Introduction to the guiding center theory

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1 Introduction

1.1 A few definitions

(see e.g. Krall and Trivelpiece 1973, Nicholson 1983, Fizpatrick 1998)

PLASMA: collection of large number of charged particles with a density sufficiently low for the electromagnetic force due to the binary short distance interactions be much weaker than that exerted by the many distant particles.

PLASMA APPROXIMATION: Regime where the kinetic energy of the particle strongly dominates the mean interparticle potential energy, in such a way that the medium displays a COLLECTIVE BEHAVIOR where each particle is subject to the continuous electromagnetic field created by the distant particles and to the ambient field (described by the Maxwell equations), with possible subdominant interactions with nearest neighbor particles, usually referred to as COLLISIONS.

DEBYE SPHERE: sphere of influence of a given particle when surrounded by particles of opposite signs that have been attracted and produce a screening effect.

DEBYE SPHERE RADIUS: $\lambda_D = \dfrac{\text{thermal velocity}}{\text{plasma frequency}} \propto \sqrt{\dfrac{T}{n}}$

(T temperature, n density of the ions or electrons),
Independent of mass and thus comparable for different species.

It turns out that the plasma approximation is satisfied as the
PLASMA PARAMETER $g = \dfrac{1}{n\lambda_D^3} \sim \dfrac{n^{1/2}}{T^{3/2}} \ll 1.$
(Requires a large number of particles be contained in a Debye sphere).

PLASMA DESCRIPTION: SCALES LARGE COMPARED TO λ_D.

In the asymptotic regime where the particles are only subject to the electro-magnetic field resulting from the mean effect of many distant particles and from external forces, the plasma is said COLLISIONLESS.

A collisionless plasma can be viewed as a collection of non interacting particles.

It is thus of interest to understand the motion of an individual particle in a given electromagnetic field, even if important effects in plasma physics often result from cooperative phenomena.

1.2 The guiding-center description

A particle trajectory is generally very complicated and cannot be analytically computed in a closed form, except in very special situations.

Solvable case: uniform static electromagnetic field. The particle gyrates in a helix about the field line, with a possible transverse drift.

Furthermore, when the magnetic field is slowly varying in space and time compared to the radius and period of the particle gyromotion, the problem is amenable of perturbation-theoretic methods, providing an approximate description of the particle motion.

This leads to the GUIDING-CENTER THEORY, also called DRIFT THEORY or DRIFT APPROXIMATION that averages out on the gyrotropic motion of the particle near a circle around a "guiding center" whose sole motion is retained.

When a large number of particles is considered, the evolution of the distribution function of the associated guiding centers can easily be derived when using an Hamiltonian representation of the particle motion.

Alternatively, the distribution function can be obtained by starting from the Vlasov equation for the distribution function of the particles, and averaging on the gyromotion.

Both approaches are successively considered in these lectures.

2 Motion of a charged particle

Equation of motion of a (classical) particle of mass m and charge q moving with a velocity v in an electric field e and a magnetic field b:

$$m\frac{dv}{dt} = q(e + v \times b)$$

(SI units are used. When using Gauss units, a factor $1/c$ where c is the light velocity arises in front of the 2nd term of the r.h.s.).

In the general case where e and b depend on space and time, this equation may not be amenable to an analytic solution.

It is nevertheless easily solved when e and b are constant. This suggests that perturbative solutions are possible when e and b display slow variations compared with the radius and the period of the circular gyration the particle would execute if the field variations were neglected.

2.1 Charged particle in a constant magnetic field

2.1.1 No electric field

$$m\frac{dv}{dt} = qv \times b$$

Velocity component $v_{||}$ along the magnetic field remains constant. Transverse velocity $v_{\perp}$ is that of a circular motion with an angular velocity vector $\vec{\omega} = -\frac{q}{m}b$.

A positively (negatively) charged particle gyrates in the left-handed (right-handed) sense with respect to the magnetic field direction with a gyro (or cyclotron) frequency $\Omega = |\vec{\omega}| = \frac{q|b|}{m}$ on the "Larmor circle" whose center is displaced parallel to the magnetic field at a fixed velocity $v_{||}$.

The resulting motion thus traces out an helix whose axis is parallel to the magnetic field and whose "Larmor radius" or "gyroradius" is $\rho_L = \frac{|v_{\perp}|}{\Omega}$.

2.1.2 Presence of a constant electric field

$$m\frac{dv_{\|}}{dt} = qe_{\|}$$
$$m\frac{dv_{\perp}}{dt} = q(e_{\perp} + v_{\perp} \times b).$$

Parallel electric field leads to a uniform acceleration of the particle along the magnetic field lines.

Transverse dynamics: it is convenient to separate $\quad v_{\perp} = v'_{\perp} + u_E$
where
⋆ the "gyro-velocity" $v'_{\perp}$ is defined as the solution of the homogeneous problem

$$m\frac{dv'_{\perp}}{dt} = qv'_{\perp} \times b$$

⋆ the transverse velocity u_E of the center of rotation (called GUIDING CENTER) is due to the transverse electric field and is taken to satisfy

$$0 = e_{\perp} + u_E \times b$$

or
$$u_E = \frac{e \times b}{|b|^2}.$$

This velocity that does not depend on the sign of the particle charge, is usually called ELECTRIC DRIFT VELOCITY or "$e \times b$ velocity".

Furthermore, a charge q describing a circular orbit of radius ρ_L with velocity v' produces a mean current $I = \dfrac{qv'}{2\pi\rho_L} = q\dfrac{\Omega}{2\pi}$ (where $\dfrac{\Omega}{2\pi}$ is the number of gyrations per second).

This leads to define the MAGNETIC MOMENT of this current I as $\vec{M} = IS\dfrac{\vec{\omega}}{\Omega}$ where $S = \pi\rho_L^2$ is the area encircled by the charge orbit.

Thus $\vec{M} = \mu\dfrac{\vec{\omega}}{\Omega}$ with $\mu = \dfrac{m|v'|^2}{2|b|} = \dfrac{q^2\rho_L^2|b|}{2m}$.

Magnetic moment associated with particle gyration points always in the direction opposite to the magnetic field direction. Gyrating charges tend to decrease the magnetic field and plasma is DIAMAGNETIC.

2.2 Charged particle in a slowly varying electromagnetic field

Assume the electromagnetic field to vary
on scales $L \gg \rho_L$ (Larmor radius) and $T \gg \dfrac{2\pi}{\Omega}$ (gyroperiod).

Small parameter exhibited by rescaling:
$x = Lr$, $t = T\tau$, $b = B_0 B$, $e = \frac{LB_0}{T} E$ (from Faraday equation), where B_0 denotes the mean value of the magnetic field. The equation rewrites

$$\epsilon \ddot{r} = E(r,t) + \dot{r} \times B(r,t)$$

where $\epsilon = \dfrac{m}{qB_0 L}\dfrac{1}{T} = \dfrac{1}{\Omega T}$ or $T = \dfrac{1}{\epsilon\Omega}$.

Initial conditions $r(0)$ and $\dot{r}(0)$ are given bounded function of ϵ.

Since the rescaled equation is identical to the primitive one, up to the replacement of m/q by ϵ, the asymptotic problem is often addressed by taking the dimensional parameter $m/q \to 0$ in the original equation.

One chooses $L = \dfrac{1}{\epsilon}\rho_L$: electromagnetic field characterized by a UNIQUE SPATIAL SCALE, assumed to be large.

Special case of the more general GYRO-KINETIC LIMIT where small amplitude perturbations of the electromagnetic field on scales comparable to the gyro-radius are permitted in the transverse direction, in addition to the long-wave longitudinal variations.

Because of the (slow) variations of the electromagnetic field, the gyromotion is not strictly mono-periodic. One expands (Kruskal 1957)

$$r = \sum_{-\infty}^{+\infty} \epsilon^{|n|} R_n(t) e^{n\frac{C(t)}{\epsilon}}$$

where the (complex) scalar and vector functions C and R_n can depend on ϵ but have finite limits as $\epsilon \to 0$.

We will here present a formal derivation of lowest order of the theory. Corrections can in principle be computed to all orders.

Rigorous proof that the above expansion provides an asymptotic representation of the exact solution of the initial value problem, valid as $\epsilon \to 0$, is given by Berkowitz and Gardner (1959).

To lowest orders

$$r = R_0 + \epsilon(R_1 e^{\frac{C}{\epsilon}} + R_{-1} e^{-\frac{C}{\epsilon}}) + \cdots,$$

(locally, a gyration on a circle centered at the GUIDING CENTER of vector coordinate R_0 and whose Larmor radius is $\epsilon\sqrt{2}|R_1|$, in units of L).

