in H^1.

In conclusion, one expects that (1) admits at least two blow up dynamics: the log-log dynamic as an almost self similar blow up which is expected to be stable; the $S(t)$ dynamic which is suspected to be unstable with respect to perturbation of the initial data.

However, the log-log rate is known to be structurally unstable in the following sense (Merle): consider in dimension $N = 2$ the Zakharov system:

$$\begin{cases} iu_t = -\Delta u + nu \\ \frac{1}{c_0^2} n_{tt} = \Delta n + \Delta |u|^2 \end{cases} \tag{14}$$

from which in the limit $c_0 \to +\infty$ we formally recover (1), then for all $c_0 > 0$, finite time blow-up solutions to (14) satisfy

$$|\nabla u(t)|_{L^2} \geq \frac{C}{T-t}. \tag{15}$$

2 STATEMENTS OF THE RESULTS

From now on, we consider :

$$\int Q^2 \leq \int |u_0|^2 \leq \int Q^2 + \alpha^* \quad \text{for} \quad \alpha^* > 0 \text{ small.}$$

Assume $u(t)$ blows up in finite time, then

$$u(t,x) = \frac{1}{\lambda(t)^{\frac{N}{2}}}(Q+\epsilon)(t, \frac{x-x(t)}{\lambda(t)})e^{i\gamma(t)}. \tag{16}$$

where $|\epsilon(t)|_{H^1} \leq \delta(\alpha^*)$.
Description of the blow up dynamics: estimating $\lambda(t) \sim \frac{1}{|\nabla u(t)|_{L^2}}$ and the behavior of $\varepsilon(t)$.

The aim of our analysis is to understand how to extract from the infinite dimensional dynamic of (1) a finite dimensional and possibly universal dynamic for the evolution of the geometrical parameters $(\lambda(t), x(t), \gamma(t))$ *which is coupled to the dispersive dynamic of* $\varepsilon(t)$*.*
Nothing for large mass... Conjectured to locally describe the generic blow-up.

In addition, we assume

$$N = 1, 2, 3, 4$$

assuming *Spectral Property* which amounts counting the number of negative eigenvalues of an operator of the form $-\Delta + V$ where $V = V(x)$ is related to Q.

Construction of example of the blow up solutions are based on fixed point arguments. Prescribe the finite dimensional dynamic of $(\lambda(t), x(t), \gamma(t))$ according to the expected rate of blow up, and then to build $\varepsilon(t)$ thanks to suitable decay estimates on the propagator of the linearized operator L close to ground state Q.
Problem : the number of degeneracies for the linearized operator

L are strictly bigger than the ones induced by H^1 symmetries and the pseudo conformal transformation, which in fact make interaction of the two dynamics preponderant.
expect existence of cancellations between finite and infinite dimension dynamics: Sinai (no), Zakharov (at the first order), Papanicolaou (at all polynomial order).

Our approach: Classification results. Looking at general constrains or properties of the equation which will lead to rigidity (i.e. valid for all blow-up solution).

- monotonicity results for $\lambda(t)$ (related to a strong maximum principle type of argument),
- or existence of a Lyapounov functional defined in L^2_{loc} which contains dispersive informations.
- Dynamic characterization the solitary wave, or $S(t)$ which are at the heart of the description of blow up dynamics.

Blow up for strictly negative energy solutions.

Let

$$E_G(u) = E(u) - \frac{1}{2}\left(\frac{Im(\int \nabla u \overline{u})}{|u|_{L^2}}\right)^2. \tag{17}$$

Theorem 1 *Assume*

$$E_G(u_0) < 0.$$

Then $u(t)$ blows up in finite time $0 < T < +\infty$ and for t close to T:

$$|\nabla u(t)|_{L^2} \leq C^* \left(\frac{\log|\log(T-t)|}{T-t}\right)^{\frac{1}{2}}. \tag{18}$$

Comments on the result

1. H^1 theory for blow-up: The blow up criterion is in H^1 and thus improves the virial result (Σ only). For $E_G(u_0) < 0$, blow up is a stable phenomenon.
(18) removes the possibility of $S(t)$ type of blow up for strictly negative energy solutions (also the ones predicted by Sinai, Zakharov...). First bound of the blow-up rate.

2. Instability of $S(t)$:Any neighborhood of $S(t)$ in H^1 contains a solution which blows up with the log-log upper bound , and thus not with rate $\frac{1}{t}$. Still open for solutions build by BW.