The ϵ in the exponentials is associated with the gyrofrequency $\Omega = 1/\epsilon T$

$$\dot{r} = \dot{R}_0 + \epsilon(\dot{R}_1 + \frac{\dot{C}R_1}{\epsilon})e^{\frac{C}{\epsilon}} + \epsilon(\dot{R}_{-1} - \frac{\dot{C}R_{-1}}{\epsilon})e^{-\frac{C}{\epsilon}} + \cdots$$

$$\ddot{r} = \ddot{R}_0 + \epsilon(\ddot{R}_1 + \frac{2\dot{C}\dot{R}_1 + \ddot{C}R_1}{\epsilon} + \frac{\dot{C}^2 R_1}{\epsilon^2})e^{\frac{C}{\epsilon}}$$

$$+\epsilon(\ddot{R}_{-1} - \frac{2\dot{C}\dot{R}_{-1} + \ddot{C}R_{-1}}{\epsilon} + \frac{\dot{C}^2 R_{-1}}{\epsilon^2})e^{-\frac{C}{\epsilon}} + \cdots.$$

$$B(r) = B(R_0) + \epsilon(R_1 e^{\frac{C}{\epsilon}} + R_{-1} e^{-\frac{C}{\epsilon}}) \cdot \nabla B(R_0) + \cdots$$

$$E(r) = E(R_0) + \epsilon(R_1 e^{\frac{C}{\epsilon}} + R_{-1} e^{-\frac{C}{\epsilon}}) \cdot \nabla E(R_0) + \cdots.$$

2.2.1 Dynamics of the guiding center

Non-oscillatory term:

$$\epsilon \ddot{R}_0 = E(R_0) + \dot{R}_0 \times B(R_0) + \epsilon \dot{C} \Big[R_1 \times R_{-1} \cdot \nabla B(R_0) - R_{-1} \times R_1 \cdot \nabla B(R_0) \Big] + O(\epsilon^2).$$

To leading order,

$$E(R_0) + \dot{R}_0 \times B(R_0) = O(\epsilon).$$

In particular

$$E(R_0) \cdot B(R_0) = O(\epsilon).$$

Furthermore, the transverse velocity of the guiding center is given by

$$\boxed{\dot{R}_{0\perp} = \frac{E(R_0) \times B(R_0)}{|B(R_0)|^2} + O(\epsilon)}$$

where, as already mentioned, this leading order contribution is called the ELECTRIC DRIFT VELOCITY or $E \times B$ drift.

In the direction of the magnetic field, one has

$$\ddot{R}_0 \cdot B(R_0) = \frac{1}{\epsilon} E(R_0) \cdot B(R_0) + \dot{C} \Big[R_1 \times R_{-1} \cdot \nabla B(R_0) - R_{-1} \times R_1 \cdot \nabla B(R_0) \Big] \cdot B(R_0) + O(\epsilon)$$

where we assumed $E_{||} = O(\epsilon)$ to prevent the dynamics to be dominated by a strong acceleration of the particle along the magnetic field line.

To simplify the writing, WE SHALL HEREAFTER NOT EXPLICIT ANYMORE THE ARGUMENTS OF THE FIELDS E AND B AND OF THEIR GRADIENTS WHEN EVALUATED AT THE GUIDING CENTER R_0.

To estimate R_1 and R_{-1}, we consider the oscillatory terms with the fundamental frequency, that obey

$$\dot{C}^2 R_1 = \dot{C} R_1 \times B + O(\epsilon)$$
$$\dot{C}^2 R_{-1} = -\dot{C} R_{-1} \times B + O(\epsilon).$$

It follows that

$$R_{-1} = R_1^*$$

and

$$\boxed{\dot{C} = i|B| + O(\epsilon)}$$

that in the primitive variables, identifies to leading order with the gyrotropic frequency evaluated at the guiding center.

Introduce the local orthonormal basis e_1, e_2, e_3 such that $B = |B|e_1$.

It follows that to leading order

$$iR_1 = R_1 \times e_1$$

or

$$R_1 = \frac{|R_1|}{\sqrt{2}}(e_2 + ie_3) \qquad\qquad |R_1| = \frac{\rho_L}{\sqrt{2}}.$$

This leads to rewrite

$$\ddot{R}_0 = \frac{1}{\epsilon}\Big(E + \dot{R}_0 \times B\Big) + |R_1|^2|B|\Big[e_2 \times (e_3 \cdot \nabla B) - e_3 \times (e_2 \cdot \nabla B)\Big] + O(\epsilon).$$

Following Northrop (1961), we write

$$
\begin{aligned}
e_2 \times (e_3 \cdot \nabla B) &= (e_3 \times e_1) \times (e_3 \cdot \nabla B) \\
&= (e_3 \cdot \nabla B \cdot e_3) e_1 - (e_3 \cdot \nabla B \cdot e_1) e_3 \\
e_3 \times (e_2 \cdot \nabla B) &= (e_1 \times e_2) \times (e_2 \cdot \nabla B) \\
&= (e_2 \cdot \nabla B \cdot e_1) e_2 - (e_2 \cdot \nabla B \cdot e_2) e_1 .
\end{aligned}
$$

Writing

$$\nabla = e_1 (e_1 \cdot \nabla) + e_2 (e_2 \cdot \nabla) + e_3 (e_3 \cdot \nabla),$$

one has

$$0 = \nabla \cdot B = e_1 \cdot \nabla B \cdot e_1 + e_2 \cdot \nabla B \cdot e_2 + e_3 \cdot \nabla B \cdot e_3 .$$

and thus

$$
\begin{aligned}
e_2 \times (e_3 \cdot \nabla B) - e_3 \times (e_2 \cdot \nabla B) &= \\
-(e_1 \cdot \nabla B \cdot e_1) e_1 - (e_2 \cdot \nabla B \cdot e_1) e_2 - (e_3 \cdot \nabla B \cdot e_1) e_3 &= -\nabla |B| .
\end{aligned}
$$

$$\boxed{\ddot{R}_0 = \frac{1}{\epsilon}\Big(E + \dot{R}_0 \times B\Big) - |R_1|^2 |B| \; \nabla |B| + O(\epsilon)}$$

It follows that $\qquad (\mu = |R_1|^2|B|)$

$$\boxed{\ddot{R}_0 \cdot e_1 = \frac{1}{\epsilon} E \cdot e_1 - |R_1|^2 |B| \; e_1 \cdot \nabla |B| + O(\epsilon)}$$

or (Kruskal 1957)

$$\boxed{\ddot{R}_0 \cdot B = \frac{1}{\epsilon} E \cdot B - \frac{1}{2} |R_1|^2 B \cdot \nabla |B|^2 + O(\epsilon)}.$$

$E_{\|} = O(\epsilon)$ to prevent rapid acceleration along the field line.

This describes the guiding-center motion parallel to magnetic field line. It involves the gyroradius that is still unknown.

2.2.2 Drifts

We have obtained $\dot{R}_{0\perp} = \dfrac{E \times B}{|B|^2} + O(\epsilon)$

where $\dfrac{E \times B}{|B|^2}$ is the ELECTRIC DRIFT VELOCITY.

When retaining the next order, we write

$\dot{R}_{0\perp} = \dfrac{E \times B}{|B|^2} + \epsilon U_1 + O(\epsilon^2)$ and thus

$$\ddot{R}_0 = \frac{1}{\epsilon}(E \cdot e_1)e_1 + U_1 \times B - |R_1|^2 |B| \ \nabla |B| + O(\epsilon).$$

Taking the cross product with $\widehat{b} = e_1$,

$$U_1 = \frac{1}{|B|} \widehat{b} \times \left(\ddot{R}_0 + |R_1|^2 |B| \ \nabla |B| \right) + O(\epsilon),$$

or using $\dot{R}_0 = v_{\|} \widehat{b} + \dfrac{E \times B}{|B|^2} + O(\epsilon), \qquad U = \dfrac{E \times B}{|B|^2}$

$$U_1 = \frac{1}{|B|} \widehat{b} \times \left(v_{\|} \frac{d\widehat{b}}{dt} + \frac{dU}{dt} + |R_1|^2 |B| \ \nabla |B| \right) + O(\epsilon).$$

In the physical units,

$$U_1 = \frac{1}{\Omega}\widehat{b} \times \left(v_{\|}\frac{d\widehat{b}}{dt} + \frac{dU}{dt} + \frac{\mu}{m}\ \nabla|B|\right).$$

This leads to define

⋆ the MAGNETIC DRIFT $\dfrac{\mu}{m\Omega}\widehat{b} \times \nabla\ |B|$,

⋆ the INERTIAL DRIFT $\dfrac{v_{\|}}{\Omega}\widehat{b} \times \dfrac{d\widehat{b}}{dt}$

$$= \frac{v_{\|}}{\Omega}\widehat{b} \times \left(\frac{\partial\widehat{b}}{\partial t} + U \cdot \nabla\right)\widehat{b} + \frac{v_{\|}^2}{\Omega}\widehat{b} \times (\widehat{b} \cdot \nabla\widehat{b}),$$

where the last term is the CURVATURE DRIFT,

⋆ the POLARIZATION DRIFT $\widehat{b} \times \dfrac{dU}{dt}$

includes the contribution $\dfrac{1}{\Omega|B|}\dfrac{dE_{\perp}}{dt}$ that in a plasma separates ions from electrons.