3. Structural instability of the log-log rate: From our proof, we can exhibit the algebraic cancellation which is responsible for existence of a blow up rate *below* the one of explicit solution $S(t)$.

See Zakharov equation.

4. Sharpness of the result: The bound is optimal. see later.

5. $\dot{H}^1$ theory: Theorem 1 is in fact a consequence of purely local estimates in $\dot{H}^1$ and L^2_{loc}. Proof is based on some monotonicity formula, or a special direction where the equa have the "maximum principle property". Give estimates on bounded region decoupled from the behavior of the radiative field at infinity. This is a "miracle" (since from the critically, the blow-up is not a local but a global problem.)

The result is sharp (for lower bound) as it also includes blow up description for self similar solutions which are in $\dot{H}^1$ but never in L^2. Indeed,let:

$$U_{b_0}(t,x) = \frac{1}{(2b_0(T-t))^{\frac{N}{4}}} Q_{b_0}\left(\frac{x}{\sqrt{2b_0(T-t)}}\right) e^{-i\frac{\ln(T-t)}{2b_0}}$$

for $b_0 > 0$ and Q_{b_0} solving the ODE:

$$\Delta Q_b - Q_{b_0} + ib_0\left(\frac{N}{2}Q_{b_0} + y\cdot\nabla Q_{b_0}\right) + Q_{b_0}|Q_{b_0}|^{\frac{4}{N}} = 0. \quad (19)$$

Such solutions are called *self similar* solutions as :

$$|\nabla U_{b_0}(t)|_{L^2} \sim \frac{1}{\sqrt{T-t}}.$$

Q_{b_0} *never belong to* L^2 as $|Q_{b_0}(y)| \sim \frac{C(b_0)}{|y|^{\frac{N}{2}}}$ as $|y| \to +\infty$

Lower bound on blow up rate and Asymptotic stability of the blow up profile.

We now turn to the question of lower bounds on blow up rate.(In H^1, better estimates than $|\nabla u(t)|_{L^2} \geq \frac{C}{\sqrt{T-t}}$?)

Even though self similar blow up indeed describes the dynamics in other settings, from the criticality, we expect for any blow up solution in H^1:

$$\sqrt{T-t}|\nabla u(t)|_{L^2} \to +\infty \quad \text{as} \quad t \to T. \tag{20}$$

First approach:

The problem is in fact implied by existence of a universal blow up profile which attracts blow up solutions as $t \to T$ (First result of Asymptotic stability of Q for power nonlinearity).

Theorem 2 *Let $u(t)$ which blows up in finite time $0 < T < +\infty$. Then there exist parameters $\lambda_0(t) = \frac{|\nabla Q|_{L2}}{|\nabla u(t)|_{L^2}}$, $x_0(t)$ and $\gamma_0(t)$ such that*

$$e^{i\gamma_0(t)}\lambda_0^{\frac{N}{2}}(t)u(t,\lambda_0(t)x + x_0(t)) \to Q \quad in \quad \dot{H}^1 \quad as \quad t \to T. \quad (21)$$

This implies the lower bound on blow up rate:

$$\lim_{t\to T} \sqrt{T-t}|\nabla u(t)|_{L^2} = +\infty. \quad (22)$$

Remark 1 *: $\varepsilon(t) \to 0$ as $t \to T$ in $\dot{H}^1$, and the profile of the solution in space is asymptotically Q. This result is optimal as strong L^2 convergence to zero is forbidden from L^2 invariance.*

To prove the non existence of self similar solutions, we need to obtain global dispersive informations in the scaling critical space L^2. The proof is based on some geometrical characterization of solitons and different notions of dispersion in L^2.

Part A: Characterization of soliton in the energy zero set.

Theorem 3 *Let*

$$E_G^0 = 0.$$

Assume that $u(t)$ is not a soliton up to the symmetries in H^1. Then $u(t)$ blows up in finite time on the left and on the right in time and upper bound (18) holds.

Characterization of the solitary wave: $Q(x)e^{it}$ *is up to the* H^1 *symmetries the only zero energy solution to (1) which lives on an infinite time interval.*

As a corollary, no KAM closed to Q.