2.2.3 Characterization of the gyroradius

The equation for the guiding center R_0 involves the gyroradius (also called Larmor radius) through the magnetic moment, that is not determined by the leading order analysis performed above.

Neglecting the $O(\epsilon)$ contributions and writing $iR_1 = R_1 \times e_1$ do not permit to determine $|R_1|$.

In order to determine $|R_1|$, we write more accurately ($\dot{C} = i|B| + O(\epsilon)$)

$$\dot{C}^2 R_1 - \dot{C} R_1 \times B = \epsilon F,$$

where

$$F = -2\dot{C}\dot{R}_1 - \ddot{C} R_1 + R_1 \cdot \nabla E + \dot{R}_1 \times B + \dot{R}_0 \times (R_1 \cdot \nabla B) + O(\epsilon),$$

and require the SOLVABILITY of this equation.

SOLVABILITY CONDITION: F orthogonal to the kernel $\mathcal{K}er(\mathcal{L}^\dagger)$ of the adjoint of the operator $\mathcal{L} = -|B| + iB\times$.

Since $\mathcal{L}$ is self-adjoint, $\quad \mathcal{K}er(\mathcal{L}^\dagger) = \{\varphi /\ iB \times \varphi = |B|\varphi\}.$

In other words, the elements φ of $\mathcal{K}er(L^\dagger)$ are the eigenvectors of the operator $\mathcal{V} = iB\times$ associated with the eigenvalue $|B|$.

We are thus led to look for the eigenvectors of $\mathcal{V}$.
For any vector w, $\quad \mathcal{V}w = iB \times w$

thus $\quad \mathcal{V}^2 w = |B|^2 w - (B \cdot w)B \quad$ and $\quad \mathcal{V}^3 w = |B|^2 \mathcal{V} w,$

leading to the characteristic equation

$$\mathcal{V}^3 - |B|^2 \mathcal{V} = 0,$$

which implies that the eigenvalues of $\mathcal{V}$ are $-|B|, 0, |B|$.

Characteristic equation, rewritten $\mathcal{V}\Big(1-\frac{\mathcal{V}^2}{|B|^2}\Big)=0$ shows that
$P_0 = 1-\frac{\mathcal{V}^2}{|B|^2}$ is the projector on the zero eigenspace of $\mathcal{V}$.
(One easily checks that it satisfies $P_0^2 = P_0$).

Determination of the projectors P_i with $i=\pm$, on the two other directions, defined by $\mathcal{V}P_i = \epsilon_i|B|P_i$ with $\epsilon_i = \pm 1$:

We write $\qquad P_i = \alpha_i\mathcal{V} + \beta_i\mathcal{V}^2,$
(higher order power are useless because of the characteristic equation).

By substitution, we get $\quad \alpha_i\mathcal{V}^2 + \beta_i\mathcal{V}^3 = \epsilon_i|B|(\alpha_i\mathcal{V} + \beta_i\mathcal{V}^2).$
It follows that $\alpha_i = \epsilon_i|B|\beta_i$ and $\beta_i|B|^2 = \epsilon_i|B|\alpha_i$.

Consequently, $\qquad P_i = \beta_i(\epsilon_i|B|\mathcal{V} + \mathcal{V}^2),$
where the condition $\quad P_i^2 = P_i \quad$ prescribes $\quad \beta_i = \frac{1}{2|B|^2}.$

In conclusion (Berkowitz and Gardner 1959),

$$P_0 = 1 - \frac{1}{|B|^2}\mathcal{V}^2$$

$$P_1 = \frac{1}{2|B|^2}(\mathcal{V}^2 + |B|\mathcal{V})$$

$$P_{-1} = \frac{1}{2|B|^2}(\mathcal{V}^2 - |B|\mathcal{V}).$$

2.2.4 Adiabatic invariance of the magnetic moment

Solvability condition: F is orthogonal to the zero eigenspace of $\mathcal{V}$ associated with the eigenvalue $|B|$.

It reads $\qquad F = P_0 F + P_{-1} F$

or $\qquad \mathcal{V}^2 F + |B|\mathcal{V}F = 0.$

Using the definition of $\mathcal{V}$, we get the condition

$$B \times (B \times F) - i|B|(B \times F) = 0$$

that is also rewritten in terms of $\dot{C}$

$$\dot{C}^2 F + (F \cdot B)B + \dot{C} F \times B = 0.$$

After substitution of F, this provides, to lowest order, a first order differential equation for R_1. Since its direction was already determined, it reduces to an equation for the Larmor radius $|R_1|$.

The quantity entering the equation of motion of the guiding center is in fact $|R_1|^2|B|$ that identifies with the magnetic moment. We now show, from the equation for $|R_1|$, that, to leading order, this quantity is a constant in time (with corrections $O(\epsilon)$).

One has to estimate

$$\frac{d}{dt}(|B||R_1|^2) = -i\frac{d}{dt}(\dot{C}|R_1|^2) = -i\Big[\ddot{C}|R_1|^2 + \dot{C} R_1 \cdot \dot{R}_1^* + \dot{C}\dot{R}_1 \cdot R_1^*\Big].$$

From

$$\dot{C}^2 F + (F \cdot B)B + \dot{C} F \times B = 0$$
$$F = -2\dot{C}\dot{R}_1 - \ddot{C} R_1 + (R_1 \cdot \nabla)E + \dot{R}_1 \times B + \dot{R}_0 \times (R_1 \cdot \nabla B),$$

we have

$$\dot{C}^2 (R_1^* \cdot F) + \dot{C} R_1^* \cdot (F \times B) = 0$$

that also rewrites

$$\dot{C}^2 (R_1^* \cdot F) - \dot{C} F \cdot (R_1^* \times B) = 0.$$

Using that to leading order $\dot{C} R_1^* = -R_1^* \times B,$ it follows that

$$\dot{C}^2 R_1^* \cdot F + \dot{C}^2 R_1^* \cdot F = 0,$$

which implies $R_1^* \cdot F = 0$ and $R_1 \cdot F^* = 0.$

These conditions rewrite

$$
\begin{aligned}
0 &= -2\dot{C}\dot{R}_1 R_1^* - \ddot{C}|R_1|^2 \\
&\quad + \Big[R_1 \cdot \nabla E + \dot{R}_1 \times B + \dot{R}_0 \times (R_1 \cdot \nabla) B \Big] \cdot R_1^* \\
0 &= 2\dot{C}\dot{R}_1^* R_1 + \ddot{C}|R_1|^2 \\
&\quad + \Big[R_1^* \cdot \nabla E + \dot{R}_1^* \times B + \dot{R}_0 \times (R_1^* \cdot \nabla) B \Big] \cdot R_1,
\end{aligned}
$$

where (to leading order $\dot{C}R_1^* = -R_1^* \times B$)

$$
\begin{aligned}
(\dot{R}_1 \times B) \cdot R_1^* &= \dot{R}_1 \cdot (B \times R_1^*) = \dot{C}\dot{R}_1 \cdot R_1^* \\
(\dot{R}_1^* \times B) \cdot R_1 &= -\dot{C}\dot{R}_1^* \cdot R_1.
\end{aligned}
$$

It follows that ($\dot{C} = i|B|$)

$$
\begin{aligned}
0 = 2i\frac{d}{dt}(|B||R_1|^2) - i|B|\frac{d}{dt}|R_1|^2 + R_1^* \cdot \nabla E \cdot R_1 - R_1 \cdot \nabla E \cdot R_1^* \\
+ R_1 \cdot (\dot{R}_0 \times (R_1^* \cdot \nabla B)) - R_1^* \cdot (\dot{R}_0 \times (R_1^* \cdot \nabla B)).
\end{aligned}
$$

We write

$$R_1^* \cdot \nabla E \cdot R_1 - R_1 \cdot \nabla E \cdot R_1^* = -R_1 \cdot [R_1^* \times (\nabla \times E)].$$

One then uses the Maxwell equation

$$\partial_t B = -\nabla \times E,$$

which gives

$$\begin{aligned}
R_1^* \cdot \nabla E \cdot R_1 - R_1 \cdot \nabla E \cdot R_1^* &= R_1 \cdot (R_1^* \times \partial_t B) \\
&= R_1 \cdot \left[R_1^* \times \left(\frac{dB}{dt} - \dot{R}_0 \cdot \nabla B\right)\right] \\
&= \frac{dB}{dt} \cdot (R_1 \times R_1^*) - (\dot{R}_0 \cdot \nabla B) \cdot (R_1 \times R_1^*) \\
&= -i|R_1|^2 e_1 \cdot \left(\frac{dB}{dt} - (\dot{R}_0 \cdot \nabla B)\right) \\
&= -i|R_1|^2 \frac{d|B|}{dt} + i|R_1|^2 (\dot{R}_0 \cdot \nabla |B|).
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
R_1 \cdot (\dot{R}_0 \times (R_1^* \cdot \nabla B)) - \text{c.c.} &= \dot{R}_0 \cdot [(R_1^* \cdot \nabla B) \times R_1 - \text{c.c.}] \\
&= \frac{|R_1|^2}{2} \dot{R}_0 \cdot \Big([(e_2 - ie_3) \cdot \nabla B] \times (e_2 + ie_3) - \text{c.c.} \Big) \\
&= i|R_1|^2 \dot{R}_0 [(e_2 \cdot \nabla B) \times e_3 - (e_3 \cdot \nabla B) \times e_2] \\
&= i|R_1|^2 \dot{R}_0 \Big[-(e_2 \cdot \nabla |B|) e_2 - (e_3 \cdot \nabla |B|) e_3 \\
&\qquad + |B| [(e_2 \cdot \nabla e_1) \times e_3 - (e_3 \cdot \nabla e_1) \times e_2] \Big].
\end{aligned}
$$

Since

$$0 = \nabla \cdot B = |B| \nabla \cdot e_1 + e_1 \cdot \nabla |B|,$$

we make the substitution

$$-(e_2 \cdot \nabla |B|) e_2 - (e_3 \cdot \nabla |B|) e_3 = -\nabla |B| + (e_1 \cdot \nabla |B|) e_1 = -\nabla |B| - |B| (\nabla \cdot e_1) e_1$$

where

$$\nabla \cdot e_1 = e_2 \cdot \nabla e_1 \cdot e_2 + e_3 \cdot \nabla e_1 \cdot e_3.$$

Furthermore,

$$
\begin{aligned}
(e_2 \cdot \nabla e_1) \times e_3 &= (e_2 \cdot \nabla e_1) \times (e_1 \times e_2) \\
&= (e_2 \cdot \nabla e_1 \cdot e_2) e_1 - (e_2 \cdot \nabla e_1 \cdot e_1) e_2 \\
&= (e_2 \cdot \nabla e_1 \cdot e_2) e_1
\end{aligned}
$$

and

$$
\begin{aligned}
(e_3 \cdot \nabla e_1) \times e_2 &= (e_3 \cdot \nabla e_1) \times (e_3 \times e_1) \\
&= (e_3 \cdot \nabla e_1 \cdot e_1) e_3 - (e_3 \cdot \nabla e_1 \cdot e_3) e_1 \\
&= -(e_3 \cdot \nabla e_1 \cdot e_3) e_1 .
\end{aligned}
$$

We thus finally obtain

$$
\boxed{\frac{d}{dt}(|B||R_1|^2) = 0}.
$$

up to $O(\epsilon)$ corrections.

2.2.5 Longitudinal dynamics of the guiding center

Coming back to the physical units, we rewrite to leading order, the equation of motion of the guiding center along the magnetic field line in the form

$$\boxed{m\ddot{R}_0 \cdot \widehat{b} = q\widehat{b} \cdot E - \mu\widehat{b} \cdot \nabla|B|}$$

$\widehat{b}$: unit vector along the local magnetic field (previously denoted by e_1).

The parameter $\mu = \dfrac{q^2}{2m^2}|B||\rho_L|^2 = \dfrac{q^2}{m^2}|B||R_1|^2$, that at the order of the computation identifies with the magnetic moment, is an "ADIABATIC INVARIANT", in the sense that its variation is negligible at the order of the retained approximation.
The exact magnetic moment (that to leading order in ϵ reduces to $\dfrac{q^2}{m}|B||R_1|^2$) is actually constant to all order in ϵ (without being exactly constant), in the sense that it deviates from a constant by a quantity that goes to zero faster than any power of ϵ.

The equation for the longitudinal dynamics of the guiding center can be rewritten in various forms.

Writing the guiding center velocity $\quad \dot{R}_0 = v = U + v_{\|}\widehat{b} \quad$ with

$V = \widehat{b} \cdot v$ and $U \cdot \widehat{b} = 0, \qquad U = u_E + 0(\epsilon) \equiv \dfrac{E \times B}{|B|^2} + 0(\epsilon)$

$$m\widehat{b} \cdot \frac{dv}{dt} = q\widehat{b} \cdot E - \mu\widehat{b} \cdot \nabla|B|.$$

Since

$$\widehat{b} \cdot \frac{dv}{dt} = \frac{dv_{\|}}{dt} - U \cdot \frac{d\widehat{b}}{dt} = \frac{dv_{\|}}{dt} + \widehat{b} \cdot \frac{dU}{dt},$$

one has (Snyder et al. 1997)

$$\boxed{m\frac{dv_{\|}}{dt} = q\widehat{b} \cdot E - \mu\widehat{b} \cdot \nabla|B| - m\widehat{b} \cdot \frac{dU}{dt}}.$$

Differently, we can rewrite

$$m\frac{dv_{\|}}{dt} = q\widehat{b}\cdot E - \mu\widehat{b}\cdot\nabla|B| + mU\cdot\Big(\frac{D\widehat{b}}{Dt} + v_{\|}\widehat{b}\cdot\nabla\widehat{b}\Big),$$

where $\quad \dfrac{D}{Dt} = \partial_t + U\cdot\nabla \qquad$ (while $\dfrac{d}{dt} = \partial_t + v\cdot\nabla$).

Furthermore, to leading order the electric drift velocity $U = \dot{R}_{0\perp}$ obeys

$$E + U\times B = 0.$$

It follows that $\quad \partial_t B = -\nabla\times E = \nabla\times(U\times B)$
or

$$\frac{DB}{Dt} = B\cdot\nabla U - (\nabla\cdot U)B.$$

This result is used to compute

$$\frac{D\widehat{b}}{Dt} = \frac{1}{|B|^2}\Big(|B|\frac{DB}{Dt} - \frac{D|B|}{Dt}B\Big) \quad \text{where} \quad \frac{D|B|}{Dt} = \frac{B}{|B|}\cdot\frac{DB}{Dt}.$$

We get

$$\frac{D\widehat{b}}{Dt} = \widehat{b} \cdot \nabla U - (\widehat{b} \cdot \nabla U \cdot \widehat{b})\widehat{b}$$

and

$$U \cdot \frac{D\widehat{b}}{Dt} = \widehat{b} \cdot \nabla U \cdot U = \widehat{b} \cdot \nabla \frac{U^2}{2}.$$

Introducing the (local) curvature $\kappa = \widehat{b} \cdot \nabla \widehat{b}$ of the magnetic field line, we obtain

$$\boxed{m\frac{dv_{\|}}{dt} = q\widehat{b} \cdot E - \mu\widehat{b} \cdot \nabla|B| + m\widehat{b} \cdot \nabla\frac{U^2}{2} + mv_{\|}(U \cdot \kappa)}.$$

Furthermore, the longitudinal electric field $\widehat{b} \cdot E$ can be expressed as the gradient of a scalar field.

In terms of the magnetic potential A, one has

$$\nabla \times E = -\partial_t B = -\partial_t \nabla \times A$$

or $\quad E = -\partial_t A - \nabla\Phi, \quad$ where Φ is a scalar function.