Part B: L^2 dispersive characterization of explicit blow up solution $S(t)$:

Theorem 4 *Let* $v(t)$ *a solution to (1) which blows up at* $0 < T < +\infty$ *and*

$$|v|^2(t) \rightharpoonup \left(\int |v(0)|^2\right) \delta_{x=0} \quad as \quad t \to T, \tag{23}$$

then

$$v(t) = S(t)$$

up to the set of H^1 *symmetries of (1).*

Proof of this result involves new type of dispersive estimates in L^2.

If v different from $S(t)$, then one show that v has a self similar behavior in norm ($|\nabla v(t)|_{L^2} \sim \frac{1}{\sqrt{T-t}}$). Then, it has to be closed in a strong sense to a stationary self similar profile, and we get a contradiction from the fact that these are not in L^2.

Second approach:

After estimates on asymptotic objects, we have able to give a direct proof. In the loglog regime in H^1, we have a sharp lower bound on blow up rate. From the complete understanding on the behavior of the solution, we are able to construct a Lioupounov functional in time which yields the result.

Theorem 5 (log-log lower bound) *Let $u(t)$ blows up in finite time $0 < T < +\infty$, then one has the following lower bound on blow up rate:*

$$|\nabla u(t)|_{L^2} \geq C_3^* \left(\frac{\log|\log(T-t)|}{T-t}\right)^{\frac{1}{2}}. \tag{24}$$

Coupled with Theorems 1 and 6, this result gives a complete description of the log-log regime

$$|\nabla u(t)|_{L^2} \sim C(N) \left(\frac{\log|\log(T-t)|}{T-t}\right)^{\frac{1}{2}}.$$

Blow up for strictly positive energy solutions.

In the positive energy case, one may have

- global solution,

- blow-up solutions (here one expect to have the 2 blow-up rate at least).

Theorem 6 *(i) Rigidity of blow up rate: Let* $E_G(u_0) > 0$, *and assume that* $u(t)$ *blows up in finite time* $T < +\infty$*, then for* t *close to* T *either*

$$|\nabla u(t)|_{L^2} \sim C^*(N) \left(\frac{\ln|\ln(T-t)|}{T-t} \right)^{\frac{1}{2}}$$

or

$$|\nabla u(t)|_{L^2} \geq \frac{C_2^*}{(T-t)\sqrt{E_G(u_0)}}. \tag{25}$$

(ii) Stability of log-log law: Moreover, the set of initial data u_0 *such that* $u(t)$ *blows up in finite time with the loglog upper bound is open in* H^1.

From this theorem, one can construct loglog blow-up solutions with negative energy (using $E = 0$).

STILL OPEN $S(t)$ blow-up solution are at the boundary of the set of blow-up solutions.

Mass quantization Property, Profile in the original variable

Question: Assume $u(t) \in H^1$ blows up in finite time. Is the solution concentrates a finite number Dirac mass for *a universal and quantized amount of mass*, and the excess of mass is then purely dispersed in L^2 outside the blow-up points.

Theorem 7 (Mass quantization) *Let $u(t)$ blows up in finite time $0 < T < +\infty$, then there exists a function $x(t) : [0,T) \to \mathbf{R}^N$ such that:*

$$\lim_{t \to T} |u(t, x + x(t))|^2 \rightharpoonup \left(\int Q^2\right) \delta_{x=0} + f \quad \textit{with} \quad f \in L^1.$$

Moreover, $x(t) \to x(T)$ finite as $t \to T$.

We have a much stronger property: For a $u^* \in L^2$,

$$u(t) - \frac{1}{\lambda(t)^{\frac{N}{2}}} Q\left(\frac{x - x(t)}{\lambda(t)}\right) e^{i\gamma(t)} \to u^* \ \text{ in } \ L^2 \ \text{ when } \ t \to T.$$

One can recover the blow-up regime from u^*.

Theorem 8 *(i) In the loglog regime,*

$$\int_{|x-x(T)|\leq R} |u^*(x)|^2 \sim \frac{C}{(\log|\log(R)|)^2} \quad \textit{when} \quad R \to 0$$

In particular,

$$u^* \notin H^1 \quad \textit{and} \quad u^* \notin L^P, \quad p > 2.$$

(ii)In the $S(t)$ regime,

$$\int_{|x-x(T)|\leq R} |u^*(x)|^2 \leq C E_0^G R^2 \quad \textit{and} \quad u^* \in H^1.$$

Explosion loglog: The blow-up is related to radiation of mass and energy conservation, with very strong coupling between singularity and regular part of the solution.

Explosion $S(t)$: Decoupling between singularity and regular part of the solution.