Using Clebsch variables (see below),
which requires zero magnetic helicity

$$B = \nabla\alpha \times \nabla\beta \quad , \quad A = \alpha\nabla\beta, \quad \text{and}$$

$$\begin{aligned} E &= -\partial_t(\alpha\nabla\beta) - \nabla\Phi \\ &= -\partial_t\alpha\nabla\beta - \alpha\nabla\partial_t\beta - \nabla\Phi \\ &= -\partial_t\alpha\nabla\beta - \nabla(\alpha\partial_t\beta) + \partial_t\beta\nabla\alpha \;\; - \nabla\Phi \\ &= E' - \nabla\Phi' \end{aligned}$$

with $\quad E' = \partial_t\beta\nabla\alpha - \partial_t\alpha\nabla\beta \quad$ and $\quad \Phi' = \Phi + \alpha\partial_t\beta.$

Since $E' \cdot B = 0,$ $\quad$ one gets $\quad \widehat{b} \cdot E = -\widehat{b} \cdot \nabla\Phi',$ $\quad$ which yields

$$\boxed{m\frac{dv_{||}}{dt} = -q\widehat{b} \cdot \nabla\Phi' - \mu\widehat{b} \cdot \nabla|B| + m\widehat{b} \cdot \nabla\frac{U^2}{2} + mv_{||}(U \cdot \kappa)}.$$

Introduce the arc length s along the field line, such as $\kappa = \dfrac{\partial \widehat{b}}{\partial s}$.

We then rewrite $v_{\|}(U \cdot \kappa) = v_{\|}\left(U \cdot \dfrac{\partial \widehat{b}}{\partial s}\right) = -v_{\|}\widehat{b} \cdot \dfrac{\partial U}{\partial s}$, which leads to (Mjølhus and Wyller 1988)

$$\boxed{m\frac{dv_{\|}}{dt} = -q\widehat{b} \cdot \nabla\Phi' - \mu\widehat{b} \cdot \nabla|B| + m\widehat{b} \cdot \nabla\frac{U^2}{2} - mv_{\|}\widehat{b} \cdot \frac{\partial U}{\partial s}}.$$

2.2.6 Application: Magnetic mirror reflection

Writing

$$v_{||}\widehat{b}\cdot\nabla\Phi' = \frac{d\Phi'}{dt} - U\cdot\nabla\Phi' - \partial_t\Phi'$$
$$v_{||}\widehat{b}\cdot\nabla|B| = \frac{d|B|}{dt} - U\cdot\nabla|B| - \partial_t|B|,$$

one gets from the longitudinal velocity equation (Northrop 1961)

$$\frac{d}{dt}\Big(\frac{mv_{||}^2}{2}+\mu|B|+q\Phi'\Big) = U\cdot\Big[\nabla(\mu|B|+\Phi')+mv_{||}(\partial_t\widehat{b}+v_{||}\partial_s\widehat{b}+U\cdot\nabla\widehat{b})\Big]+\partial_t(\mu|B|+\Phi')$$

IMPORTANT SPECIAL CASE : to leading order, the electromagnetic field reduces to a STATIC MAGNETIC FIELD (small electric field). The above equation reduces to conservation of kinetic energy $\frac{m}{2}(v_{||}^2+|v_\perp|^2)$.

In this context, the adiabatic invariance of the magnetic moment μ plays an essential role for particle trapping, using "MAGNETIC MIRRORS" (see e.g. Chen 1984).

As a particle moves from a weak-field region to a strong-field region, it sees an increasing $|B|$. In order to keep μ constant, $|v_\perp|^2$ must increase. The conservation of the particle kinetic energy then implies that $|v_\parallel|$ should necessary decrease.

If $|B|$ reaches large enough values for the longitudinal velocity to vanish, the particle will be reflected back to the weak-field region.

The non uniform field of a pair of coils forms two magnetic mirrors between which a plasma can be trapped (MAGNETIC BOTTLE).

The trapping is however not perfect.

Question: Conditions for a particle to be trapped or to escape.

Let B_{min} be the minimum magnetic field that is present in the midplane and B_{max} the maximum magnetic field.

A particle with transverse velocity $v_{\perp 0}$ and longitudinal velocity $v_{||0}$ at the midplane will have a zero longitudinal velocity and a transverse velocity $v'_{\perp}$ at the turning point where the field amplitude is B'.

From the invariance of $\mu = \frac{1}{2}m|v_{\perp 0}|^2/B_{min} = \frac{1}{2}m|v'_{\perp}|^2/B'$ and the energy conservation $|v'_{\perp}|^2 = |v_{\perp 0}|^2 + v_{||0}^2$, one has

$$\frac{B_{min}}{B'} = \frac{|v_{\perp 0}|^2}{|v'_{\perp}|^2} = \frac{|v_{\perp 0}|^2}{|v_{\perp 0}|^2 + v_{||0}^2} \equiv \sin^2\theta$$

where θ is the PITCH ANGLE of the orbit in the weak-field region. Particles with smaller θ will mirror in regions of higher B. If θ is too small, B' exceeds B_{max}, and the particle does not mirror at all. The smallest θ of a confined particle is given by

$$\sin^2\theta_m = \frac{B_0}{B_{max}} = \frac{1}{R_m} \qquad (\mathrm{R_m} \text{ is the mirror ratio}).$$

The angle θ_m defines in the velocity space the boundary of the "loss cone", whose axis is along the longitudinal velocity.

Particles with velocity within the loss cone are not confined.

The magnetic field of the earth, being strong at the poles and weak at the equator forms a natural mirror with rather large R_m.

The oscillation (or bouncing) of a particle between the two mirror points M_1 and M_2 leads to define in this case of a static magnetic field, a second adiabatic invariant called the "LONGITUDINAL INVARIANT"
$$I = \int_{M_1}^{M_2} m v_{||} ds$$
(see e.g. Ferraro and Plumpton 1966).

For a detailed discussion of adiabatic invariants, see Northrop (1963, Chapter 3).

The number of adiabatic invariants (that is less or equal to the number of degrees of freedom) is determined by the number of periodicities.

For example in the case where the magnetic field is nowhere large enough to reflect the particle, the particle displays a periodic gyration about the magnetic field line but no periodicity in the motion along this line. The only adiabatic invariant is the magnetic moment.

In contrast, if the field is such that a particle is always trapped and oscillates between two mirrors, there will be a "second" or "longitudinal (also called parallel) invariant" associated with the parallel motion.

Finally, if the drift from line to line as the particle oscillates between mirrors carries the particle repeatedly around a close surface, there is a third periodicity associated with this motion and, as a consequence, a "third" or "flux adiabatic invariant" will exist.

2.2.7 A natural example of plasma confinement

THE (TWO) VAN ALLEN RADIATION BELTS: tori of charged particles around Earth, trapped by Earth's magnetic field.

Particles trapped in the VAN ALLEN RADIATION BELTS possess the three above periodicities and the associated adiabatic invariants (Northrop and Teller 1960): GYRATION around the geomagnetic field lines (typically thousands of time per second), NORTH-SOUTH BOUNCING motion along the field line (typically lasting 1/10 second), slow DRIFT by which they move from one field line to another one nearby, slightly rotated around the Earth's magnetic axis (typical time to circle the Earth, a few minutes). Viewed from the north pole, a positive ion gradually rotates clockwise, an electron counter-clockwise.

The drift of positive ions and negative electrons in opposite directions results in an electric current, called RING CURRENT that circulates clockwise around the Earth when viewed from north.

The magnetic field produced by the ring current contributes (rather slightly) to the magnetic field observed at the surface of the Earth.

However during MAGNETIC STORMS, charged particles are massively injected into the Van Allen belts from the outer magnetosphere, giving rise to a sharp increase in the ring current, and a corresponding decrease in the Earth's equatorial magnetic field.

These particles eventually precipitate out of the magnetosphere into the upper atmosphere at high lattitudes, giving rise to intense AURORAL ACTIVITY, with serious interference in electromagnetic communications and, in extreme cases, disruption of the electric power grids. The magnetic storm of March 13, 1989 was so severe that it tripped out the whole Hydro Quebec electric distribution system (see Fitzpatrick 1998 for a short discussion).

2.2.8 Guiding center dynamics along a magnetic field line

CLEBSCH VARIABLES α and β , such that $B = \nabla\alpha \times \nabla\beta$ (Kulsrud 1983).

If such scalars exist, B here defined is divergence free.

One also has $B \cdot \nabla\alpha = 0$ and $B \cdot \nabla\beta = 0$, which implies that α and β are constant along the magnetic field lines.
FIXING α AND β CHARACTERIZES A MAGNETIC FIELD LINE.

Jacobian of the transformation from coordinates $r = (x, y, z)$ to coordinates (α, β, s) is $J = \nabla s \cdot (\nabla\alpha \times \nabla\beta)$.

Since $\nabla s = \widehat{b}$, $J = |B|$ and $d\alpha d\beta$ represents the element of flux:
If a surface S cutting the field lines is parameterized by the coordinates α and β, $d\alpha d\beta$ is the flux through the corresponding element of area.

THE VARIABLES α, β ARE NOT UNIQUELY DEFINED.

2.2.9 Determination of a couple of Clebsch coordinates

Choose an arbitrary parameterization α, β' of the surface S and extend them through all space so that to satisfy $B \cdot \nabla\alpha = 0$ and $B \cdot \nabla\beta' = 0$, that is to say by keeping them constant on the magnetic field lines. Then,

$$B \times (\nabla\alpha \times \nabla\beta') = (B \cdot \nabla\beta')\nabla\alpha - (B \cdot \nabla\alpha)\nabla\beta' = 0.$$

So $B = g(\nabla\alpha \times \nabla\beta')$, where g is a scalar.

From $\nabla \cdot B = 0$, we have $0 = (\nabla\alpha \times \nabla\beta') \cdot \nabla g = \frac{1}{g}(B \cdot \nabla g)$,

so g is constant along the magnetic field lines and is thus a function of α and β'.

Now, choose β to satisfy $\frac{d\beta}{d\beta'} = g(\alpha, \beta')$ and get

$$B = \nabla\alpha \times \frac{d\beta}{d\beta'}\nabla\beta' = \nabla\alpha \times \nabla\beta.$$

After the variables α, β have been characterized to represent B at a given time, they can be specified at later times by prescribing

$$\partial_t\alpha + U \cdot \nabla\alpha = 0$$
$$\partial_t\beta + U \cdot \nabla\beta = 0.$$

One indeed has

$$\begin{aligned}
&\partial_t(\nabla\alpha \times \nabla\beta) - \nabla \times (U \times (\nabla\alpha \times \nabla\beta) = \\
&\nabla(\partial_t\alpha) \times \nabla\beta + \nabla\alpha \times \nabla(\partial_t\beta) - \nabla \times [(U \cdot \nabla\beta)\nabla\alpha - (U \cdot \nabla\alpha)\nabla\beta] = \\
&-\nabla(U \cdot \nabla\alpha) \times \nabla\beta - \nabla\alpha \times \nabla(U \cdot \nabla\beta) \\
&\qquad\qquad -\nabla(U \cdot \nabla\beta) \times \nabla\alpha + \nabla(U \cdot \nabla\alpha) \times \nabla\beta = 0.
\end{aligned}$$

2.2.10 Hamiltonian description

Instead of the arc length s along the magnetic field line, let us now introduce the coordinate σ such that σ has a constant value at a point moving at the electric drift velocity U. This implies

$$\partial_t \sigma + U \cdot \nabla \sigma = 0.$$

Changes in σ thus correspond to the true guiding center motion along the magnetic field line characterized by (α, β). Let $\chi = \dfrac{d\sigma}{ds}$, that represents the stretching of the magnetic field line. One has

$$\partial_t \chi + U \cdot \nabla \chi + \partial_s U \cdot \nabla \sigma = 0$$

or, since $\nabla \sigma = \chi \widehat{b}$, $\quad \frac{D\chi}{Dt} = -\chi \widehat{b} \cdot \partial_s U.$

The equation for the longitudinal velocity of the guiding center $m\dfrac{dv_{\|}}{dt} = -q\widehat{b} \cdot \nabla \Phi' - \mu \widehat{b} \cdot \nabla |B| + m\widehat{b} \cdot \nabla \dfrac{U^2}{2} - mv_{\|}\widehat{b} \cdot \partial_s U$ rewrites

$$m\frac{dv_{\|}}{dt} = \partial_s \Big(- q\Phi' - \mu |B| + m\frac{U^2}{2} \Big) + \frac{mv_{\|}}{\chi} \frac{D\chi}{Dt},$$

or $\quad (\frac{d}{dt} = \frac{D}{Dt} + v_{\|}\widehat{b}\cdot\nabla)$

$$m\frac{Dv_{\|}}{Dt} + mv_{\|}\widehat{b}\cdot\nabla v_{\|} = \partial_s\Big(- q\Phi' - \mu|B| + m\frac{U^2}{2}\Big) + \frac{mv_{\|}}{\chi}\frac{D\chi}{Dt},$$

where we can substitute

$$v_{\|}\widehat{b}\cdot\nabla v_{\|} = v_{\|}\partial_s v_{\|} = \frac{1}{2}\partial_s v_{\|}^2$$

to get

$$m\Big(\frac{Dv_{\|}}{Dt} - \frac{v_{\|}}{\chi}\frac{D\chi}{Dt}\Big) = \partial_s\Big(- q\Phi' - \mu|B| + m\frac{U^2}{2} - m\frac{v_{\|}^2}{2}\Big).$$

Defining $p = \dfrac{mv_{\|}}{\chi}$, we obtain

$$\frac{Dp}{Dt} = \partial_\sigma\Big(- q\Phi' - \mu|B| + m\frac{U^2}{2} - \frac{\chi^2 p^2}{2m}\Big),$$

that describes the motion of a guiding center of mass m and charge q along a moving field line.

On a magnetic field line (fixed α and β),

$$\dot{p} = \partial_\sigma \Big(- q\Phi' - \mu|B| + m\frac{U^2}{2} - \frac{\chi^2 p^2}{2m} \Big).$$

Similarly, under the same constraint,

$$\dot{\sigma} = \frac{d\sigma}{ds}\dot{s} = \chi v_{||} = \frac{\chi^2 p}{m} = \partial_p \Big(\frac{\chi^2 p^2}{2m} \Big).$$

Defining the Hamiltonian

$$\boxed{H = q\Phi' + \mu|B| - m\frac{U^2}{2} + \frac{1}{2m}\chi^2 p^2},$$

we write the guiding center equations in the Hamiltonian form

$$\boxed{\dot{p} = -\partial_\sigma H \quad , \quad \dot{\sigma} = \partial_p H}.$$

2.2.11 Guiding center distribution function

By Liouville theorem, we get that the guiding center distribution function $F(\alpha, \beta, \sigma, p, \mu, t)$ obeys

$$\partial_t F + \partial_p H \partial_\sigma F - \partial_\sigma H \partial_p F = 0,$$

that rewrites (Grad 1966, Mjølhus and Wyller 1988)

$$\boxed{\partial_t F + \frac{\chi^2 p}{m} \partial_\sigma F - \partial_\sigma \left(q\Phi' + \mu|B| - \frac{m}{2} U^2 + \frac{1}{2m} \chi^2 p^2 \right) \partial_p F = 0},$$

an equation to be written for each particle species.

3 Guiding center limit of Vlasov equation

Vlasov equation for the distribution function in velocity and position spaces of a given particle species

$$\partial_t f + v \cdot \nabla f + \frac{q}{m}\Big(E + v \times B\Big) \cdot \nabla_v f = 0.$$

$f d^3 v d^3 x$ is the fraction of particles in the phase-space volume $d^3 v d^3 x$ centered at velocity v and position x.

We concentrate on the variations of the distribution function with frequencies low compared to the gyrofrequency of the particles and scales large compared to their Larmor radius (Kulsrud 1983, see also Frieman, Davidson and Lanngdon 1966, Volvov 1966).

We are thus looking for a description where the small-scale dynamics is averaged out.

3.1 The guiding-center ordering

We assume that both the time scale and the length scales parallel and perpendicular to the local magnetic field are of order ϵ^{-1}.

This regime is a limiting case of the more general "gyrotropic ordering" that permits small-magnitude perturbations with typical transverse scale comparable with the Larmor radius.

The last term in Vlasov equation (that contains no space or time derivative) is dominant by a factor ϵ^{-1}.

One expands $f = f_0 + f_1 + \cdots$ where f_n is of order ϵ^n.

To leading order, $$\Big(E + v \times B\Big) \cdot \nabla_v f_0 = 0,$$

It is convenient to rewrite the velocity of individual particles:

$v = U + v' + v_{||}\widehat{b}$ where $U = \dfrac{E \times B}{|B|^2}$ is the electric drift velocity.

One easily checks that

$$E + v \times B = E_{||}\widehat{b} + v' \times B.$$

The leading order equation rewrites

$$E_{||}\partial_{v_{||}} f_0 + (v' \times B) \cdot \nabla_{v_\perp} f_0 = 0.$$

We now introduce the velocity cylindrical coordinates by writing

$$v' = v_\perp \cos\phi \; e_2 + v_\perp \sin\phi \; e_3.$$

Consequently,

$$\nabla_{v'} = e_2 \Big(\cos\phi \partial_{v_\perp} - \frac{\sin\phi}{v_\perp}\partial_\phi \Big) + e_3 \Big(\sin\phi \partial_{v_\perp} + \frac{\cos\phi}{v_\perp}\partial_\phi \Big).$$

It follows that

$$(v' \times B) \cdot \nabla_{v'} = -B\partial_\phi.$$

One thus has

$$E_{||}\partial_{v_{||}} f_0 - |B|\partial_\phi f_0 = 0.$$

If $E_{||} = O(1)$, then f_0 is constant along an helix in velocity space extending to infinite velocities, and f_0 cannot approach zero as $v_{||} \to \infty$.

Consequently, $E_{||} = O(\epsilon)$. Thus,

$$\boxed{\partial_\phi f_0 = 0}$$

and $f_0 = F_0(r, w, v_{||}, t)$ where $r = (x, y, z)$ and $w = \dfrac{v_\perp^2}{2}$.

Up to this axial symmetry in velocity space, f_0 is undetermined. It will be characterized by considering the next order in the expansion.

At the next order, we have

$$\boxed{\frac{q|B|}{m}\partial_\phi f_1 = \partial_t f_0 + v\cdot\nabla f_0 + \frac{q}{m}E_{||}\partial_{v_{||}} f_0}.$$

Since f_1 is periodic in ϕ, this equation requires the
SOLVABILITY CONDITION

$$\int_0^{2\pi}\Big(\partial_t f_0 + v\cdot\nabla f_0 + \frac{q}{m}E_{||}\partial_{v_{||}} f_0\Big)d\phi = 0.$$

When replacing $f_0(x,v,t)$ by $F_0(x,w,v_{||}\,t)$, one must take into account that w and $v_{||}$ now depend on space and time.

One has $\quad \partial_t f_0 = \partial_t F_0 + \partial_w F_0 \partial_t w + \partial_{v_{||}} F_0 \partial_t v_{||} \quad$ with

$$\partial_t w = v'\cdot\partial_t(-U - v_{||}\widehat{b}) = -v'\cdot\partial_t U - v_{||}v'\cdot\partial_t\widehat{b}$$
$$\partial_t v_{||} = \partial_t(v\cdot\widehat{b}) = (U+v')\cdot\partial_t\widehat{b}.$$

Consequently,

$$\int_0^{2\pi} \partial_t f_0 d\phi = 2\pi\Big(\partial_t F_0 + \partial_{v_\|} F_0\, U \cdot \partial_t \widehat{b}\Big).$$

One also directly has

$$\frac{q}{m}\int_0^{2\pi} E_\| \partial_{v_\|} f_0 d\phi = 2\pi \frac{q}{m} E_\| \partial_{v_\|} f_0.$$

Estimating $\displaystyle\int_0^{2\pi} v\cdot\nabla f_0 d\phi$ where $\nabla f_0 = \nabla F_0 + \partial_w F_0 \nabla w + \partial_{v_\|} F_0 \nabla v_\|$, requires to compute

$$\begin{aligned}
&\int_0^{2\pi} v\cdot\nabla w d\phi = \int_0^{2\pi} (U + v_\|\widehat{b} + v')\cdot[\nabla(v - U - v_\|\widehat{b})]\cdot v' d\phi \\
&= -\int_0^{2\pi} (v_\perp \cos\phi\ e_2 + v_\perp \sin\phi\ e_3)\cdot\nabla(U + v_\|\widehat{b})\cdot(v_\perp\cos\phi\ e_2 + v_\perp \sin\phi\ e_3) d\phi \\
&= -2\pi w[e_2\cdot\nabla(U + v_\|\widehat{b})\cdot e_2 + e_3\cdot\nabla(U + v_\|\widehat{b})\cdot e_3] \\
&= -2\pi w\Big(\nabla\cdot U - \widehat{b}\cdot\nabla U\cdot\widehat{b} + v_\|\nabla\cdot\widehat{b}\Big)
\end{aligned}$$

and

$$\begin{aligned}
\int_0^{2\pi} v\cdot\nabla v_{\|} d\phi &= \int_0^{2\pi}(v\cdot\nabla)(v\cdot\widehat{b})\,d\phi = \int_0^{2\pi} v\cdot\nabla\widehat{b}\cdot v\,d\phi \\
&= \int_0^{2\pi}(U+v_{\|}\widehat{b})\cdot\nabla\widehat{b}\cdot(U+v_{\|}\widehat{b})\,d\phi + \int_0^{2\pi} v'\cdot\nabla\widehat{b}\cdot v'\,d\phi \\
&= 2\pi\Big(U\cdot\nabla\widehat{b}\cdot U + v_{\|}\widehat{b}\cdot\nabla\widehat{b}\cdot U + w\nabla\cdot\widehat{b}\Big).
\end{aligned}$$

The solvability condition thus reads

$$\begin{aligned}
&\partial_t F_0 + (U+v_{\|}\widehat{b})\cdot\nabla F_0 - w\Big(\nabla\cdot U - \widehat{b}\cdot\nabla U\cdot\widehat{b} + v_{\|}\nabla\cdot\widehat{b}\Big)\partial_w F_0 \\
&\qquad +\Big(U\cdot\nabla\widehat{b}\cdot U + v_{\|}\widehat{b}\cdot\nabla\widehat{b}\cdot U + w\nabla\cdot\widehat{b} + U\cdot\partial_t\widehat{b} + \frac{q}{m}E_{\|}\Big)\partial_{v_{\|}} F_0 = 0.
\end{aligned}$$

Defining $\quad \dfrac{d}{dt} = \partial_t + (U + v_{\|}\widehat{b}) \cdot \nabla, \quad$ one rewrites

$$
\begin{aligned}
U \cdot \nabla\widehat{b} \cdot U + v_{\|}\widehat{b} \cdot \nabla\widehat{b} \cdot U + w\nabla \cdot \widehat{b} + U \cdot \partial_t\widehat{b} + \frac{q}{m}E_{\|} \quad &= \quad U \cdot \frac{d\widehat{b}}{dt} + w\nabla \cdot \widehat{b} + \frac{q}{m}E_{\|} \\
&= \quad -\widehat{b} \cdot \frac{dU}{dt} + \frac{v_\perp^2}{2}\nabla \cdot \widehat{b} + \frac{q}{m}E_{\|}.
\end{aligned}
$$

Furthermore $\quad w\partial_w = \dfrac{1}{2}v_\perp\partial_{v_\perp}. \quad$ One thus obtains

$$
\begin{aligned}
&\partial_t F_0 + (U + v_{\|}\widehat{b}) \cdot \nabla F_0 - \frac{v_\perp}{2}\Big(\nabla \cdot U - \widehat{b} \cdot \nabla U \cdot \widehat{b} + v_{\|}\nabla \cdot \widehat{b}\Big)\partial_{v_\perp} F_0 \\
&\qquad + \Big(- \widehat{b} \cdot \frac{dU}{dt} + \frac{v_\perp^2}{2}\nabla \cdot \widehat{b} + \frac{q}{m}E_{\|}\Big)\partial_{v_{\|}} F_0 = 0,
\end{aligned}
$$

or in terms of the magnetic moment $\mu = \dfrac{m|v_\perp|^2}{2|B|}$ (keeping the same notation for the distribution function for the sake of simplicity)

$$\partial_t F_0 + (U + v_{||}\widehat{b}) \cdot \nabla F_0 - \mu\Big(\nabla \cdot U - \widehat{b} \cdot \nabla U \cdot \widehat{b} + v_{||} \nabla \cdot \widehat{b}\Big)\partial_\mu F_0$$
$$+\Big(- \widehat{b} \cdot \frac{dU}{dt} + \frac{\mu}{m}|B|\nabla \cdot \widehat{b} + \frac{q}{m}E_{||}\Big)\partial_{v_{||}} F_0 = 0.$$

When considering the characteristics of this equations, the coefficients of $\partial_\mu F_0$ identifies with $\dot{\mu}$. Since, the magnetic moment is invariant at the order of the expansion, this coefficient should vanish.

One thus gets (Snyder et al. 1997)

$$\partial_t F_0 + (U + v_{||}\widehat{b}) \cdot \nabla F_0 + \Big(- \widehat{b} \cdot \frac{dU}{dt} + \frac{\mu}{m}|B|\nabla \cdot \widehat{b} + \frac{q}{m}E_{||}\Big)\partial_{v_{||}} F_0 = 0,$$

or using $0 = \nabla \cdot B = |B|\nabla \cdot \widehat{b} + \widehat{b} \cdot \nabla|B|$,

$$\boxed{\partial_t F_0 + (U + v_{||}\widehat{b}) \cdot \nabla F_0 + \Big(- \widehat{b} \cdot \frac{dU}{dt} - \frac{\mu}{m}\widehat{b} \cdot \nabla|B| + \frac{q}{m}E_{||}\Big)\partial_{v_{||}} F_0 = 0}.$$

When introducing the coordinate system (α, β, σ) such that

$$\begin{aligned} \partial_t \alpha + U \cdot \nabla \alpha &= 0 \\ \partial_t \beta + U \cdot \nabla \beta &= 0 \\ \partial_t \sigma + U \cdot \nabla \sigma &= 0, \end{aligned}$$

we define $F(\alpha, \beta, \sigma, \mu, v_{\|}, t) = F_0(x, y, z, \mu, v_{\|}, t)$, where μ is constant.

One easily checks that $\quad \partial_t F_0 + U \cdot \nabla F_0 = \partial_t F.$

Furthermore, $\quad \widehat{b} \cdot \nabla F_0 = \partial_s F = \dfrac{d\sigma}{ds} \partial_\sigma F$
where s is the arc length along the magnetic field line.

One gets

$$\boxed{\partial_t F + v_{\parallel} \frac{d\sigma}{ds} \partial_\sigma F + \Big(- \widehat{b} \cdot \frac{dU}{dt} - \frac{\mu}{m} \widehat{b} \cdot \nabla |B| + \frac{q}{m} E_{\parallel} \Big) \partial_{v_{\parallel}} F = 0}.$$

Using the previously established relation

$$\widehat{b} \cdot \frac{dU}{dt} = -U \cdot \frac{d\widehat{b}}{dt} = -\Big(\widehat{b} \cdot \nabla \frac{U^2}{2} \Big)$$

and replacing the variable $v_{\parallel}$ by $p = \dfrac{m v_{\parallel}}{\chi}$ with $\chi = \dfrac{d\sigma}{ds}$, one recovers the kinetic equation derived from the Hamiltonian description of the guiding center of an isolated particle.

3.2 Conservative form of the guiding center kinetic equation

Let us define

$$\boxed{g_{||} = -\widehat{b} \cdot \frac{dU}{dt} - \frac{\mu}{m}\widehat{b} \cdot \nabla|B| + \frac{q}{m}E_{||}}$$

and rewrite the kinetic equation in the form

$$\partial_t F + (U + v_{||}\widehat{b}) \cdot \nabla F + g_{||}\partial_{v_{||}} F = 0.$$

The induction equation $\partial_t B + U \cdot \nabla B = B \cdot \nabla U - (\nabla \cdot U)B$ gives

$$\begin{aligned} \partial_t|B| &= \frac{1}{|B|} B \cdot \partial_t B = \widehat{b} \cdot \partial_t B \\ &= -U \cdot \nabla B \cdot \widehat{b} + B \cdot \nabla U \cdot \widehat{b} - |B|(\nabla \cdot U). \end{aligned}$$

Since $\partial_{v_{||}} g_{||} = -\widehat{b} \cdot \nabla U \cdot \widehat{b}$, one writes the guiding center kinetic equation in the phase conservative form

$$\boxed{\partial_t(|B|F) + \nabla \cdot \Big(|B|F(U + v_{||}\widehat{b})\Big) + \partial_{v_{||}}\Big(|B|F g_{||}\Big) = 0}.$$

3.3 Hydrodynamic description

We now consider a plasma constituted of protons and electrons.

We denote by a subscript r the particle species, when considered in general, and by i or e when referring specifically to ions or electrons.

$u_r = \dfrac{\int v f_r d^3v}{\int f_r d^3v}$: hydrodynamic velocity of the particles of species r.

In particular, $\quad u_{\|r} = \dfrac{\int v_{\|} f_r d^3v}{\int f_r d^3v}$: parallel velocity.

The definition of these hydrodynamic quantities is at this step purely formal since we have not shown that a collisionless plasma can be regarded as a continuous medium. For example, the hydrodynamic velocity is the mean velocity (per unit volume) of an ensemble of non interacting particles rather than the velocity of an elementary volume of matter, as it is the case in usual hydrodynamics.

Using a hydrodynamic description for a neutral collisionless gas would be meaningless.

It makes sense in the case of a magnetized plasma because of the important role played by the self-consistent fields in plasma phenomena. These fields replace collisions for binding particles together and make it difficult for individual particle to act independently (see e.g. Volkov 1966). As seen below, electron velocities are very closely tied to the ion velocities, as the result of the self-consistent magnetic field.

At the leading order considered in the present theory, the distribution functions is approximated by that of the guiding center and is thus gyrotropic, in the sense that the transverse velocity is isotropically distributed.

It follows that when writing $v = U + v_{||} + v'$, one gets $u_r = U + u_{r||}$.

Let us now consider, the Maxwell equations where we neglect the displacement current (the velocities of the considered disturbances are much smaller than the speed of light) in the form

$$\begin{aligned}
&\nabla \cdot E = 4\pi \sum_r q_r n_r \int f_r d^3v \\
&\nabla \times B = 4\pi \sum_r q_r n_r \int v f_r d^3v \\
&\partial_t B = -\nabla \times E \\
&\nabla \cdot B = 0,
\end{aligned}$$

where $\displaystyle\int f_r d^3v = 1$.

Note that the equation for the distribution function is written in terms of the guiding system coordinates such as $d^3v = \dfrac{2\pi}{m_r}|B| d\mu_r dv_{\|}$, which makes the conservative form specially convenient.

Because of the assumption of slow variation of the electromagnetic field, the gradient operator involves a small parameter.

For an proton-electron plasma (where $q_i = -q_e = q$), one has to leading order

$$
\begin{aligned}
q(n_i - n_e) &= 0 \\
q(n_i u_i - n_e u_e) &= 0,
\end{aligned}
$$

that implies $n_i = n_e \equiv n$ and $u_i = u_e \equiv u$.

In particular, the parallel components satisfy $u_{\|i} = u_{\|e} \equiv u_{\|}$.

As a consequence, we write the plasma hydrodynamic velocity in the form $u = U + u_{\|}\widehat{b}$ where U is the electric drift velocity.

From the definition of the electric drift velocity, and the subdominant character of the parallel electric field component, one has $E = -U \times B$. Consequently, the induction equation has the usual form of ideal MHD

$$\partial_t B = \nabla \times (u \times B),$$

since the parallel velocity does not contribute to the cross product.

Furthermore, in the present asymptotics where both particle species have the same hydrodynamic velocity, one easily derives from the Vlasov equation, the usual equations obeyed by the plasma density $\rho = (m_i + m_e)n \approx m_i n$ and the plasma velocity u:

$$\begin{aligned} &\partial_t \rho + \nabla \cdot (\rho u) = 0 \\ &\rho(\partial_t u + u \nabla u) = j \times B - \nabla \cdot P. \end{aligned}$$

The current j is given by $j = \dfrac{1}{4\pi} \nabla \times B$.

The distribution functions being gyrotropic to leading order,
the pressure tensor $P = \sum_r m_r n_r \int (v-u) \otimes (u-v) f_r d^3v$
is also gyrotropic and rewrites $P = p_\perp (I - \widehat{b} \otimes \widehat{b}) + p_{||} \widehat{b} \otimes \widehat{b}$

with $p_\perp = \sum_r \frac{m_r}{2} \int v'^2 f_r d^3v$ and $p_{||} = \sum_r m_r \int (v_{||} - u_{||})^2 f_r d^3v$.

An equation for the pressure tensor can also be dirived from the Vlasov equation, that reduces to the so called CGL (for Chew, Goldberger and Low 1956) equations for the perpendicular and parallel pressures.

Retaining corrections to the guiding-center distribution function generates non-gyrotropic contribution to the pressure tensor, often called finite Larmor radius corrections (Friedman et al. 1966, Volkov 1966, Yajima 1966). The complexity of the calculation increases nevertheless rapidly with the order, when no additional assumption is made.

CGL equations for the perpendicular and parallel pressures involve a heat flux tensor that in the present asymptotics is also gyrotropic and characterized by two scalar functions.

Equations for higher order moments of the distribution functions can be written leading to a hierarchy that is usually unclosed, except in specific regimes such as the adiabatic regime that involves no heat fluxes.

In more general regimes, heuristic CLOSURES have been proposed. They involve the description of the heat fluxes and include Landau damping (Snyder et al. 1997, Passot and Sulem 2003).

Extensions also retaining deviations from gyrotropic pressure and heat flux tensors, associated with finite Larmor radius (FLR) corrections, were recentlty proposed in a weakly nonlinear regime where reductive perturbative expansions of Vlasov-Maxwell system lead to exact closures for MHD waves in the long-wave limit (Passot and Sulem 2004).

A more general kinetic theory called the GYROKINETIC DESCRIPTION has been developed during the last decades. Retaining small amplitude perturbations with transverse scales comparable to the particle gyroradius, it takes FLR effects into account.

A GYRO-FLUID DESCRIPTION is obtained when equations for hydrodynamic moments are derived from the gyrokinetic distribution function.

Such descriptions are extensively used in fusion plasmas.

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